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**CONVOLUTION IN FOURIER-WIENER TRANSFORM**

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## CONVOLUTION IN FOURIER-WIENER TRANSFORM

J. YEH

Let  $C$  be the Wiener space and  $K$  be the space of complex valued continuous functions on  $0 \leq t \leq 1$  which vanish at  $t = 0$ . The Fourier-Wiener transform of a functional  $F[x]$ ,  $x \in K$ , is by definition

$$G[y] = \int_{\sigma}^w F[x + iy] d_w x, \quad y \in K.$$

Let  $E_0$  be the class of functionals  $F[x]$  of the type

$$F[x] = \Phi_F \left[ \int_0^1 \alpha_1(t) dx(t), \dots, \int_0^1 \alpha_n(t) dx(t) \right]$$

where  $\Phi_F(\zeta_1, \dots, \zeta_n)$  is an entire function of the  $n$  complex variables  $\{\zeta_j\}$  of the exponential type and  $\{\alpha_j\}$  are  $n$  linearly independent real functions of bounded variation on  $0 \leq t \leq 1$ . Let  $E_m$  be the class of functionals which are mean continuous, entire and of mean exponential type.

We define the convolution of two functionals  $F_1, F_2$  to be

$$(F_1 * F_2)[x] = \int_{\sigma}^w F_1 \left[ \frac{y+x}{2^{1/2}} \right] F_2 \left[ \frac{y-x}{2^{1/2}} \right] d_w y, \quad x \in K.$$

Then if  $F_1, F_2 \in E_0$  or  $F_1, F_2 \in E_m$ , the convolution of  $F_1, F_2$  exists for every  $x \in K$  and furthermore

$$G_{F_1 * F_2}[z] = G_{F_1} \left[ \frac{z}{2^{1/2}} \right] G_{F_2} \left[ -\frac{z}{2^{1/2}} \right], \quad z \in K.$$

Let  $K$  be the space of complex-valued continuous functions defined on  $0 \leq t \leq 1$  which vanish at  $t = 0$  and let  $C$  be the Wiener space, namely the subspace of  $K$  which consists of real-valued elements of  $K$ . Let  $F[x] = F[x(\cdot)]$  be a functional which is defined throughout  $K$ . If it exists, the functional

$$(1.1) \quad G[y] = \int_{\sigma}^w F[x + iy] d_w x, \quad y \in K$$

is called the Fourier-Wiener transform of  $F[x]$ .

The first class  $E_0$  of functionals is defined as follows: A functional  $F[x]$  belongs to  $E_0$  if

$$(1.2) \quad F[x] = \Phi_F \left[ \int_0^1 \alpha_1(t) dx(t), \dots, \int_0^1 \alpha_n(t) dx(t) \right]$$

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where  $\Phi_F(\zeta_1, \dots, \zeta_n)$  is an entire function of the  $n$  complex variables  $\{\zeta_j\}$  of exponential type

$$(1.3) \quad |\Phi_F(\zeta_1, \dots, \zeta_n)| < Me^{\alpha(|\zeta_1| + \dots + |\zeta_n|)}$$

and  $\alpha_j(t)$  are  $n$  linearly independent real functions of bounded variation on  $0 \leq t \leq 1$ . The function  $\Phi_F$  as well as the constants  $M$  and  $\alpha$  depend on  $F$ .

The second class  $E_m$  consists of functionals  $F[x]$  which are mean continuous, entire and of mean exponential type: that is,  $E_m$  is the class of functionals satisfying the following three conditions:

1°  $\lim_{n \rightarrow \infty} F[x^{(n)}] = F[x]$  holds for all  $x$  and  $x^{(n)}$  in  $K$  for which  $\lim_{n \rightarrow \infty} \int_0^1 |x^{(n)}(t) - x(t)|^2 dt = 0$ .

2°  $F[x + \lambda y]$  is an entire function of the complex variable  $\lambda$  for all  $x$  and  $y$  in  $K$ ; and

3° there exist positive constants  $A_F$  and  $B_F$  depending on  $F$  such that

$$(1.4) \quad |F[x]| \leq A_F \exp \left\{ B_F \left( \int_0^1 |x(t)|^2 dt \right)^{1/2} \right\} \quad \text{for all } x \in K.$$

According to Theorems 1 and A, [3], if  $F[x]$  belongs to  $E_0$  or  $E_m$ , its transform  $G[y]$  exists for all  $y \in K$  and belongs to the same class.

