EXTREME OPERATORS INTO $C(K)$

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If $X$ and $Y$ are real Banach spaces let $S(X, Y)$ denote the convex set of all linear operators from $X$ into $Y$ having norm less than or equal to 1. The main theorem is this: If $K_1$ and $K_2$ are compact Hausdorff spaces with $K_1$ metrizable and if $T$ is an extreme point of $S(C(K_1), C(K_2))$, then there are continuous functions $\phi: K_2 \to K_1$ and $\lambda$ in $C(K_2)$ with $|\lambda| = 1$ such that $(Tf)(k) = \lambda(k)f(\phi(k))$ for all $k$ in $K_2$ and $f$ in $C(K_1)$. There are several additional theorems which discuss the possibility of replacing $C(K_1)$ in this theorem by an arbitrary Banach space.

Suppose that $K_1$ is a compact Hausdorff space and that $C(K_1)$ is the Banach space of real-valued continuous functions on $K_1$, with supremum norm. Denote by $S^*$ the unit ball of $C(K_1)^*$; then $S^*$ is a weak$^*$ compact convex set and therefore the set ext $S^*$ of its extreme points is nonempty, by the Krein-Milman theorem. Arens and Kelley [1] (cf. [4, p. 441]) showed that these extreme points are precisely those functionals of the form $f \mapsto \lambda f(k)(f \in C(K_1))$, where $k \in K_1$ and $\lambda = 1$ or $\lambda = -1$. [We denote the functional “evaluation at $k$” by $\varphi_k$, so the extreme points of $S^*$ are the functionals $\lambda \varphi_k$, $|\lambda| = 1$.] If we restrict our attention to the “positive face” of $S^*$ (those functionals $\varphi$ such that $\varphi(1) = 1$), then this is a weak$^*$ compact convex subset of $S^*$ and its extreme points are those obtained from ext $S^*$ by taking $\lambda = 1$. Now, the members of $C(K_1)^*$ can be regarded as continuous linear operators from $C(K_1)$ into $C(K_2)$, where $K_2$ consists of a single point. Thus, it is natural to consider the possible extension of the above results to the more general situation when $K_2$ is an arbitrary compact Hausdorff space, and $S^*$ is replaced by the unit ball $S = S(C(K_1), C(K_2))$ of the Banach space $B = B(C(K_1), C(K_2))$ of all bounded linear operators from $C(K_1)$ into $C(K_2)$, with the usual operator norm. Corresponding to the positive face of $S^*$ we have the convex subset $S_1$ of $S$, consisting of those $T$ in $S$ such that $T1 = 1$. It was shown by A. and C. Ionescu Tulcea [5] (and generalized in [8]) that ext $S_1$ consists of the “composition operators”, i.e., those of the form $(Tf)(k) = f(\psi(k))$ ($f \in C(K_1)$, $k \in K_2$) where $\psi$ is a continuous function from $K_2$ into $K_1$. It is easily seen that these operators are also extreme in $S$; more generally, if $\psi: K_2 \to K_1$ is continuous and $\lambda \in C(K_2)$ with $|\lambda| = 1,$

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then every operator $T$ of the form

$$(*) \quad Tf = \lambda \cdot f \circ \psi,$$

$f \in C(K_i)$

is an extreme point of $S$. Indeed, if $k$ is a point of $K_2$, then the functional on $C(K_1)$ defined by $f \mapsto (Tf)(k) = \lambda(k)f(\psi(k))$ is the extreme point $\lambda(k)\varphi(\psi(k))$ of $S^* \subset C(K_i)^*$. The above assertion follows easily from this remark. The main result of the present paper is the converse of this assertion (Theorem 1), under the additional hypothesis that $K_1$ be metrizable. The proof is not elementary, in sharp contrast to the simplicity of the proofs for the corresponding statement for $S_1$. The metrizability hypothesis may be unnecessary; we use it only to enable us to apply one of Michael’s selection theorems [6, 7]. If we consider the case of complex-valued continuous functions on $K_1$ and $K_2$, then the obvious analogues of the Arens-Kelley and Ionescu Tulcea results are valid ([4, p. 441] and [8]), and it is still true that every operator of the form $(*)$ is extreme. We do not know whether the converse is true, however, even with the metrizability hypothesis.

