REDUCTION OF SETS OF MATRICES TO A TRIANGULAR FORM

INDRANAND SINHA
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A TRIANGULAR FORM

I. Sinha

A set $\Omega$ of $n \times n$ matrices is said to have Property $T$ if the following two conditions are satisfied: (i) If $\Omega$ is looked upon as a set of linear transformations of a vector space $V$ of dimension $n$ then $V$ has an $\Omega$-decomposition into primary components; i.e. $V = V_1 \oplus \cdots \oplus V_i$, where the restrictions of the elements of $\Omega$ to any $V_i$ are primary linear transformations; and (ii) $V$ has an $\Omega$-composition series with 1-dimen-
ionsal composition-factors.

Our aim in this paper will be to characterize sets of non-
singular linear transformations having Property $T$.

The latter condition (ii) has been called Property $P$ for $\Omega$. It is known that Property $P$ is equivalent to simultaneous triangularisation of the elements of $\Omega$, and also to the existence of common characteristic vectors for all of $\Omega$ and to the fact that the additive commutators $AB - BA$ of pairs $A, B$ from $\Omega$ belong to the radical of the enveloping associative algebra generated by $\Omega$ ([5], page 592–600).

It is also known that $\Omega$ has Property $T$ if it is a commutative set of matrices ([1], page 41). Also for a Lie algebra of linear transformations of a finite dimensional vector space, it is known that Property $T$ is equivalent to the nilpotency (in the Lie-sense) of the Lie algebra ([2], page 878–879).

Throughout we shall identify $\Omega$ with a set of nonsingular linear transformations of a finite dimensional vectorspace $V$ over an algebraically closed field $F$ of characteristic zero.

2. DEFINITION. Let $A$ be any nonsingular linear transformation. Then it is known that $A$ can be factorized uniquely as $A = SU$, where $U$ is a unipotent linear transformation and $S$ is semi-simple, and $SU = US$, ([1], page 41). $U$ is called the unipotent part of $A$ and $S$ is called the semi-simple part of $A$. This will be referred to as the Jordan-
multiplicative decomposition of $A$.

$S$ can also be characterized by the fact that the module determined by it is completely reducible; and hence, over an algebraically closed field, $S$ is representable as an $n \times n$ diagonal matrix.

We shall let $\Omega_s$ be the set of the semi-simple parts of the elements

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of $\Omega$, and $\Omega_t$ be the set of their unipotent parts.

We prove,

**Theorem 1.** Let $V$ have an $\Omega$-composition-series with 1-dimensional composition-factors. Then a necessary and sufficient condition for $\Omega$ to have Property $T$ is that $\Omega_s$ commutes with $\Omega$ elementwise.

**Proof.** For the necessity part of the theorem, we observe that if $\Omega$ has Property $T$, then the matrices in $\Omega$ can be assumed to be a direct sum of triangular blocks, each of which is in the triangular form having a single characteristic root along the diagonal. Thus any element $A$ in $\Omega$ can be supposed to be of the form,

$$A = \begin{bmatrix}
\lambda & * \\
\vdots & \ddots \\
0 & \cdots & \lambda
\end{bmatrix},$$

where * denotes possible nonzero entries. Then $A = (\lambda \cdot I) \cdot (\lambda^{-1} \cdot A) = A_s \cdot A_u$, where $A_s = \lambda \cdot I$ and $A_u = \lambda^{-1} A$, so that $A_s A_u = A_u A_s$, and $A_s$ is semi-simple while $A_u$ is unipotent. Then from the uniqueness of the Jordan multiplicative decomposition, we conclude that $A_s$ is in $\Omega_s$ and $A_u$ is in $\Omega_u$. Thus $\Omega_s$ consists of scalar matrices only and hence commutes with $\Omega$ elementwise.

