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CONVOLUTION IN FOURIER-WIENER TRANSFORM

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Let C be the Wiener space and K be the space of complex valued continuous functions on $0 \leq t \leq 1$ which vanish at $t = 0$. The Fourier-Wiener transform of a functional $F[x]$, $x \in K$, is by definition

$$G[y] = \int_{\sigma}^w F[x + iy] d_w x, \quad y \in K.$$

Let E_0 be the class of functionals $F[x]$ of the type

$$F[x] = \Phi_x \left[\int_0^1 \alpha_1(t) dx(t), \dots, \int_0^1 \alpha_n(t) dx(t) \right]$$

where $\Phi_x(\zeta_1, \dots, \zeta_n)$ is an entire function of the n complex variables $\{\zeta_j\}$ of the exponential type and $\{\alpha_j\}$ are n linearly independent real functions of bounded variation on $0 \leq t \leq 1$. Let E_m be the class of functionals which are mean continuous, entire and of mean exponential type.

We define the convolution of two functionals F_1, F_2 to be

$$(F_1 * F_2)[x] = \int_{\sigma}^w F_1 \left[\frac{y+x}{2^{1/2}} \right] F_2 \left[\frac{y-x}{2^{1/2}} \right] d_w y, \quad x \in K.$$

Then if $F_1, F_2 \in E_0$ or $F_1, F_2 \in E_m$, the convolution of F_1, F_2 exists for every $x \in K$ and furthermore

$$G_{F_1 * F_2}[z] = G_{F_1} \left[\frac{z}{2^{1/2}} \right] G_{F_2} \left[-\frac{z}{2^{1/2}} \right], \quad z \in K.$$

Let K be the space of complex-valued continuous functions defined on $0 \leq t \leq 1$ which vanish at $t = 0$ and let C be the Wiener space, namely the subspace of K which consists of real-valued elements of K . Let $F[x] = F[x(\cdot)]$ be a functional which is defined throughout K . If it exists, the functional

$$(1.1) \quad G[y] = \int_{\sigma}^w F[x + iy] d_w x, \quad y \in K$$

is called the Fourier-Wiener transform of $F[x]$.

The first class E_0 of functionals is defined as follows: A functional $F[x]$ belongs to E_0 if

$$(1.2) \quad F[x] = \Phi_x \left[\int_0^1 \alpha_1(t) dx(t), \dots, \int_0^1 \alpha_n(t) dx(t) \right]$$

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where $\Phi_F(\zeta_1, \dots, \zeta_n)$ is an entire function of the n complex variables $\{\zeta_j\}$ of exponential type

$$(1.3) \quad |\Phi_F(\zeta_1, \dots, \zeta_n)| < Me^{\alpha(|\zeta_1| + \dots + |\zeta_n|)}$$

and $\alpha_j(t)$ are n linearly independent real functions of bounded variation on $0 \leq t \leq 1$. The function Φ_F as well as the constants M and α depend on F .

The second class E_m consists of functionals $F[x]$ which are mean continuous, entire and of mean exponential type: that is, E_m is the class of functionals satisfying the following three conditions:

1° $\lim_{n \rightarrow \infty} F[x^{(n)}] = F[x]$ holds for all x and $x^{(n)}$ in K for which $\lim_{n \rightarrow \infty} \int_0^1 |x^{(n)}(t) - x(t)|^2 dt = 0$.

2° $F[x + \lambda y]$ is an entire function of the complex variable λ for all x and y in K ; and

3° there exist positive constants A_F and B_F depending on F such that

$$(1.4) \quad |F[x]| \leq A_F \exp \left\{ B_F \left(\int_0^1 |x(t)|^2 dt \right)^{1/2} \right\} \quad \text{for all } x \in K.$$

According to Theorems 1 and A, [3], if $F[x]$ belongs to E_0 or E_m , its transform $G[y]$ exists for all $y \in K$ and belongs to the same class.

