Pacific Journal of Mathematics

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Vol. 15, No. 2 October 1965

PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

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A set Ω of $n \times n$ matrices is said to have Property T if the following two conditions are satisfied: (i) If Ω is looked upon as a set of linear transformations of a vector space V of dimension n then V has an Ω-decomposition into primary components; i.e. $V = V_1 \oplus \cdots \oplus V_t$, where the restrictions of the elements of Ω to any V_i are primary linear transformations; and (ii) V has an Ω-composition series with 1-dimensional composition-factors.

Our aim in this paper will be to characterize sets of nonsingular linear transformations having Property T.

The latter condition (ii) has been called Property P for Ω. It is known that Property P is equivalent to simultaneous triangularisation of the elements of Ω, and also to the existence of common characteristic vectors for all of Ω and to the fact that the additive commutators $AB - BA$ of pairs A, B from Ω belong to the radical of the enveloping associative algebra generated by Ω ([5], page 592–600).

It is also known that Ω has Property T if it is a commutative set of matrices ([1], page 41). Also for a Lie algebra of linear transformations of a finite dimensional vector space, is known that Property T is equivalent to the nilpotency (in the Lie-sense) of the Lie algebra ([2], page 878–879).

Throughout we shall identify Ω with a set of nonsingular linear transformations of a finite dimensional vectorspace V over an algebraically closed field F of characteristic zero.

2. DEFINITION. Let A be any nonsingular linear transformation. Then it is known that A can be factorized uniquely as $A = SU$, where U is a unipotent linear transformation and S is semi-simple, and $SU = US$, ([1], page 41). U is called the unipotent part of A and S is called the semi-simple part of A. This will be referred to as the Jordan-multiplicative decomposition of A.

S can also be characterized by the fact that the module determined by it is completely reducible; and hence, over an algebraically closed field, S is representable as an $n \times n$ diagonal matrix.

We shall let Ω_s be the set of the semi-simple parts of the elements

Received December 3, 1963, and in revised form March 25, 1964. The author wishes to express his gratitude to Prof. C. W. Curtis for the help received in this piece of research.

of Ω, and Ω_t be the set of their unipotent parts.

We prove,

THEOREM 1. *Let V have an Ω-composition-series with 1-dimensional composition-factors. Then a necessary and sufficient condition for Ω to have Property T is that Ω_s commutes with Ω elementwise.*

Proof. For the necessity part of the theorem, we observe that if Ω has Property T, then the matrices in Ω can be assumed to be a direct sum of triangular blocks, each of which is in the triangular form having a single characteristic root along the diagonal. Thus any element A in Ω can be supposed to be of the form,

$$A = \begin{bmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{bmatrix},$$

where * denotes possible nonzero entries. Then $A = (\lambda \cdot I) \cdot (\lambda^{-1} \cdot A) = A_s \cdot A_u$, where $A_s = \lambda \cdot I$ and $A_u = \lambda^{-1} A$, so that $A_s A_u = A_u A_s$, and A_s is semi-simple while A_u is unipotent. Then from the uniqueness of the Jordan multiplicative decomposition, we conclude that A_s is in Ω_s and A_u is in Ω_u. Thus Ω_s consists of scaler matrices only and hence commutes with Ω elementwise.

For the sufficiency part of the theorem, let A be any element of Ω, and $A = A_s A_u$ be its Jordan-multiplicative decomposition, so that A_s is in Ω_s and A_u is in Ω_u. Let $V = V_{\lambda_1, A} \oplus \cdots \oplus V_{\lambda_t, A}$, where λ_i are the distinct characteristic roots of A_s and hence of A, be a decomposition of V into primary components with respect to A_s. Since A_s is semi-simple, so for any vector $\mathscr{U}$ in $V_{\lambda_k, A}$, we have,

$$\mathscr{U} \cdot (A_s - \lambda_k I) = 0.$$

If B is an arbitrary element of Ω, then $(\mathscr{U} B)(A_s - \lambda_k I) = \mathscr{U}(A_s - \lambda_k I) B = 0$, since Ω_s is assumed to commute with Ω elementwise. Thus each component $V_{\lambda_k, A}$ is invariant with respect to the whole of Ω. Also the restriction of A_u to any $V_{\lambda_k, A}$ is itself unipotent. Therefore the restriction of A to each $V_{\lambda_k, A}$ is primary.

If some C in Ω is not primary on any of the $V_{\lambda_k, A}$, we repeat the process with C in place of A, so that we can conclude that V has an Ω-decomposition $V = V_1 \oplus \cdots \oplus V_t$, such that the restrictions of the elements of Ω to any V_i are primary.

Combined with the hypothesis on the existence of Ω-composition series with 1-dimensional composition factors, the above conclusion gives Property T for Ω.

The following analogue of McCoy's result in ([5], page 593) can be easily verified.