It is possible to formulate our characterization of the extreme operators in $S$ in another way: An operator $T$ in $S(C(K_1), C(K_2))$ ($K_1$ metrizable) is extreme in $S$ if and only if the functional $F_T(k): f \mapsto (Tf)(k) \; (f \in C(K_1))$ is extreme in $S^* \subset C(K_i)^*$ for a dense set of $k$‘s in $K_2$. (Simply use the fact that $\text{ext } S^*$ is weak* compact and that $k \mapsto F_T(k)$ is continuous.) In this form, the statement makes sense for operators $T$ from an arbitrary Banach space $X$ into $C(K)$, so that it is conceivable that the following is true: Suppose that $X$ is a Banach space, that $K$ is a compact Hausdorff space and that $T$ is an operator in $S = S(X, C(K))$, the unit ball of bounded operators from $X$ to $C(K)$. Then:

(A) If $T$ is extreme in $S$, then the functional $F_T(k)$ is extreme in the unit ball $S^*$ of $X^*$ for a dense set of points $k$ in $K$.

The converse of this assertion is clearly true. The assertion itself, however, is false; Theorem 2 shows that (in particular) if $K$ has no isolated points, then there exists a space $X$ and an extreme operator $T$ in $S(X, C(K))$ such that for each $k$ in $K$, $F_T(k)$ is not extreme in $S^*$. The hypothesis concerning isolated points is unavoidable, since it is easily verified that if $T$ is extreme in $S(X, C(K))$ and $k$ is an isolated point of $K$, then $F_T(k)$ is extreme in $S^*$. (Thus, if $K$ has a dense set of isolated points, then assertion (A) is true. This observation let Arterburn and Whitley [2] to conjecture that it is generally true.) Assertion (A) is true if some strong restrictions are made on $X$; as noted in [2], (A) holds if every point in the boundary of $S^*$ is extreme (i.e. if $X^*$ strictly convex). Assertion (A) is also true if $X$ is three-dimensional (cf. Theorem 3). An example shows that (A) is
In order to prove Theorem 1 we make the usual identification between $C(K)^*$ and the space of finite signed Borel measures $\mu$ on $K$. For $f$ in $C(K)$, $\mu(f)$ denotes the integral of $f$ with respect to $\mu$. Recall that $\mu = \mu^+ - \mu^-$ where $\mu^+, \mu^-$ are nonnegative and have disjoint Borel supports; furthermore, $||\mu|| = \mu^+(1) + \mu^-(1)$. Now, to each operator $T$ in $B(C(K_1), C(K_2))$ there corresponds a unique function $k \to \mu_k$ from $K_2$ into $C(K_1)^*$ such that

$$(Tf)(k) = \mu_k(f) \quad (k \in K_2, f \in C(K_1))$$

and

$$||T|| = \sup \{||\mu_k|| : k \in K_2\}.$$ 

This function is continuous from $K_2$ into $C(K_1)^*$ in its weak* topology (briefly: $k \to \mu_k$ is weak* continuous). Any weak* continuous function $k \to \nu_k$ for which $||\nu_k||$ is bounded defines an operator in $B$ (by means of the above equation) with norm equal to $\sup ||\nu_k||$. This correspondence between norm bounded weak* continuous functions and operators in $B$ is linear; in particular, we have the following useful fact.

If $k \to \mu_k$ is related to $T$ as above, then $T$ is extreme in $S(C(K_1), C(K_2))$ if and only if the following is true: Whenever $k \to \nu_k$ is weak* continuous and $||\mu_k \pm \nu_k|| \leq 1$ for each $k$ in $K_2$, then $\nu_k = 0$ for all $k$.

We can now prove our main result.

**Theorem 1.** Suppose that $K_1$ and $K_2$ are compact Hausdorff spaces, that $K_1$ is metrizable, and that $T$ is an operator in $S(C(K_1), C(K_2))$. Then $T$ is an extreme point of this set if and only if there exist continuous functions $\psi: K_2 \to K_1$ and $\lambda$ in $C(K_2)$, $|\lambda| = 1$, such that

$$(Tf)(k) = \lambda(k)f(\psi) \quad (k \in K_2, f \in C(K_1)).$$

**Proof.** We have already noted that any operator of the above form is extreme, so suppose $T$ is extreme in $S(C(K_1), C(K_2))$ and let $k \to \mu_k$ be the corresponding weak* continuous map from $K_2$ into the weak* compact unit ball $S^*$ of $C(K_1)^*$. We distinguish two cases:

**Case I.** For each $k$ in $K_2$, either $\mu_k \geq 0$ or $\mu_k \leq 0$.

**Case II.** For some $k$ in $K_2$, the decomposition $\mu_k = \mu_k^+ - \mu_k^-$ is nontrivial.

We take care of Case I with the help of the Ionescu Tulcea theorem. Note that in this case, $||\mu_k|| = |\mu_k(1)|$. Furthermore, the
real valued function \( k \rightarrow |\mu_k(1)| \) is continuous on \( K_2 \), hence if \( \mu \) is any nontrivial measure in \( S^* \), then the function \( k \rightarrow (1 - ||\mu_k||)\mu \) is weak* continuous from \( K_2 \) into \( S^* \). Since