For the sufficiency part of the theorem, let $A$ be any element of $\Omega$, and $A = A_s A_u$ be its Jordan-multiplicative decomposition, so that $A_s$ is in $\Omega_s$ and $A_u$ is in $\Omega_u$. Let $V = V_{\lambda_1 A} \oplus \cdots \oplus V_{\lambda_n A}$, where $\lambda_i$ are the distinct characteristic roots of $A_s$ and hence of $A$, be a decomposition of $V$ into primary components with respect to $A_s$. Since $A_s$ is semi-simple, so for any vector $\mathcal{U}$ in $V_{\lambda_i A}$, we have,

$$\mathcal{U} \cdot (A_s - \lambda_i I) = 0.$$

If $B$ is an arbitrary element of $\Omega$, then $(\mathcal{U} B)(A_s - \lambda_i I) = \mathcal{U}(A_s - \lambda_i I)B = 0$, since $\Omega_s$ is assumed to commute with $\Omega$ elementwise. Thus each component $V_{\lambda_i A}$ is invariant with respect to the whole of $\Omega$. Also the restriction of $A_u$ to any $V_{\lambda_i A}$ is itself unipotent. Therefore the restriction of $A$ to each $V_{\lambda_i A}$ is primary.

If some $C$ in $\Omega$ is not primary on any of the $V_{\lambda_i A}$, we repeat the process with $C$ in place of $A$, so that we can conclude that $V$ has an $\Omega$-decomposition $V = V_1 \oplus \cdots \oplus V_r$, such that the restrictions of the elements of $\Omega$ to any $V_i$ are primary.

Combined with the hypothesis on the existence of $\Omega$-composition series with 1-dimensional composition factors, the above conclusion gives Property $T$ for $\Omega$. 
The following analogue of McCoy's result in ([5], page 593) can be easily verified.

**Lemma 1.** $\Omega$ has Property $P$ if and only if for each pair $A, B$ of elements in $\Omega$, $ABA^{-1}B^{-1} - I$ lies in the radical of the enveloping associative algebra $\bar{\Omega}$ generated by $\Omega$.

Using this we conclude at once,

**Theorem 2.** A set of necessary and sufficient conditions for $\Omega$ to have property $T$ is that,

(i) $\Omega$, commute with $\Omega$ elementwise, and

(ii) for every pair $A, B$ of elements in $\Omega$, $ABA^{-1}B^{-1} - I$ lies in the radical of the enveloping associative algebra $\bar{\Omega}$ generated by $\Omega$.

3. In this section we limit $\Omega$ to be an algebraic group ([1], page 29). The following results are well-known and the proofs are omitted here.

**Lemma 2 (Lie-Kolchin).** A connected algebraic group $\Omega$ has Property $P$ if and only if it is solvable: ([3], page 30).

**Lemma 3.** If $\Omega$ is a connected nilpotent algebraic group, then $\Omega_s$ is contained in the centre ([1], page Theorem 11.1).

**Lemma 4.** If $N$ is an invariant commutative algebraic subgroup of a connected algebraic group $\Omega$, and consists of semi-simple elements only, then $N$ is contained in the centre of $\Omega$ ([1], page 45, Proposition 7-9).

It may be relevant to recall that connectivity is taken here in the sense of the Zariski-Topology in $\Omega([3], page 26)$.

**Theorem 3.** A necessary and sufficient condition for a connected algebraic group $\Omega$ to have Property $T$ is that $\Omega$ be nilpotent.

**Proof.** For the sufficiency we observe that by Lemma 3 $\Omega_s$ commutes with $\Omega$ elementwise. Then by Lemma 2, $\Omega$ has Property $P$. Thus, Theorem 1 implies that $\Omega$ has Property $T$.

For the necessity, let $\Omega$ have Property $T$. Again we can assume that any element $A$ of $\Omega$ has the form,$\quad A = \begin{bmatrix} \lambda & * \\ \cdot & \cdot \\ 0 & \lambda \end{bmatrix} = (\lambda \cdot I) \cdot (\lambda^{-1}A) = A_s \cdot A_u.$
If \( F^* \) denotes the multiplicative group of the nonzero elements of the ground field \( F \), then \( \Omega \) is isomorphic to the external direct product, \( F^* \times U \), where \( U \) is the group of unipotent matrices (\( \lambda^{-1}A \)).

\( U \) is a group of unipotent matrices in triangular form, and such groups are known to be nilpotent, so \( \Omega \) being a product of two nilpotent groups, is itself nilpotent.

Another characterization of Property \( T \) can be obtained in,

**Theorem 4.** A necessary and sufficient condition for a connected algebraic group \( \Omega \) to have Property \( T \) is that \( \ Omega_s \) be an algebraic subgroup contained in the centre of \( \Omega \).