LEMMA 1. *Ω has Property P if and only if for each pair A, B of elements in Ω, $ABA^{-1}B^{-1} - I$ lies in the radical of the enveloping associative algebra $\bar{\Omega}$ generated by Ω.*

Using this we conclude at once,

THEOREM 2. *A set of necessary and sufficient conditions for Ω to have property T is that,*

(i) *Ω_s commute with Ω elementwise, and*

(ii) *for every pair A, B of elements in Ω, $ABA^{-1}B^{-1} - I$ lies in the radical of the enveloping associative algebra $\bar{\Omega}$ generated by Ω.*

3. In this section we limit Ω to be an algebraic group ([1], page 29). The following results are well-known and the proofs are omitted here.

LEMMA 2 (Lie-Kolchin). *A connected algebraic group Ω has Property P if and only if it is solvable:* ([3], page 30).

LEMMA 3. *If Ω is a connected nilpotent algebraic group, then Ω_s is contained in the centre* ([1], page *Theorem* 11.1).

LEMMA 4. *If N is an invariant commutative algebraic subgroup of a connected algebraic group Ω, and consists of semi-simple elements only, then N is contained in the centre of Ω* ([1], page 45, *Proposition* 7·9).

It may be relevant to recall that connectivity is taken here in the sense of the Zariski-Topology in Ω([3], page 26).

THEOREM 3. *A necessary and sufficient condition for a connected algebraic group Ω to have Property T is that Ω be nilpotent.*

Proof. For the sufficiency we observe that by Lemma 3 Ω_s commutes with Ω elementwise. Then by Lemma 2, Ω has Property P. Thus, Theorem 1 implies that Ω has Property T.

For the necessity, let Ω have Property T. Again we can assume that any element A of Ω has the form,

$$A = \begin{bmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} = (\lambda \cdot I)\cdot(\lambda^{-1}A) = A_s \cdot A_u \cdot$$

If F^* denotes the multiplicative group of the nonzero elements of the ground field F, then Ω is isomorphic to the external direct product, $F^* \times U$, where U is the group of unipotent matrices $(\lambda^{-1} \cdot A)$.

U is a group of unipotent matrices in triangular form, and such groups are known to be nilpotent, so Ω being a product of two nilpotent groups, is itself nilpotent.

Another characterization of Property T can be obtained in,

THEOREM 4. *A necessary and sufficient condition for a connected algebraic group Ω to have Property T is that Ω_s be an algebraic subgroup contained in the centre of Ω.*

Proof. If Ω has Property T, then by Theorem 3, Ω is nilpotent, so that Ω_s is an algebraic subgroup of the centre, ([1], page 53, Theorem 11·1).

Conversely, let Ω_s be an algebraic subgroup of the centre of Ω. Then it can be shown that Ω is equal to the internal direct product $\Omega_s \times \Omega_u$ ([1], page 53, Theorem 11·1). Therefore, we at once have that $\Omega/\Omega_u \cong \Omega_s$ is Abelian and hence $\Omega_u \supseteq$ the commutator-subgroup of Ω. From this it follows at once that Ω has Property P. (See for example, the proof of Theorem 4·11 in [3], page 31).

Now Ω_s commutes with Ω elementwise, and Ω has Property P. Therefore, by virtue of Theorem 1, Ω has Property T.

The converse part of the above theorem has an interesting generalization to arbitrary subgroups of the general linear group $GL(n, F)$. In order to exhibit it, we shall use the notation $\langle \Omega_s \rangle$ for the group generated by Ω_s in $GL(n, F)$. This is necessitated by the fact that we now drop the restriction of algebraic connectivity for Ω, so that Ω_s may no longer be a part of Ω. We now state.

THEOREM 5. *Let Ω be a subgroup of $GL(n, F)$ such that Ω_s commutes with Ω elementwise. Then Ω has Property T and is nilpotent of class at most $(n-1)$.*[1]

Proof. We divide the proof in four parts.

(i) First observe that if the underlying vector space V is irreducible under $\Omega \cup \Omega_s$, then V is irreducible under Ω. For, suppose to the contrary that V_1 is a proper minimal invariant Ω-subspace of V. Then

[1] The author is highly indebted to the referee for conveying this theorem and its proof to him.

for each u in $\langle \Omega_s \rangle$, $V_1 u$ is a minimal invariant Ω-subspace, because u commutes with every element of Ω. Now $\sum V_1 u$, where the summation is over all u in $\langle \Omega_s \rangle$, is invariant under $\Omega \cup \Omega_s$ and is therefore the whole of V in view of the irreducibility of V with respect to $\Omega \cup \Omega_s$. Therefore, $V = W_1 \oplus \cdots \oplus W_k$, where $W_i = V_1 u_i$, (u_i in $\langle \Omega_s \rangle$), and the W_i are Ω-invariant.