\[
||\mu_k \pm (1 - ||\mu_k||)\mu|| \leq ||\mu_k|| + (1 - ||\mu_k||)||\mu|| \leq 1 ,
\]

we see that for each \( k \), \( |\mu_k(1)| = ||\mu_k|| = 1 \). Thus, if we let \( \lambda(k) = \mu_k(1) \), then \( \lambda \in C(K_2) \) and \( |\lambda| = 1 \). Consider the operator \( U: C(K_1) \rightarrow C(K_1) \) defined by \( (Uf)(k) = \lambda(k)^{-1}(Tf)(k) \). It is easily verified that \( U \) is extreme in \( S \) and that \( U1 = 1 \). It follows from the Ionescu Tulcea theorem [5; 8, Theorem 1] that there exists a continuous function \( \psi: K_2 \rightarrow K_1 \) such that \( Uf = f \circ \psi \), and hence \( T \) has the required form.

We next show that if Case II holds, then \( T \) can not be extreme. Assume, then, that there exists a point \( k_0 \) in \( K_2 \) such that \( \mu_{k_0} = \mu_{k_0}^+ - \mu_{k_0}^- \) and \( \mu_{k_0}^+ \neq 0, \mu_{k_0}^- \neq 0 \). For each \( k \) in \( K_2 \), let

\[
\Sigma(k) = \{\mu: \mu \in S^*, 0 \leq \mu \leq \mu_k\} .
\]

This is a nonempty convex weak* closed subset of \( S^* \), hence is weak* compact. We shall prove that the map \( k \rightarrow \Sigma(k) \) from \( K_2 \) into the set of all subsets of \( C(K_1)^* \) is lower semicontinuous in the sense of Michael [6] (where we take the weak* topology in \( C(K_1)^* \)). Thus, we must show the following: Given a point \( k_1 \) in \( K_2 \), a measure \( \mu_1 \) in \( \Sigma(k_1) \) and a weak* neighborhood \( V \) of \( \mu_1 \), then there is a neighborhood \( U \) of \( k_1 \) such that \( V \cap \Sigma(k) \) is empty for each \( k \) in \( U \). We can assume that \( V = \{\mu: \mu \in C(K_1)^*, |\mu(f_i) - \mu_1(f_i)| \leq 1, f_i \in C(K_1), i = 1, 2, \ldots, n\} \). Suppose that there is a net \( k_\alpha \) converging to \( k_1 \) such that \( V \cap \Sigma(k_\alpha) \) is empty for each \( \alpha \). Since \( \mu_{k_\alpha}^+ \in S^* \) for each \( \alpha \) we may assume (by taking a subnet if necessary) that \( \mu_{k_\alpha}^+ \rightarrow \mu_2 \) for some \( \mu_2 \geq 0 \) in \( S^* \); similarly \( \mu_{k_\alpha}^- \rightarrow \mu_3 \geq 0 \). Since \( \mu_{k_\alpha}^+ \rightarrow \mu_{k_1}^+ \) we have \( \mu_{k_1} = \mu_2 - \mu_3 = \mu_{k_1}^+ - \mu_{k_1}^- \) and it follows from the definition of the Hahn decomposition that \( \mu_2 \geq \mu_{k_1}^- \). Now, \( \mu_{k_1}^+ \geq \mu_1 \geq 0 \), so by the Radon-Nikodym theorem we can write \( d\mu_1 = g_1d\mu_2 \), where \( g_1 \) is a Borel function on \( K_1 \) with \( 0 \leq g_1 \leq 1 \). We can choose a continuous function \( g \) on \( K_1 \) such that \( 0 \leq g \leq 1 \) and

\[
\int_{K_1} |g - g_1| d\mu_2 \leq (2 \max \{||f_i||: 1 \leq i \leq n\})^{-1} .
\]

If we define \( \mu_\alpha \) by \( d\mu_\alpha = g_\alpha d\mu_\alpha^+ \) then \( \mu_\alpha \in \Sigma(k_\alpha) \) for each \( \alpha \). Furthermore, for each \( i \) we have

\[
\mu_\alpha(f_i) = \int_{K_1} f_id\mu_\alpha = \int_{K_1} f_id\mu_\alpha^+ \rightarrow \int_{K_1} f_id\mu_2
\]

and hence

\[
\lim sup |\mu_\alpha(f_i) - \mu_1(f_i)| \leq \int_{K_1} |f_ig - f_ig_1| d\mu_2 \leq 1/2 ,
\]
so that $\mu_a$ is eventually in $V \cap \Sigma(k_a)$, a contradiction.