We now define the convolution of two functionals  $F_1[x]$  and  $F_2[x]$  to be

$$(1.5) \quad (F_1 * F_2)[x] = \int_0^w F_1 \left[ \frac{y+x}{2^{1/2}} \right] F_2 \left[ \frac{y-x}{2^{1/2}} \right] d_w y, \quad x \in K$$

if the integral in the right side exists.

The result of this paper is stated in the following two theorems:

**THEOREM I.** *If  $F_1[x], F_2[x] \in E_0$ , the convolution (1.5) exists for every  $x \in K$ . Moreover, the Fourier-Wiener transform  $G_{F_1 * F_2}[z]$  of (1.5) exists and satisfies*

$$(1.6) \quad G_{F_1 * F_2}[z] = G_{F_1} \left[ \frac{z}{2^{1/2}} \right] G_{F_2} \left[ -\frac{z}{2^{1/2}} \right] \quad \text{for every } z \in K.$$

**THEOREM II.** *Exactly the same as in Theorem I holds for any two functionals belonging to  $E_m$ .*

Theorem I and II will be proved in §2 and §3 respectively. From these theorems follows the Parseval relation of [3].

2. NOTATION. We introduce the notation  $\Phi([\zeta_j]_n)$  for the function  $\Phi(\zeta_1, \dots, \zeta_n)$  of  $n$  complex variables,  $\Phi([\zeta_j]_n, [\zeta'_j]_m)$  for the function  $\Phi(\zeta_1, \dots, \zeta_n, \zeta'_1, \dots, \zeta'_m)$  of  $n + m$  complex variables. In particular,  $\Phi([\zeta_j]_n, \zeta')$  stands for the function  $\Phi(\zeta_1, \dots, \zeta_n, \zeta')$  of  $n + 1$  complex variables.

We first make a few remarks on the entire functions of exponential type.

REMARK 1. If  $\Phi_1([\zeta_j]_n), \Phi_2([\zeta_j]_n)$  are two entire functions of exponential type, the two factors in the right hand and consequently the left hand of

$$(2.1) \quad \Phi([\zeta_j]_n, [\zeta'_j]_n) = \Phi_1([2^{-1/2}(\zeta_j + \zeta'_j)]_n)\Phi_2([2^{-1/2}(\zeta_j - \zeta'_j)]_n)$$

are entire functions of exponential type of the  $n$  complex variables  $\zeta_1, \dots, \zeta_n$  for fixed  $\zeta'_1, \dots, \zeta'_n$  and, similarly, of the  $n$  complex variables  $\zeta'_1, \dots, \zeta'_n$  for fixed  $\zeta_1, \dots, \zeta_n$ .

REMARK 2. If  $\varphi(u_1, \dots, u_n, \zeta)$  is continuous in the  $n + 1$  variables for  $-\infty < u_j < \infty, j = 1, 2, \dots, n$  and  $\zeta \in R$ , a region in the complex plane, and is analytic in  $\zeta \in R$  for fixed  $u_1, \dots, u_n$ , the uniform convergence over  $R$  of the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varphi(u_1, \dots, u_n, \zeta) du_1 \dots du_n$$

implies that the integral is an analytic function of  $\zeta \in R$ .

REMARK 3. If  $\Phi([\zeta_j]_n, [\zeta'_j]_n)$  is an entire function of exponential type of  $2n$  complex variables, the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Phi([\zeta_j]_n, [\zeta'_j]_n) \exp \{-\zeta_1^2 - \dots - \zeta_n^2\} d\zeta_1 \dots d\zeta_n$$

is an entire function of exponential type of the  $n$  complex variables  $\zeta'_1, \dots, \zeta'_n$ .