We now define the convolution of two functionals $F_1[x]$ and $F_2[x]$ to be

$$(1.5) \quad (F_1 * F_2)[x] = \int_0^w F_1 \left[\frac{y+x}{2^{1/2}} \right] F_2 \left[\frac{y-x}{2^{1/2}} \right] d_w y, \quad x \in K$$

if the integral in the right side exists.

The result of this paper is stated in the following two theorems:

THEOREM I. *If $F_1[x], F_2[x] \in E_0$, the convolution (1.5) exists for every $x \in K$. Moreover, the Fourier-Wiener transform $G_{F_1 * F_2}[z]$ of (1.5) exists and satisfies*

$$(1.6) \quad G_{F_1 * F_2}[z] = G_{F_1} \left[\frac{z}{2^{1/2}} \right] G_{F_2} \left[-\frac{z}{2^{1/2}} \right] \quad \text{for every } z \in K.$$

THEOREM II. *Exactly the same as in Theorem I holds for any two functionals belonging to E_m .*

Theorem I and II will be proved in § 2 and § 3 respectively. From these theorems follows the Parseval relation of [3].

2. NOTATION. We introduce the notation $\Phi([\zeta_j]_n)$ for the function $\Phi(\zeta_1, \dots, \zeta_n)$ of n complex variables, $\Phi([\zeta_j]_n, [\zeta'_j]_m)$ for the function $\Phi(\zeta_1, \dots, \zeta_n, \zeta'_1, \dots, \zeta'_m)$ of $n + m$ complex variables. In particular, $\Phi([\zeta_j]_n, \zeta')$ stands for the function $\Phi(\zeta_1, \dots, \zeta_n, \zeta')$ of $n + 1$ complex variables.

We first make a few remarks on the entire functions of exponential type.

REMARK 1. If $\Phi_1([\zeta_j]_n), \Phi_2([\zeta_j]_n)$ are two entire functions of exponential type, the two factors in the right hand and consequently the left hand of

$$(2.1) \quad \Phi([\zeta_j]_n, [\zeta'_j]_n) = \Phi_1([2^{-1/2}(\zeta_j + \zeta'_j)]_n)\Phi_2([2^{-1/2}(\zeta_j - \zeta'_j)]_n)$$

are entire functions of exponential type of the n complex variables ζ_1, \dots, ζ_n for fixed $\zeta'_1, \dots, \zeta'_n$ and, similarly, of the n complex variables $\zeta'_1, \dots, \zeta'_n$ for fixed ζ_1, \dots, ζ_n .

REMARK 2. If $\varphi(u_1, \dots, u_n, \zeta)$ is continuous in the $n + 1$ variables for $-\infty < u_j < \infty, j = 1, 2, \dots, n$ and $\zeta \in R$, a region in the complex plane, and is analytic in $\zeta \in R$ for fixed u_1, \dots, u_n , the uniform convergence over R of the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varphi(u_1, \dots, u_n, \zeta) du_1 \dots du_n$$

implies that the integral is an analytic function of $\zeta \in R$.

REMARK 3. If $\Phi([\zeta_j]_n, [\zeta'_j]_n)$ is an entire function of exponential type of $2n$ complex variables, the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Phi([\zeta_j]_n, [\zeta'_j]_n) \exp\{-\zeta_1^2 - \dots - \zeta_n^2\} d\zeta_1 \dots d\zeta_n$$

is an entire function of exponential type of the n complex variables $\zeta'_1, \dots, \zeta'_n$.