**Proof.** If \( \Omega \) has Property \( T \), then by Theorem 3, \( \Omega \) is nilpotent, so that \( \Omega_s \) is an algebraic subgroup of the centre, ([1], page 53, Theorem 11.1).

Conversely, let \( \Omega_s \) be an algebraic subgroup of the centre of \( \Omega \). Then it can be shown that \( \Omega \) is equal to the internal direct product \( \Omega_s \times \Omega_u \) ([1], page 53, Theorem 11.1). Therefore, we at once have that \( \Omega/\Omega_u \cong \Omega_s \) is Abelian and hence \( \Omega_s \cong \) the commutator-subgroup of \( \Omega \). From this it follows at once that \( \Omega \) has Property \( P \). (See for example, the proof of Theorem 4.11 in [3], page 31).

Now \( \Omega_s \) commutes with \( \Omega \) elementwise, and \( \Omega \) has Property \( P \). Therefore, by virtue of Theorem 1, \( \Omega \) has Property \( T \).

The converse part of the above theorem has an interesting generalization to arbitrary subgroups of the general linear group \( GL(n, F) \). In order to exhibit it, we shall use the notation \( \langle \Omega_s \rangle \) for the group generated by \( \Omega_s \) in \( GL(n, F) \). This is necessitated by the fact that we now drop the restriction of algebraic connectivity for \( \Omega \), so that \( \Omega_s \) may no longer be a part of \( \Omega \). We now state.

**Theorem 5.** Let \( \Omega \) be a subgroup of \( GL(n, F) \) such that \( \Omega_s \) commutes with \( \Omega \) elementwise. Then \( \Omega \) has Property \( T \) and is nilpotent of class at most \( (n - 1) \).

**Proof.** We divide the proof in four parts.

(i) First observe that if the underlying vector space \( V \) is irreducible under \( \Omega \cup \Omega_u \), then \( V \) is irreducible under \( \Omega \). For, suppose to the contrary that \( V_1 \) is a proper minimal invariant \( \Omega \)-subspace of \( V \). Then

\[ \text{The author is highly indebted to the referee for conveying this theorem and its proof to him.} \]
for each \( u \in \langle \Omega_s \rangle \), \( V_iu \) is a minimal invariant \( \Omega \)-subspace, because \( u \) commutes with every element of \( \Omega \). Now \( \sum V_iu \), where the summation is over all \( u \) in \( \langle \Omega_s \rangle \), is invariant under \( \Omega \cup \Omega_s \) and is therefore the whole of \( V \) in view of the irreducibility of \( V \) with respect to \( \Omega \cup \Omega_s \). Therefore, \( V = W_1 \oplus \cdots \oplus W_k \), where \( W_i = V_iu_i \), \( (u_i \in \langle \Omega_s \rangle) \), and the \( W_i \) are \( \Omega \)-invariant.

Corresponding to this decomposition, there is a basis for \( V \) such that the matrix \( X \) in \( \Omega \) has the block-decomposition,

\[
X = \begin{bmatrix}
X_1 & 0 \\
X_2 & \ddots \\
\vdots & \ddots & \ddots \\
0 & \cdots & X_k
\end{bmatrix},
\]

where the \( X_i \) are square-blocks of dimension \( n/k \). The corresponding matrix for \( X_s \) in \( \Omega_s \) is clearly,

\[
X_s = \begin{bmatrix}
(X_1)_s & 0 \\
(X_2)_s & \ddots \\
\vdots & \ddots & \ddots \\
0 & \cdots & (X_k)_s
\end{bmatrix},
\]

where, as usual, \( (X_i)_s \) denotes the semisimple part of \( X_i \).

However, this implies that \( V \) is reducible with respect to \( \Omega \cup \Omega_s \), contrary to the hypothesis.

(ii) Next, we assert that if \( V \) is irreducible with respect \( \Omega \cup \Omega_s \), then \( V \) has dimension 1. For, by (i) we can assume that \( V \) is \( \Omega \)-irreducible. Since \( \Omega_s \) commutes with \( \Omega \) elementwise and \( F \) is algebraically closed, it follows by Schur's Lemma, ([6]; Theorem 27·3), that \( \Omega_s \) is a set of scalars; i.e., \( \Omega_s \subseteq F^* \). \( I \), where \( I \) is the identity matrix, and \( F^* \) denotes the multiplicative group of nonzero elements of the base field \( F \).