*Proof of Theorem I.* For  $F_1[x], F_2[x] \in E_0$ ,

$$(2.2) \quad F_i[x] = \Phi_i\left(\left[\int_0^1 \alpha_j(t) dx(t)\right]_n\right), \quad i = 1, 2$$

where  $\Phi_i([\zeta_j]_n), i = 1, 2$ , are two entire functions of exponential type of  $n$  complex variables. We first prove the theorem for the special case where  $\{\alpha_j(t)\}$  are an orthonormal set on  $0 \leq t \leq 1$ . We quote a result by Paley and Wiener [7] which states that for any orthonormal set of real functions  $\{\alpha_j(t)\}$  of bounded variation on  $0 \leq t \leq 1$ , the equality

$$(2.3) \quad \int_{\sigma}^w \Psi \left( \left[ \int_0^1 \alpha_j(t) dx(t) \right]_n \right) d_w x = \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Psi([u_j]_n) \\ \times \exp \{-u_1^2 - \cdots - u_n^2\} du_1 \cdots du_n$$

holds for every function  $\Psi([u_j]_n)$  for which the integral on the right side exists as an absolutely convergent Lebesgue integral. By (1.5), (2.2), (2.1), (2.3),

$$(2.4) \quad (F_1 * F_2)[x] = \int_{\sigma}^w \Phi \left( \left[ \int_0^1 \alpha_j(t) dy(t) \right]_n, \left[ \int_0^1 \alpha_j(t) dx(t) \right]_n \right) d_w y \\ = \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([u_j]_n, \left[ \int_0^1 \alpha_j(t) dx(t) \right]_n) \\ \times \exp \{-u_1^2 - \cdots - u_n^2\} du_1 \cdots du_n$$

for every  $x \in K$ , where the last integral exists because  $\Phi([\zeta_j]_n, [\zeta'_j]_n)$  is an entire function of exponential type in  $\{\zeta_j\}$  for fixed  $\{\zeta'_j\}$  according to Remark 1. This proves the existence of  $(F_1 * F_2)[x]$  for every  $x \in K$ .

Now according to Remark 3,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([\zeta_j]_n, [\zeta'_j]_n) \exp \{-\zeta_1^2 - \cdots - \zeta_n^2\} d\zeta_1 \cdots d\zeta_n$$

is an entire function of exponential type of  $\{\zeta'_j\}$ , and hence, Theorem 1, [3] applies to the last member of (2.4). Thus the Fourier-Wiener transform of  $(F_1 * F_2)[x]$  namely  $G_{F_1 * F_2}[z]$ , exists for every  $z \in K$  and is given by (1.1) as

$$(2.5) \quad G_{F_1 * F_2}[z] = \int_{\sigma}^w \frac{1}{\pi^{n/2}} \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([u_j]_n, \left[ \int_0^1 \alpha_j(t) dx(t) + i \int_0^1 \alpha_j(t) dz(t) \right]_n) \right. \\ \left. \times \exp \{-u_1^2 - \cdots - u_n^2\} du_1 \cdots du_n \right\} d_w x.$$

Now since

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([\zeta_j]_n, [\zeta'_j + \zeta''_j]_n) \exp \{-\zeta_1^2 - \cdots - \zeta_n^2\} d\zeta_1 \cdots d\zeta_n$$

is an entire function of exponential type of  $\{\zeta'_j\}$  for fixed  $\{\zeta''_j\}$ , (2.3) is applicable to the last integral of (2.5). Thus

$$G_{F_1 * F_2}[z] = \frac{1}{\pi^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi([u_j]_n, \left[ v_j + i \int_0^1 \alpha_j(t) dz(t) \right]_n) \\ \times \exp \{-u_1^2 - v_1^2 - \cdots - u_n^2 - v_n^2\} du_1 \cdots du_n dv_1 \cdots dv_n \\ = \frac{1}{\pi^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_1 \left( \left[ 2^{-1/2} \left( u_j + v_j + i \int_0^1 \alpha_j(t) dz(t) \right) \right]_n \right) \\ \times \Phi_2 \left( \left[ 2^{-1/2} \left( u_j - v_j - i \int_0^1 \alpha_j(t) dz(t) \right) \right]_n \right) \\ \times \exp \{-u_1^2 - v_1^2 - \cdots - u_n^2 - v_n^2\} du_1 \cdots du_n dv_1 \cdots dv_n,$$