Proof of Theorem I. For $F_1[x], F_2[x] \in E_0$,

$$(2.2) \quad F_i[x] = \Phi_i\left(\left[\int_0^1 \alpha_j(t) dx(t)\right]_n\right), \quad i = 1, 2$$

where $\Phi_i([\zeta_j]_n), i = 1, 2$, are two entire functions of exponential type of n complex variables. We first prove the theorem for the special case where $\{\alpha_j(t)\}$ are an orthonormal set on $0 \leq t \leq 1$. We quote a result by Paley and Wiener [7] which states that for any orthonormal set of real functions $\{\alpha_j(t)\}$ of bounded variation on $0 \leq t \leq 1$, the equality

$$(2.3) \quad \int_{\sigma}^w \Psi \left(\left[\int_0^1 \alpha_j(t) dx(t) \right]_n \right) d_w x = \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Psi([u_j]_n) \\ \times \exp \{ -u_1^2 - \cdots - u_n^2 \} du_1 \cdots du_n$$

holds for every function $\Psi([u_j]_n)$ for which the integral on the right side exists as an absolutely convergent Lebesgue integral. By (1.5), (2.2), (2.1), (2.3),

$$(2.4) \quad (F_1 * F_2)[x] = \int_{\sigma}^w \Phi \left(\left[\int_0^1 \alpha_j(t) dy(t) \right]_n, \left[\int_0^1 \alpha_j(t) dx(t) \right]_n \right) d_w y \\ = \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([u_j]_n, \left[\int_0^1 \alpha_j(t) dx(t) \right]_n) \\ \times \exp \{ -u_1^2 - \cdots - u_n^2 \} du_1 \cdots du_n$$

for every $x \in K$, where the last integral exists because $\Phi([\zeta_j]_n, [\zeta'_j]_n)$ is an entire function of exponential type in $\{\zeta'_j\}$ for fixed $\{\zeta_j\}$ according to Remark 1. This proves the existence of $(F_1 * F_2)[x]$ for every $x \in K$.

Now according to Remark 3,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([\zeta_j]_n, [\zeta'_j]_n) \exp \{ -\zeta_1^2 - \cdots - \zeta_n^2 \} d\zeta_1 \cdots d\zeta_n$$

is an entire function of exponential type of $\{\zeta'_j\}$, and hence, Theorem 1, [3] applies to the last member of (2.4). Thus the Fourier-Wiener transform of $(F_1 * F_2)[x]$ namely $G_{F_1 * F_2}[z]$, exists for every $z \in K$ and is given by (1.1) as

$$(2.5) \quad G_{F_1 * F_2}[z] = \int_{\sigma}^w \frac{1}{\pi^{n/2}} \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi \left([u_j]_n, \left[\int_0^1 \alpha_j(t) dx(t) + i \int_0^1 \alpha_j(t) dz(t) \right]_n \right) \right. \\ \left. \times \exp \{ -u_1^2 - \cdots - u_n^2 \} du_1 \cdots du_n \right\} d_w x .$$

Now since

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([\zeta_j]_n, [\zeta'_j + \zeta''_j]_n) \exp \{ -\zeta_1^2 - \cdots - \zeta_n^2 \} d\zeta_1 \cdots d\zeta_n$$

is an entire function of exponential type of $\{\zeta'_j\}$ for fixed $\{\zeta''_j\}$, (2.3) is applicable to the last integral of (2.5). Thus

$$G_{F_1 * F_2}[z] = \frac{1}{\pi^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi \left([u_j]_n, \left[v_j + i \int_0^1 \alpha_j(t) dz(t) \right]_n \right) \\ \times \exp \{ -u_1^2 - v_1^2 - \cdots - u_n^2 - v_n^2 \} du_1 \cdots du_n dv_1 \cdots dv_n \\ = \frac{1}{\pi^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_1 \left(\left[2^{-1/2} \left(u_j + v_j + i \int_0^1 \alpha_j(t) dz(t) \right) \right]_n \right) \\ \times \Phi_2 \left(\left[2^{-1/2} \left(u_j - v_j - i \int_0^1 \alpha_j(t) dz(t) \right) \right]_n \right) \\ \times \exp \{ -u_1^2 - v_1^2 - \cdots - u_n^2 - v_n^2 \} du_1 \cdots du_n dv_1 \cdots dv_n ,$$