Thus, the set-valued map $k \mapsto \Sigma(k)$ is lower semicontinuous and since $C(K_i)$ is separable—this is the only place where we use the metrizability of $K_i$—we may apply a selection theorem of Michael ([6, Th. 3.2*], [6, Ex. 1.3*] and [7]) to conclude the existence of a continuous selection for this map taking at $k_0$ the preassigned value $\mu^+_{k_0}$, that is, there exists $\sigma: K_i \rightarrow C(K_i)^*$ which is weak* continuous such that $\sigma_{k_0} = \mu^+_{k_0}$ and $\sigma_k \in \Sigma(k)$ for each $k$, i.e. $0 \leq \sigma_k \leq \mu^+_{k}$. Similarly, there exists a continuous map $\tau: K_i \rightarrow C(K_i)^*$ such that $\tau_{k_0} = \mu^{-}_{k_0}$ and $0 \leq \tau_k \leq \mu^-_{k}$ for each $k$. We define the map $\nu: K_i \rightarrow C(K_i)^*$ by

$$
\nu_k = \sigma_k \left\| \tau_k \right\| + \tau_k \left\| \sigma_k \right\| .
$$

Since $\| \sigma_k \| = \sigma_k(1)$, the function $k \mapsto \| \sigma_k \|$ is continuous (similarly for $\tau$) and hence $k \mapsto \nu_k$ is weak* continuous. Furthermore, $\nu_{k_0} \neq 0$. (This last assertion is the reason we needed two cases; in case I, $\nu_k \equiv 0$.) Finally, $\| \mu_k \pm \nu_k \| \leq 1$ for each $k$. Indeed,

$$
\| \mu_k + \nu_k \| = \| (\mu^+_{k} + \sigma_k \| \tau_k \|) + (\mu^-_{k} - \tau_k \| \sigma_k \|) \|
\begin{align*}
&= \| \mu^+_{k} \| + \| \sigma_k \| \| \tau_k \| + \| \mu^-_{k} \| - \| \sigma_k \| \| \tau_k \|
&= \| \mu_k \| \leq 1 ;
\end{align*}
$$
similarly for $\| \mu_k - \nu_k \|$. It follows that $T$ is not extreme, a contradiction which completes the proof.

It is clear that the above proof would be valid without the hypothesis that $K_i$ be metrizable if there were a selection theorem for the more general situation. Unfortunately, however, it is known (see e.g. [3]) that there exist compact Hausdorff space $K_1$ and $K_2$ ($K_1$ not metrizable, $K_2$ metrizable) and a weak* lower semicontinuous map from $K_2$ into the weak* compact convex subsets of the unit ball of $C(K_2)^*$ which does not have a continuous selection.

It is worth noting that we can eliminate the metrizability hypothesis in Theorem 1 if we put an additional hypothesis on the operator $T$. Namely, if $T$ is extreme in $S(C(K_i), C(K_j))$ and if the function $k \mapsto \| \mu_k \|$ is continuous, then $T$ is of the form (*). We will sketch a proof of this fact. Since the proof of Theorem 1 used metrizability only in Case II, we will show that this case cannot occur. First, note that the continuity of $k \mapsto \| \mu_k \|$ implies the weak* continuity of the functions $k \mapsto \mu^+_{k}$ and $k \mapsto \mu^-_{k}$. (This may be proved by contradiction, using the weak* compactness of $S^*$, the weak* continuity of $k \mapsto \mu_k$ and the fact that if $\mu_k = \mu_1 - \mu_2$, $\mu_1, \mu_2 \geq 0$, then $\mu_1 \geq \mu^+_{k}$, $\mu_2 \geq \mu^-_{k}$.) It follows that $k \mapsto \nu_k = \mu^+_{k}(1)\mu^+_{k} + \mu^-_{k}(1)\mu^-_{k}$ is weak* continuous, and $\| \mu_k \pm \nu_k \| \leq 1$ for each $k$. Since $T$ is extreme, $\nu_k \equiv 0$; in particular, $0 = \nu_k(1) = 2\mu^+_{k}(1)\mu^+_{k}(1)$ for each $k$, which implies that Case I holds.

The above hypothesis on $T$ is fulfilled in case $T$ is a compact
operator; indeed, $T$ is compact if and only if $k \to \mu_k$ is norm continuous [4, p. 490]. (Examination of the above proof and that in [8, Theorem 1] shows that in this case it suffices to assume only that $T$ is extreme in the unit ball of compact operators.)

Suppose, now, that $X$ is a Banach space and that $T$ is an extreme point of $S(X, C(K))$ ($K$ compact Hausdorff). As noted in the introduction, the functional $F_\tau(k)$ is extreme in the unit ball $S^*$ of $X^*$ whenever $k$ is an isolated point of $K$. The next result shows that for any $K$ there exists a space $X$ such that these are the only such functionals which are extreme.

**Theorem 2.** Suppose that $K$ is a compact Hausdorff space. Then there exists a Banach space $X$ and an extreme operator $T$ in $S(X, C(K))$ such that $F_\tau(k) \in \text{ext } S^*$ (if and only if $k$ is an isolated point of $K$.