Let

$$\begin{aligned} u'_j &= 2^{-1/2}(u_j + v_j), \\ v'_j &= 2^{-1/2}(u_j - v_j), \end{aligned} \quad j = 1, 2, \dots, n$$

and apply (2.3) to the result of this transformation. By (2.2), (1.1) we have

$$\begin{aligned} G_{F_1 * F_2}[z] &= \left\{ \int_0^w \Phi_1 \left( \left[ \int_0^1 \alpha_j(t) dx(t) + \frac{i}{2^{1/2}} \int_0^1 \alpha_j(t) dz(t) \right]_n \right) d_w x \right\} \\ &\quad \times \left\{ \int_0^w \Phi_2 \left( \left[ \int_0^1 \alpha_j(t) dx(t) - \frac{i}{2^{1/2}} \int_0^1 \alpha_j(t) dz(t) \right]_n \right) d_w x \right\} \\ &= G_{F_1} \left[ \frac{z}{2^{1/2}} \right] G_{F_2} \left[ -\frac{z}{2^{1/2}} \right]. \end{aligned}$$

This proves Theorem I for the special case.

In the general ease where  $\alpha_j(t)$  are  $n$  linearly independent real valued functions of bounded variation on  $0 \leq t \leq 1$ , according to the argument on p. 493, [3], we can write  $F_i[x]$ ,  $i = 1, 2$  defined by (2.2) as

$$F_i[x] = \Phi_i^* \left( \left[ \int_0^1 \alpha'_j(t) dx(t) \right]_n \right), \quad i = 1, 2$$

where  $\Phi_i^*([\zeta_j]_n)$  are entire functions of exponential type of  $\{\zeta_j\}$  and  $\alpha'_j(t)$  are  $n$  orthonormal functions of bounded variation on  $0 \leq t \leq 1$ . Now the result for the special case applies and the theorem is proved.

3. LEMMA. Let  $\{F_{1,n}[x]\}$ ,  $F_1[x]$ ,  $\{F_{2,n}[x]\}$ ,  $F_2[x]$  be such that

$$1^\circ \quad (3.1) \quad \lim_{n \rightarrow \infty} F_{i,n}[x] = F_i[x] \text{ for every } x \in K, \quad i = 1, 2.$$

2° the Fourier-Wiener transform exists for every  $F_{i,n}[x]$   $n = 1, 2, \dots, i = 1, 2$ ; the convolution  $(F_{1,n} * F_{2,n})[x]$  exists, its Fourier-Wiener transform also exists and satisfies

$$(3.2) \quad G_{F_{1,n} * F_{2,n}}[z] = G_{F_{1,n}} \left[ \frac{z}{2^{1/2}} \right] G_{F_{2,n}} \left[ -\frac{z}{2^{1/2}} \right],$$

for every  $z \in K$ , for  $n = 1, 2, \dots$ ; and

3° (3.3)  $|F_{i,n}[x]| \leq A \exp \{B |||x|||^{2-\varepsilon}\}$ ,  $n = 1, 2, \dots, i = 1, 2$  where  $A, B, > 0, 2 > \varepsilon > 0$  and  $|||x||| = \max_{0 \leq t \leq 1} |x(t)|$ . Then the Fourier-Wiener transforms of  $F_1[x]$ ,  $F_2[x]$ , the convolution of  $F_1[x]$ ,  $F_2[x]$  and the Fourier-Wiener transform of the convolution exist and (1.6) holds.