Let

$$\begin{aligned} u'_j &= 2^{-1/2}(u_j + v_j), \\ v'_j &= 2^{-1/2}(u_j - v_j), \end{aligned} \quad j = 1, 2, \dots, n$$

and apply (2.3) to the result of this transformation. By (2.2), (1.1) we have

$$\begin{aligned} G_{F_1 * F_2}[z] &= \left\{ \int_{\mathcal{C}}^w \Phi_1 \left(\left[\int_0^1 \alpha_j(t) dx(t) + \frac{i}{2^{1/2}} \int_0^1 \alpha_j(t) dz(t) \right]_n \right) d_w x \right\} \\ &\quad \times \left\{ \int_{\mathcal{C}}^w \Phi_2 \left(\left[\int_0^1 \alpha_j(t) dx(t) - \frac{i}{2^{1/2}} \int_0^1 \alpha_j(t) dz(t) \right]_n \right) d_w x \right\} \\ &= G_{F_1} \left[\frac{z}{2^{1/2}} \right] G_{F_2} \left[-\frac{z}{2^{1/2}} \right]. \end{aligned}$$

This proves Theorem I for the special case.

In the general case where $\alpha_j(t)$ are n linearly independent real valued functions of bounded variation on $0 \leq t \leq 1$, according to the argument on p. 493, [3], we can write $F_i[x]$, $i = 1, 2$ defined by (2.2) as

$$F_i[x] = \Phi_i^* \left(\left[\int_0^1 \alpha'_j(t) dx(t) \right]_n \right), \quad i = 1, 2$$

where $\Phi_i^*(\{\zeta_j\}_n)$ are entire functions of exponential type of $\{\zeta_j\}$ and $\alpha'_j(t)$ are n orthonormal functions of bounded variation on $0 \leq t \leq 1$. Now the result for the special case applies and the theorem is proved.

3. LEMMA. Let $\{F_{1,n}[x]\}, F_1[x], \{F_{2,n}[x]\}, F_2[x]$ be such that

$$1^\circ \quad (3.1) \quad \lim_{n \rightarrow \infty} F_{i,n}[x] = F_i[x] \text{ for every } x \in K, \quad i = 1, 2.$$

2° the Fourier-Wiener transform exists for every $F_{i,n}[x]$ $n = 1, 2, \dots, i = 1, 2$; the convolution $(F_{1,n} * F_{2,n})[x]$ exists, its Fourier-Wiener transform also exists and satisfies

$$(3.2) \quad G_{F_{1,n} * F_{2,n}}[z] = G_{F_{1,n}} \left[\frac{z}{2^{1/2}} \right] G_{F_{2,n}} \left[-\frac{z}{2^{1/2}} \right],$$

for every $z \in K$, for $n = 1, 2, \dots$; and

3° (3.3) $|F_{i,n}[x]| \leq A \exp \{B |||x|||^{2-\varepsilon}\}, \quad n = 1, 2, \dots, i = 1, 2$ where $A, B, > 0, 2 > \varepsilon > 0$ and $|||x||| = \max_{0 \leq t \leq 1} |x(t)|$. Then the Fourier-Wiener transforms of $F_1[x], F_2[x]$, the convolution of $F_1[x], F_2[x]$ and the Fourier-Wiener transform of the convolution exist and (1.6) holds.

Proof of the lemma. By (1.5), (1.1), the equality (3.2) can be written as

$$(3.4) \quad \int_{\mathcal{C}}^w \left\{ \int_{\mathcal{C}}^w F_{1,n} \left[\frac{y+x+iz}{2^{1/2}} \right] F_{2,n} \left[\frac{y-x-iz}{2^{1/2}} \right] d_w y \right\} d_w x \\ = \left\{ \int_{\mathcal{C}}^w F_{1,n} \left[x + \frac{iz}{2^{1/2}} \right] d_w x \right\} \left\{ \int_{\mathcal{C}}^w F_{2,n} \left[x - \frac{iz}{2^{1/2}} \right] d_w x \right\}, \quad n = 1, 2, \dots$$