**Proof.** Let $M$ be the compact set of all points in $K$ which are not isolated and let $l_2(M)$ be the Banach space of all real-valued functions $y$ on $M$ such that $\|y\|^2 = \sum_{k \in M} |\hat{y}(k)|^2 < \infty$. Let $X_0 = C(K) \oplus l_2(M)$, with norm $\|(f, y)\| = \max (\|f\|, \|y\|)$ ($f \in C(K), y \in l_2(M)$). As is well known, the conjugate space $X_0^*$ of such a product space is linearly isometric to $C(K)^* \oplus l_2(M)^*$, with $\|(\mu, y^*)\| = \|\mu\| + \|y^*\|$. Since $l_2(M)$ is a Hilbert space, we may identify $l_2(M)^*$ and $l_2(M)$. The space $X_0^*$ in its weak* topology is linearly homeomorphic with the product space $C(K)^* \times l_2(M)^*$ (where we take the weak* topologies in each of these spaces). Let $S_1 = S_{\sigma, C(K)}^* \times \{0\}$ and $S_2 = \{0\} \times S_{\sigma, l_2(M)}^*$; these sets are weak* compact and convex, and the set $\text{conv} (S_1 \cup S_2)$ is the unit ball of $X_0^*$. (Consider $(\mu, y^*) = \|\mu\| (\|\mu\|, 0) + \|y^*\| (0, y^*/\|y^*\|)$.) For $k$ in $M$, let $e_k \in l_2(M)$ be the function which is 1 at $k$, 0 elsewhere. Let $Q = \{\pm (\varphi_k, \pm e_k) : k \in M\} \cup \{\pm (\varphi_k, 0) : k \in M\}$ (as usual, $\varphi_k \in C(K)^*$ is evaluation at $k$). Note that $Q$ is weak* compact. Indeed, it follows from the definition of $l_2(M)$ that any $y$ in $l_2(M)$ gets arbitrarily small outside finite sets, so if $\{k_n\}$ is a net in $M$ with infinite range, then the set $\{e_{k_n}\}$ has a subnet which converges to 0 in the weak* topology. This fact (together with compactness of $M$ and weak* continuity of $k \to \varphi_k$) makes it possible to show that $Q$ is weak* compact. Let $A$ be the weak* closed convex hull of $Q$; then $A$ is norm bounded, hence weak* compact, and the set $S' = \text{conv} (S_1 \cup S_2 \cup A)$ is weak* compact and convex. Furthermore, $S'$ is symmetric with respect to 0 and contains the unit ball $\text{conv} (S_1 \cup S_2)$ of $X_0^*$. From standard results on duality, then, $S'$ is the polar of a convex body in $X_0$ which defines a norm $\| \cdot \|'$ on $X_0$, equivalent to the original norm. Furthermore, if we let $X$ denote $X_0$ in the norm $\| \cdot \|'$, then $S'$ is the unit ball of the space $X^*$. We have defined our space $X$; to define the operator
Let $F_\tau(k) = (\varphi_k, 0)$, $k \in K$. It is clear that $F_\tau$ is weak* continuous; our next task is to show that $T$ has the required properties.

Suppose, first, that $k \in K$ and that $(\varphi_k, 0) \pm (\mu, y^*) \in S'$. It follows immediately that $||\varphi_k \pm \mu|| \leq 1$, so that $\mu = 0$ and hence $(\varphi_k, \pm y^*) \in S'$. We will show that if $k \in K \sim M$, then $y^* = 0$, while if $k \in M$, then $y^* = \lambda e_k$ for some $|\lambda| \leq 1$. To this end, let $\Phi_k = \{f : f \in C(K), f(k) = 1 = ||f||\}$. If $f \in \Phi_k$, let $H_f = \{(\nu, z^*) : (\nu, z^*) \in X_0^*, \nu(f) = 1\}$; this is a weak* closed hyperplane and, since $(\nu, z^*) \in S'$ implies $||\nu|| \leq 1$ (hence $\nu(f) \leq 1$), $H_f$ is a supporting hyperplane to $S'$. In particular, $H_f \cap S'$ is an extremal subset of $S'$.