*Proof of the lemma.* By (1.5), (1.1), the equality (3.2) can be written as

$$(3.4) \quad \int_{\sigma}^w \left\{ \int_{\sigma}^w F_{1,n} \left[ \frac{y+x+iz}{2^{1/2}} \right] F_{2,n} \left[ \frac{y-x-iz}{2^{1/2}} \right] d_w y \right\} d_w x \\ = \left\{ \int_{\sigma}^w F_{1,n} \left[ x + \frac{iz}{2^{1/2}} \right] d_w x \right\} \left\{ \int_{\sigma}^w F_{2,n} \left[ x - \frac{iz}{2^{1/2}} \right] d_w x \right\}, \quad n = 1, 2, \dots$$

We prove the lemma by justifying the passing to the limit under the integral signs on both sides of (3.4). To do this, we observe that for any  $p$  complex numbers  $\zeta_1, \dots, \zeta_p$ ,

$$(3.5) \quad \left| \sum_{k=1}^p \zeta_k \right|^{2-\varepsilon} \leq \left( p \max_k \{ |\zeta_1|, \dots, |\zeta_p| \} \right)^{2-\varepsilon} \leq p^2 \sum_{k=1}^p |\zeta_k|^{2-\varepsilon}.$$

An estimate of the first integrand on the right hand side of (3.4) is given by (3.3) and (3.5) with  $p = 2$ :

$$(3.6) \quad \left| F_{1,n} \left[ x + \frac{iz}{2^{1/2}} \right] \right| \leq A \exp \{ 4B (||| x |||^{2-\varepsilon} + ||| z |||^{2-\varepsilon}) \}.$$

Since  $\int_{\sigma}^w \exp \{ 4B (||| x |||^{2-\varepsilon}) \} d_w x$  is finite according to [4], the right side of (3.6) is integrable with respect to  $x$  over the entire Wiener space for fixed  $z$ . By (3.1) with dominated convergence and by (1.1)

$$(3.7) \quad \lim_{n \rightarrow \infty} \int_{\sigma}^w F_{1,n} \left[ x + \frac{iz}{2^{1/2}} \right] d_w x = G_{F_1} \left[ \frac{z}{2^{1/2}} \right]$$

for every  $z \in K$  and similarly

$$(3.8) \quad \lim_{n \rightarrow \infty} \int_{\sigma}^w F_{2,n} \left[ x - \frac{iz}{2^{1/2}} \right] d_w x = G_{F_2} \left[ -\frac{z}{2^{1/2}} \right],$$

for every  $z \in K$ . From (3.3) and (3.5) with  $p = 3$ , the integrand of the left side of (3.4) is seen to be bounded by  $A^2 \exp \{ 18B (||| x |||^{2-\varepsilon} + ||| y |||^{2-\varepsilon} + ||| z |||^{2-\varepsilon}) \}$ . The repeated integral of the above expression with respect to  $y$  and then with respect to  $x$  over the entire Wiener space is finite for every  $z \in K$ . Thus by (3.1) with dominated convergence and by (1.5), (1.1),

$$(3.9) \quad \lim_{n \rightarrow \infty} \int_{\sigma}^w \left\{ \int_{\sigma}^w F_{1,n} \left[ \frac{y+x+iz}{2^{1/2}} \right] F_{2,n} \left[ \frac{y-z-iz}{2^{1/2}} \right] d_w y \right\} d_w x = G_{F_1 \circ F_2} [z]$$

for every  $z \in K$ . By letting  $n \rightarrow \infty$  on both sides of (3.4) and by (3.7), (3.8) and (3.9), the lemma is established.

*Proof of Theorem II.* Let  $F_i[x] \in E_m, i = 1, 2$ , and let  $\varphi_1(t), \varphi_2(t), \dots$  be a complete orthonormal set of real valued continuous functions on the interval  $0 \leq t \leq 1$  which vanish when  $t = 0$ . Let

$$(3.10) \quad F_{i,n}[z] = F_i \left[ \sum_{j=1}^n \varphi_j(\cdot) \int_0^1 x(t) \varphi_j(t) dt \right] \quad n = 1, 2, \dots, i = 1, 2,$$

and let

$$x^{(n)} = \sum_{j=1}^n \varphi_j(\cdot) \int_0^1 x(t) \varphi_j(t) dt, \quad n = 1, 2, \dots$$

By 1° in the definition of  $E_m$ ,

$$(3.11) \quad \lim_{n \rightarrow \infty} F_{i,n}[x] = F_i[x],$$

for every  $x \in K, i = 1, 2$ , and  $F_{i,n}[x], i = 1, 2$ , satisfy 1° of the lemma.