Let \( \Omega_1 = \{ X \in \Omega \cdot F^* | \det X = 1 \} \). As \( F \) is algebraically closed, so \( \Omega \subseteq \Omega_1 \cdot F^* \), and therefore \( \Omega_1 \) is also an irreducible group. Since \( (\Omega \cdot F^*)_s = \Omega_s \cdot F^* = F^* \cdot I \), so each \( X \in \Omega_1 \) has a unique characteristic root of multiplicity \( n \). For every \( X \in \Omega_1 \), we have trace \( X = \text{trace} X_s \), so that the set \( \{ \text{trace} X | X \in \Omega_1 \} \subseteq \{ \lambda \in F^* | \lambda^n = 1 \} \), and so is finite. But, by an argument of Burnside, an irreducible group with only a finite set of trace-values is finite ([6], Theorem 36·1). Thus \( \Omega_1 \) is a finite irreducible group. Since characteristic of \( F \) is zero, so every element of \( \Omega_1 \) is semi-simple and \( \Omega_1 = (\Omega_1)_s \subseteq (\Omega \cdot F^*)_s = F^* \cdot I \). So \( \Omega_1 \) is an irreducible group of scalars which is possible only when the dimension \( n = 1 \).
(iii) Now we prove that $\Omega$ has Property $P$. For, let $V$ have a basis with respect to which $\Omega \cup \Omega_s$ has the form,

$$X = \begin{bmatrix} X_1 & * \\ X_2 \\ \vdots \\ 0 & X_n \end{bmatrix},$$

with diagonal-blocks $X_i$, and possible nonzero entries only above these diagonal blocks. Suppose $X_i$'s cannot be reduced any further. Then the mapping $X \rightarrow X_i$ defines a homomorphism of $\langle \Omega \cup \Omega_s \rangle$ for each $i$, such that the images of $\Omega$ and $\Omega_s$ are, say, $\Omega^{(i)}$ and $\Omega_s^{(i)}$ respectively, for a fixed $i$. Clearly, $(X_i)_i = (X)_i$, and so $\Omega_s^{(i)} = (\Omega^{(i)})_s$. Since $\Omega \cup \Omega_s$ cannot be further reduced, $\Omega^{(i)} \cup \Omega_s^{(i)}$ is irreducible. Also $\Omega^{(i)}$ is a group and $\Omega_s^{(i)}$ commutes with $\Omega^{(i)}$ elementwise. Hence by (ii), each block $X_i$ must be of dimension 1.

(iv) Finally, by Theorem 1, combined with (iii), we immediately conclude that $\Omega$ has Property $T$. Also from the proof of Theorem 3, it then follows that $\Omega$ is nilpotent of class at-most $(n - 1)$, for $n \geq 2$, since the group of all upper triangular unipotent matrices is known to be nilpotent of class $(n - 1)$.

We remark that Theorem 5 shows that a nilpotent connected algebraic group has nilpotency class $\leq (n - 1)$. On the other hand, matrix groups of degree $n$ and arbitrary nilpotency-class $k \geq 1$ are known to exist, ([7], page 57). Thus we observe that such groups cannot have Property $T$ for $k > n$. Thus, for a general matrix-group nilpotency does not imply Property $T$.

**Corollary 1.** If $\Omega$ is a connected algebraic group, then a necessary and sufficient condition for $\Omega$ to have Property $T$ is that $\Omega_s$ be an invariant commutative algebraic subgroup of $\Omega$.

This follows at once from Lemma 4 and Theorem 4.

Combining the above results we deduce the following equivalence of propositions.

**Corollary 2.** For a connected algebraic group $\Omega$, the followings are equivalent,

(i) $\Omega$ is nilpotent,
(ii) $\Omega$ has Property $T$,
(iii) $\Omega$ is an invariant commutative algebraic subgroup of $\Omega$, 
(iv) $\Omega$ is an algebraic subgroup in the centre of $\Omega$.

Lastly we note that connectivity is an essential part in our hypothesis as can be seen by taking $\Omega$ to be a non abelian finite nilpotent group. Then $\Omega$ can have Property $P$ or Property $T$ if and only if it is commutative. This follows at once from the Theorem of Maschke about the complete reducibility of finite groups: ([4]).

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