Since the intersection of extremal sets is an extremal set, the set $J_k = \bigcap_{f \in \Phi_k} H_f$ is an extremal subset of $S'$; furthermore, it is clear that $(\varphi_k, y^*) \in S' \cap J_k$. Now, since $J_k$ is extremal, $\text{ext}(J_k \cap S') \subset J_k \cap \text{ext} S'$, and from the definition of $S'$ we see that $\text{ext} S' \subset \text{ext} S_1 \cup \text{ext} S_2 \cup \text{ext} A$. Also, since $Q$ is compact, the Milman theorem implies that $\text{ext} A \subset Q$. Since $J_k \cap S_2$ is empty, we have $\text{ext}(J_k \cap S') \subset (J_k \cap \text{ext} S_1) \cup (Q \cap J_k)$. Recall that if $(\nu, z^*) \in J_k$, $||\nu|| \leq 1$, then $\nu(f) = 1$ for all $f \in \Phi_k$, so $\nu = \varphi_k$. Thus, $J_k \cap S_1 = \{(\varphi_k, 0)\}$, and $Q \cap J_k = \{(\varphi_k, \pm e_k), (\varphi_k, 0)\}$ if $k \in M$; otherwise $Q \cap J_k$ is empty. Now, $(\varphi_k, y^*) \in J_k \cap S'$ and the latter is the closed convex hull of its extreme points. From what we have just shown, $(\varphi_k, y^*) = (\varphi_k, 0)$ if $k \in K \sim M$, while $(\varphi_k, y^*)$ is a convex combination of $(\varphi_k, 0)$ and $(\varphi_k, \pm e_k)$ if $k \in M$; these remarks show that $y^*$ has the form indicated earlier. It is clear that $(\varphi_k, 0) \in \text{ext} S'$ if and only if $k \in K \sim M$.

It remains to show that if there exists a weak* continuous map $G : K \to X_0^*$ such that $F_\tau(k) \pm G(k) \in S'$ for each $k$, then $G = 0$. Such a map would have the form $G(k) = (\mu_k, y_k^*)$, where $k \to \mu_k, k \to y_k^*$ are weak* continuous in the appropriate spaces, and we would have $(\varphi_k, 0) \pm (\mu_k, y_k^*) \in S'$. From what we have shown above, $\mu_k = 0$ for all $k$, $y_k^* = 0$ for $k \in K \sim M$, and $y_k^* = \lambda(k)e_k$ for some $|\lambda(k)| \leq 1$ if $k \in M$. Since $k \to y_k^*$ is continuous, $y_k^* = 0$ for $k$ in the closure of $K \sim M$. Suppose that $k$ is an interior point of $M$ and choose an infinite net of distinct points $k_\alpha$ in $M$ with $k_\alpha \to k$. We know that $e_{k_\alpha} \to 0$ in the weak* topology, so $y^*_{k_\alpha} = \lambda(k_\alpha)e_{k_\alpha} \to 0$ and hence $y_k^* = 0$. Thus, $y_k^* = 0$ for all $k$ in $K$ and the proof is complete.

Our next result shows that the conjecture of Arterburn and Whitley is true for two or three dimensional spaces $X$, as well as for certain other finite dimensional spaces.

**Theorem 3.** Suppose that $X$ is a finite dimensional normed linear space such that

(i) $\dim X \leq 3$

or

(ii) the unit ball of $X$ is a polyhedron.
If $K$ is a compact Hausdorff space and $T$ is an extreme operator in $S(X, C(K))$, then \{ $k : k \in K \text{ and } F_\tau(k) \in \text{ext } S^*$ \} is dense in $K$.