To show that 2° of the lemma is satisfied, let us define  $\Phi_{i,n}([\zeta_j]_n)$  by

$$(3.12) \quad \Phi_{i,n}([\zeta_j]_n) = F_i \left[ \sum_{j=1}^n \zeta_j \varphi_j(\cdot) \right], \quad n = 1, 2, \dots, i = 1, 2.$$

To show that each  $\Phi_{i,n}$  is an entire function of exponential type of  $n$  complex variables, we set

$$x(t) = \zeta_1 \varphi_1(t) + \dots + \zeta_{j-1} \varphi_{j-1}(t) + \zeta_{j+1} \varphi_{j+1}(t) + \dots + \zeta_n \varphi_n(t),$$

$$y(t) = \varphi_j(t).$$

From (3.12) it follows that  $\Phi_{i,n}([\zeta_j]_n) = F_i[x(t) + \zeta_j y(t)]$  and by 2° in the definition of  $E_m, \Phi_{i,n}$  is an entire function of  $\zeta_j$ . From the arbitrariness of the choice of  $\zeta_j$  from  $\{\zeta_j\}$  and by Hartogs' regularity theorem,  $\Phi_{i,n}$  is an entire function of the  $n$  complex variables  $\{\zeta_j\}$  for  $n = 1, 2, \dots, i = 1, 2$ . That  $\Phi_{i,n}$  is of exponential type follows from (3.12) and 3° of the definition of  $E_m$ :

$$|\Phi_{i,n}([\zeta_j]_n)| \leq A_{F_i} \exp \left\{ B_{F_i} \left( \int_0^1 \left| \sum_{j=1}^n \zeta_j \varphi_j(t) \right|^2 dt \right)^{1/2} \right\}$$

$$\leq A_{F_i} \exp \left\{ B_{F_i} \left( \sum_{j=1}^n |\zeta_j|^2 \right)^{1/2} \right\}$$

$$\leq A_{F_i} \exp \left\{ B_{F_i} \sum_{j=1}^n |\zeta_j| \right\}.$$

This proves the asserted property of  $\Phi_{i,n}$ . On the other hand from (3.10), (3.12)

$$(3.13) \quad F_{i,n}[x] = \Phi_{i,n} \left( \left[ \int_0^1 x(t) \varphi_j(t) dt \right]_n \right), \quad n = 1, 2, \dots, i = 1, 2.$$

Now if we let  $\alpha_j(t) = \int_0^t \varphi_j(t) dt, n = 1, 2, \dots$ , then by integration by parts  $\int_0^1 x(t) \varphi_j(t) dt = \int_0^1 \alpha_j(t) dx(t)$ , and (3.13) becomes

$$F_{i,n}[x] = \Phi_{i,n} \left( \left[ \int_0^1 \alpha_j(t) dx(t) \right]_n \right), \quad n = 1, 2, \dots, i = 1, 2$$



where by definition  $\alpha_j(t)$  are of bounded variation on  $0 \leq t \leq 1$ . Therefore each  $F_{i,n}[x]$  satisfies the conditions of Theorem I, [3] and hence its Fourier-Wiener transform exists. Moreover by Theorem I the convolution  $(F_{i,n} * F_{2,n})[x]$  exists and satisfies (3.2) for every  $z \in K$  for  $n = 1, 2, \dots$ . Thus 2° of the lemma is satisfied.

Finally, let  $A$  be the greater of  $A_{F_1}, A_{F_2}$  and  $B$  be the greater of  $B_{F_1}, B_{F_2}$  in 3° of the definition of  $E_m$ . By (3.10), (3.14)

$$\begin{aligned} |F_{i,n}[x]| &\leq A \exp \left\{ B \left( \int_0^1 \left| \sum_{j=1}^n \varphi_j(s) \int_0^1 x(t) \varphi_j(t) dt \right|^2 ds \right)^{1/2} \right\} \\ &\leq A \exp \left\{ B \left( \int_0^1 |x(t)|^2 dt \right)^{1/2} \right\} \\ &\leq A \exp \{ B \|x\|^{2-\varepsilon} \} \end{aligned}$$

for  $1 > \varepsilon > 0$  and 3° of the lemma is satisfied.

By the conclusion of the lemma, Theorem II is proved.

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