We prove the lemma by justifying the passing to the limit under the integral signs on both sides of (3.4). To do this, we observe that for any p complex numbers ζ_1, \dots, ζ_p ,

$$(3.5) \quad \left| \sum_{k=1}^p \zeta_k \right|^{2-\varepsilon} \leq \left(p \max_k \{ |\zeta_1|, \dots, |\zeta_p| \} \right)^{2-\varepsilon} \leq p^3 \sum_{k=1}^p |\zeta_k|^{2-\varepsilon}.$$

An estimate of the first integrand on the right hand side of (3.4) is given by (3.3) and (3.5) with $p = 2$:

$$(3.6) \quad \left| F_{1,n} \left[x + \frac{iz}{2^{1/2}} \right] \right| \leq A \exp \{ 4B (\| \| x \| \| \|^{2-\varepsilon} + \| \| z \| \| \|^{2-\varepsilon}) \}.$$

Since $\int_{\mathcal{C}}^w \exp \{ 4B \| \| x \| \| \|^{2-\varepsilon} \} d_w x$ is finite according to [4], the right side of (3.6) is integrable with respect to x over the entire Wiener space for fixed z . By (3.1) with dominated convergence and by (1.1)

$$(3.7) \quad \lim_{n \rightarrow \infty} \int_{\mathcal{C}}^w F_{1,n} \left[x + \frac{iz}{2^{1/2}} \right] d_w x = G_{F_1} \left[\frac{z}{2^{1/2}} \right]$$

for every $z \in K$ and similarly

$$(3.8) \quad \lim_{n \rightarrow \infty} \int_{\mathcal{C}}^w F_{2,n} \left[x - \frac{iz}{2^{1/2}} \right] d_w x = G_{F_2} \left[-\frac{z}{2^{1/2}} \right],$$

for every $z \in K$. From (3.3) and (3.5) with $p = 3$, the integrand of the left side of (3.4) is seen to be bounded by $A^3 \exp \{ 18B (\| \| x \| \| \|^{2-\varepsilon} + \| \| y \| \| \|^{2-\varepsilon} + \| \| z \| \| \|^{2-\varepsilon}) \}$. The repeated integral of the above expression with respect to y and then with respect to x over the entire Wiener space is finite for every $z \in K$. Thus by (3.1) with dominated convergence and by (1.5), (1.1),

$$(3.9) \quad \lim_{n \rightarrow \infty} \int_{\mathcal{C}}^w \left\{ \int_{\mathcal{C}}^w F_{1,n} \left[\frac{y+x+iz}{2^{1/2}} \right] F_{2,n} \left[\frac{y-z-iz}{2^{1/2}} \right] d_w y \right\} d_w x = G_{F_1 * F_2} [z]$$

for every $z \in K$. By letting $n \rightarrow \infty$ on both sides of (3.4) and by (3.7), (3.8) and (3.9), the lemma is established.

Proof of Theorem II. Let $F_i[x] \in E_m, i = 1, 2$, and let $\varphi_1(t), \varphi_2(t), \dots$ be a complete orthonormal set of real valued continuous functions on the interval $0 \leq t \leq 1$ which vanish when $t = 0$. Let

$$(3.10) \quad F_{i,n}[z] = F_i \left[\sum_{j=1}^n \varphi_j(\cdot) \int_0^1 x(t) \varphi_j(t) dt \right] \quad n = 1, 2, \dots, i = 1, 2,$$

and let

$$x^{(n)} = \sum_{j=1}^n \varphi_j(\cdot) \int_0^1 x(t) \varphi_j(t) dt, \quad n = 1, 2, \dots$$

By 1° in the definition of E_m ,

$$(3.11) \quad \lim_{n \rightarrow \infty} F_{i,n}[x] = F_i[x],$$

for every $x \in K$, $i = 1, 2$, and $F_{i,n}[x]$, $i = 1, 2$, satisfy 1° of the lemma.