Proof. We shall give the proof for the case $\dim X = 3$, since the proof for the remaining cases is similar but much simpler. The idea of the proof (in any of the three cases) is this: We show that for each $k$, $F_\tau(k)$ must lie on the surface of the unit ball $S^*$ of $X^*$. We then show that $F_\tau(k)$ cannot be in the (relative) interior of any face of $S^*$. In the two dimensional case, this completes the proof, while in the polyhedral case we can apply the same argument to faces of one less dimension, and the proof proceeds by induction. (Recall that in case (ii), the dual ball $S^*$ is also a polyhedron.) In the three dimensional case we need a special argument, since the interior of a one dimensional face need not be isolated from other one dimensional faces. Suppose, then, that $\dim X = 3$ and that $T \in \text{ext } S(X, C(K))$. Let $y^* \neq 0$ be a fixed element of $S^*$. The map $k \rightarrow (1 - ||F_\tau(k)||) y^*$ is continuous, and $||F_\tau(k) \pm (1 - ||F_\tau(k)||) y^*|| \leq ||F_\tau(k)|| + (1 - ||F_\tau(k)||) ||y^*|| \leq 1$; since $T$ is extreme, this implies that $||F_\tau(k)|| = 1$ for each $k$, i.e., $F_\tau(k)$ is on the boundary of $S^*$ for each $k$. We next show that $F_\tau(k)$ cannot be an interior point of a two dimensional face of $S^*$, for any $k$ in $K$. Indeed, if $F_\tau(k_0)$ were an interior point of such a face, the same would be true for all points of $k$ in some neighborhood of $k_0$. We could then easily find a continuous real valued function $g$ on $K$, not identically zero, and $x^* \in X^*$, $x^* \neq 0$, such the $||F_\tau(k) \pm g(k)x^*|| \leq 1$ for each $k$, contradicting the fact that $T$ is extreme. Suppose, finally, that there exists nonempty open set $U \subset K$ such that $F_\tau(k) \in \text{ext } S^*$ for each $k$ in $U$. From what we have shown it follows that for each $x^*$ in $F_\tau(U)$ there exists a vector, $u(x^*)$ in $X^*$, unique up to sign, such that $||u(x^*)|| = 1$ and $||x^* \pm \varepsilon u(x^*)|| = 1$ for some $\varepsilon > 0$. For every $x^*$ in $F_\tau(U)$, let $\lambda(x^*) = \sup \{ \varepsilon : ||x^* \pm \varepsilon u(x^*)|| = 1 \}$. Suppose that $x^*_n, x^*$ are in $F_\tau(U)$ and that $x^*_n \rightarrow x^*$. The sequence $u(x^*_n)$ has a subsequence converging to some $y^*$ in $S^*$. If $\lambda(x^*_n) \geq \delta$ for some $\delta > 0$ and all $n$ we see that $||x^* \pm \delta y^*|| = 1$ and hence $y^* = \pm u(x^*)$ and $\lambda(x^*) \geq \delta$. This shows that $\lambda$ and therefore $\lambda \circ F_\tau$ are upper semicontinuous; in particular, $\lambda \circ F_\tau$ has points of continuity and hence there exist a nonempty open subset $M$ of $U$ and an $\epsilon > 0$ such that $\lambda(x^*) \geq \epsilon$ for $x^*$ in $F_\tau(M)$. We may assume that for some $x$ in $X$, $|(u(x^*), x)| > 0$ for all $x^*$ in $F_\tau(M)$ (otherwise replace $M$ by $M \cap F_\tau^{-1}\{x^*; |(u(x^*), x)| > 0\}$, where $x$ is chosen so that this intersection is nonempty). For $x^*$ in $F_\tau(M)$ let $e(x^*)$ be the sign (i.e. 1 or $-1$) such that $e(x^*)(u(x^*), x) > 0$. The compactness argument used above shows that the map $x^* \rightarrow$
$e(x^*)u(x^*)$ is continuous on $F_T(M)$. Choose a real valued continuous function $g$ on $K$ such that $g \neq 0$, $0 \leq g \leq \varepsilon$ and $g(k) = 0$ if $k \notin M$. Then for all $k$,

$$|| F_T(k) \pm g(k)e(F_T(k))u(F_T(k)) || \leq 1 .$$

(We define $g(k)e(F_T(k))u(F_T(k))$ to be zero for $k \notin M$.) Since $T$ is extreme, we conclude that the set $U$ must have been empty and the proof is complete.

Our final result is an example which shows that the above theorem fails for a certain four dimensional space $X$.

**EXAMPLE.** There exists a four dimensional normed linear space $X$ and an extreme operator $T$ in $S(X, C[0, 1])$ such that $F_T(t) \notin \text{ext } S^*$ for each $t$ in $[0, 1]$.

**Proof.** Let $g$ be a monotonic increasing function on $[0, 1]$ such that $g$ is discontinuous at every rational point $t$ and continuous elsewhere. Let $\Sigma$ be the subset of $R^4$ consisting of the points

$$\pm (\cos t, \sin t, \cos g(t + 0), \sin g(t + 0))$$
$$\pm (\cos t, \sin t, -\cos g(t + 0), -\sin g(t + 0))$$
$$\pm (\cos t, \sin t, \cos g(t - 0), \sin g(t - 0))$$
$$\pm (\cos t, \sin t, -\cos g(t - 0), -\sin g(t - 0)) ,$$

where $0 \leq t \leq 1$. The set $\Sigma$ is symmetric with respect to the origin and it is also compact: If $\{t_n\}$ is a sequence in $[0, 1]$ there is a subsequence $\{t_{n_k}\}$ such that either $t_{n_k} \uparrow t$ or $t_{n_k} \downarrow t$ for some $t$ in $[0, 1]$.

Let $S'$ be the convex hull of $\Sigma$; then $S'$ is a compact, convex, symmetric subset of $R^4$ having nonempty interior, and hence is the unit ball of $X^*$, where $X$ is the four-dimensional Banach space which has the polar $S^o$ of $S'$ as its unit ball. Suppose that $t$ is an irrational number in $[0, 1]$. The point $x = (\cos t, \sin t, 0, 0)$ is in $S^o$; indeed, for each $(u, v, w, z)$ in $\Sigma$, $u \cos t + v \sin t \leq 1$ and equality occurs only if $u = \cos t$, $v = \sin t$ and $(w, z) = \pm (\cos g(t), \sin g(t))$. It follows that if $(\cos t, \sin t, 0, 0) \pm (u, v, w, z) \in S'$, then (since the inner product of these two points with $x = (\cos t, \sin t, 0, 0)$ equals 1) $u = v = 0$ and the point $(\cos t, \sin t, w, z)$ is in the convex hull of the subset of $\Sigma$ whose inner product with $x$ equals 1, i.e. $w = \lambda \cos g(t)$, $z = \lambda \sin g(t)$, $|\lambda| \leq 1$.