Corresponding to this decomposition, there is a basis for V such that the matrix X in Ω has the block-decomposition,

$$X = \begin{bmatrix} X_1 & & & 0 \\ & X_2 & & \\ & & \ddots & \\ 0 & & & X_k \end{bmatrix},$$

where the X_i are square-blocks of dimension n/k. The corresponding matrix for X_s in Ω_s is clearly,

$$X_s = \begin{bmatrix} (X_1)_s & & & 0 \\ & (X_2)_s & & \\ & & \ddots & \\ 0 & & & (X_h)_s \end{bmatrix},$$

where, as usual, $(X_i)_s$ denotes the semisimple part of X_i.

However, this implies that V is reducible with respect to $\Omega \cup \Omega_s$, contrary to the hypothesis.

(ii) Next, we assert that if V is irreducible with respect $\Omega \cup \Omega_s$, then V has dimension 1. For, by (i) we can assume that V is Ω-irreducible. Since Ω_s commutes with Ω elementwise and F is algebraically closed, it follows by Schur's Lemma, ([6]; Theorem 27·3), that Ω_s is a set of scalars; i.e., $\Omega_s \subseteq F^*$. I, where I is the identity matrix, and F^* denotes the multiplicative group of nonzero elements of the base field F.

Let $\Omega_1 = \{X \text{ in } \Omega \cdot F^* \mid \det X = 1\}$. As F is algebraically closed, so $\Omega \subseteq \Omega_1 \cdot F^*$, and therefore Ω_1 is also an irreducible group. Since $(\Omega \cdot F^*)_s = \Omega_s \cdot F^* = F^* \cdot I$, so each X in Ω_1 has a unique characteristic root of multiplicity n. For every X in Ω_1, we have trace $X =$ trace X_s, so that the set $\{\text{trace } X \mid X \text{ in } \Omega_1\} \subseteq \{\lambda \text{ in } F^* \mid \lambda^n = 1\}$, and so is finite. But, by an argument of Burnside, an irreducible group with only a finite set of trace-values is finite ([6], Theorem 36·1). Thus Ω_1 is a finite irreducible group. Since characteristic of F is zero, so every element of Ω_1 is semi-simple and $\Omega_1 = (\Omega_1)_s \subseteq (\Omega \cdot F^*)_s = F^* \cdot I$. So Ω_1 is an irreducible group of scalars which is possible only when the dimension $n = 1$.

(iii) Now we prove that Ω has Property P. For, let V have a basis with respect to which $\Omega \cup \Omega_s$ has the form,

$$X = \begin{bmatrix} X_1 & & & * \\ & X_2 & & \\ & & \ddots & \\ 0 & & & X_n \end{bmatrix},$$

with diagonal-blocks X_i, and possible nonzero entries only above these diagonal blocks. Suppose X_i's cannot be reduced any further. Then the mapping $X \to X_i$ defines a homomorphism of $\langle \Omega \cup \Omega_s \rangle$ for each i, such that the images of Ω and Ω_s are, say, $\Omega^{(i)}$ and $\Omega_s^{(i)}$ respectively, for a fixed i. Clearly, $(X_i)_s = (X_s)_i$, and so $\Omega_s^{(i)} = (\Omega^{(i)})_s$. Since $\Omega \cup \Omega_s$ cannot be further reduced, $\Omega^{(i)} \cup \Omega_s^{(i)}$ is irreducible. Also $\Omega^{(i)}$ is a group and $\Omega_s^{(i)}$ commutes with $\Omega^{(i)}$ elementwise. Hence by (ii), each block X_i must be of dimension 1.

(iv) Finally, by Theorem 1, combined with (iii), we immediately conclude that Ω has Property T. Also from the proof of Theorem 3, it then follows that Ω is nilpotent of class at-most $(n-1)$, for $n \geq 2$, since the group of all upper triangular unipotent matrices is known to be nilpotent of class $(n-1)$.

We remark that Theorem 5 shows that a nilpotent connected algebraic group has nilpotency class $\leq (n-1)$. On the other hand, matrix groups of degree n and arbitrary nilpotency-class $k \geq 1$ are known to exist, ([7], page 57). Thus we observe that such groups cannot have Property T for $k \geq n$. Thus, for a general matrix-group nilpotency does not imply Property T.

COROLLARY 1. *If Ω is a connected algebraic group, then a necessary and sufficient condition for Ω to have Property T is that Ω_s be an invariant commutative algebraic subgroup of Ω.*

This follows at once from Lemma 4 and Theorem 4.

Combining the above results we deduce the following equivalence of propostions.

COROLLARY 2. *For a connected algebraic group Ω, the followings are equivalent,*

(i) *Ω is nilpotent,*
(ii) *Ω has Property T,*
(iii) *Ω_s is an invariant commutative algebraic subgroup of Ω,*

(iv) *Ω_s is an algebraic subgroup in the centre of Ω.*

Lastly we note that connectivity is an essential part in our hypothesis as can be seen by taking Ω to be a non abelian finite nilpotent group. Then Ω can have Property P or Property T if and only if it is commutative. This follows at once from the Theorem of Maschke about the complete reducibility of finite groups: ([4]).

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 15, No. 2 October, 1965