To show that 2° of the lemma is satisfied, let us define $\Phi_{i,n}([\zeta_j]_n)$ by

$$(3.12) \quad \Phi_{i,n}([\zeta_j]_n) = F_i \left[\sum_{j=1}^n \zeta_j \varphi_j(\cdot) \right], \quad n = 1, 2, \dots, i = 1, 2.$$

To show that each $\Phi_{i,n}$ is an entire function of exponential type of n complex variables, we set

$$\begin{aligned} x(t) &= \zeta_1 \varphi_1(t) + \dots + \zeta_{j-1} \varphi_{j-1}(t) + \zeta_{j+1} \varphi_{j+1}(t) + \dots + \zeta_n \varphi_n(t), \\ y(t) &= \varphi_j(t). \end{aligned}$$

From (3.12) it follows that $\Phi_{i,n}([\zeta_j]_n) = F_i[x(t) + \zeta_j y(t)]$ and by 2° in the definition of E_m , $\Phi_{i,n}$ is an entire function of ζ_j . From the arbitrariness of the choice of ζ_j from $\{\zeta_j\}$ and by Hartogs' regularity theorem, $\Phi_{i,n}$ is an entire function of the n complex variables $\{\zeta_j\}$ for $n = 1, 2, \dots, i = 1, 2$. That $\Phi_{i,n}$ is of exponential type follows from (3.12) and 3° of the definition of E_m :

$$\begin{aligned} |\Phi_{i,n}([\zeta_j]_n)| &\leq A_{F_i} \exp \left\{ B_{F_i} \left(\int_0^1 \left| \sum_{j=1}^n \zeta_j \varphi_j(t) \right|^2 dt \right)^{1/2} \right\} \\ &\leq A_{F_i} \exp \left\{ B_{F_i} \left(\sum_{j=1}^n |\zeta_j|^2 \right)^{1/2} \right\} \\ &\leq A_{F_i} \exp \left\{ B_{F_i} \sum_{j=1}^n |\zeta_j| \right\}. \end{aligned}$$

This proves the asserted property of $\Phi_{i,n}$. On the other hand from (3.10), (3.12)

$$(3.13) \quad F_{i,n}[x] = \Phi_{i,n} \left(\left[\int_0^1 x(t) \varphi_j(t) dt \right]_n \right), \quad n = 1, 2, \dots, i = 1, 2.$$

Now if we let $\alpha_j(t) = \int_t^1 \varphi_j(t) dt$, $n = 1, 2, \dots$, then by integration by parts $\int_0^1 x(t) \varphi_j(t) dt = \int_0^1 \alpha_j(t) dx(t)$, and (3.13) becomes

$$F_{i,n}[x] = \Phi_{i,n} \left(\left[\int_0^1 \alpha_j(t) dx(t) \right]_n \right), \quad n = 1, 2, \dots, i = 1, 2$$

where by definition $\alpha_j(t)$ are of bounded variation on $0 \leq t \leq 1$. Therefore each $F_{i,n}[x]$ satisfies the conditions of Theorem I, [3] and hence its Fourier-Wiener transform exists. Moreover by Theorem I the convolution $(F_{i,n} * F_{2,n})[x]$ exists and satisfies (3.2) for every $z \in K$ for $n = 1, 2, \dots$. Thus 2° of the lemma is satisfied.

Finally, let A be the greater of A_{F_1}, A_{F_2} and B be the greater of B_{F_1}, B_{F_2} in 3° of the definition of E_m . By (3.10), (3.14)

$$\begin{aligned} |F_{i,n}[x]| &\leq A \exp \left\{ B \left(\int_0^1 \left| \sum_{j=1}^n \varphi_j(s) \int_0^1 x(t) \varphi_j(t) dt \right|^2 ds \right)^{1/2} \right\} \\ &\leq A \exp \left\{ B \left(\int_0^1 |x(t)|^2 dt \right)^{1/2} \right\} \\ &\leq A \exp \{ B \|x\|^{2-\varepsilon} \} \end{aligned}$$

for $1 > \varepsilon > 0$ and 3° of the lemma is satisfied.

By the conclusion of the lemma, Theorem II is proved.

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