Let $T$ be the operator from $X$ into $C[0, 1]$ defined by $F_T(t) = (\cos t, \sin t, 0, 0)$, $t \in [0, 1]$. For each $t$, $F_T(t) \pm (0, 0, \cos g(t), \sin g(t)) \in \Sigma$, and hence $F_T(t) \in S'$ but $F_T(t) \notin \text{ext } S'$. Suppose, however, that there exists $G: [0, 1] \rightarrow X^*$ which is continuous and satisfies $F_T(t) \pm G(t) \in S'$ for every $t$. Then, as we have seen, for every irrational $t$, $G(t) =
(0, 0, \lambda(t) \cos g(t), \lambda(t) \sin g(t)) with |\lambda(t)| \leq 1. Let t_0 be a rational point in (0, 1). Since G is continuous, \lim_{t \to t_0} G(t) = \lim_{t \to t_0} G(t) and since g is discontinuous at t_0 it follows that if t_n \to t_0 with t_n irrational, then \lambda(t_n) \to 0. Thus G(t_0) = 0, and therefore G vanishes on the rationals in (0, 1) and hence identically on [0, 1]. It follows that T is extreme and the proof is complete.

BIBLIOGRAPHY

2. David Arterburn and Robert Whitley, Extreme linear operators, dittoed notes.
5. A. Ionescu Tulcea and C. Ionescu Tulcea, A note on extreme points, dittoed notes.

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David R. Arterburn and Robert James Whitley, *Projections in the space of bounded linear operators* .......................................................... 739

Robert McCallum Blumenthal, Joram Lindenstrauss and Robert Ralph Phelps, *Extreme operators into C(K)* .................................................... 747

L. Carlitz, *A note on multiple exponential sums* ........................................ 757

Joseph A. Cima, *A nonnormal Blaschke-quotient* ...................................... 767

Paul Civin and Bertram Yood, *Lie and Jordan structures in Banach algebras* .... 775

Luther Elic Claborn, *Dedekind domains: Overrings and semi-prime elements* .... 799

Luther Elic Claborn, *Note generalizing a result of Samuel’s* ....................... 805

George Bernard Dantzig, E. Eisenberg and Richard Warren Cottle, *Symmetric dual nonlinear programs* ......................................................... 809

Philip J. Davis, *Simple quadratures in the complex plane* ............................ 813

Edward Richard Fadell, *On a coincidence theorem of F. B. Fuller* ............... 825

Delbert Ray Fulkerson and Oliver Gross, *Incidence matrices and interval graphs* .... 835

Larry Charles Grove, *Tensor products over H*-algebras* .............................. 857

Deborah Tepper Haimo, *L2 expansions in terms of generalized heat polynomials and of their Appell transforms* ........................................... 865

I. Martin (Irving) Isaacs and Donald Steven Passman, *A characterization of groups in terms of the degrees of their characters* ......................... 877

Donald Gordon James, *Integral invariants for vectors over local fields* .......... 905

Fred Krakowski, *A remark on the lemma of Gauss* ..................................... 917

Marvin David Marcus and H. Mine, *A subdeterminant inequality* .................. 921

Kevin Mor McCrimmon, *Norms and noncommutative Jordan algebras* .......... 925

Donald Earl Myers, *Topologies for Laplace transform spaces* ....................... 957

Olav Njstad, *On some classes of nearly open sets* ..................................... 961

Milton Philip Olson, *A characterization of conditional probability* ................ 971

Barbara Ososky, *A counter-example to a lemma of Skornjakov* ...................... 985

Sidney Charles Port, *Ratio limit theorems for Markov chains* ...................... 989

George A. Reid, *A generalisation of \( W^* \)-algebras* ................................. 1019

Robert Wells Ritchie, *Classes of recursive functions based on Ackermann’s function* ............................................................... 1027

Thomas Lawrence Sherman, *Properties of solutions of nth order linear differential equations* ........................................................... 1045

Ernst Snapper, *Inflation and deflation for all dimensions* ............................ 1061

Kondagunta Sundaresan, *On the strict and uniform convexity of certain Banach spaces* ............................................................. 1083

Frank J. Wagner, *Maximal convex filters in a locally convex space* ............. 1087

Joseph Albert Wolf, *Translation-invariant function algebras on compact groups* .............................................................. 1093