A NOTE ON MULTIPLE EXPONENTIAL SUMS

L. CARLITZ

Put

\[ S(c) = \sum_{x,y = 1}^{p-1} e(x + y + cx'y') , \]

where \( e(x) = e^{2\pi i x/p} \) and \( xx' \equiv yy' \equiv 1 \pmod{p} \), Mordell has conjectured that \( S(c) = O(p) \). The writer shows first, by an elementary argument that \( S(c) = O(p^{3/2}) \). Next he proves, using a theorem of Lang and Weil that \( S(c) = O(p^{11/8}) \). Finally he proves that \( S(c) = O(p^{5/4}) \); the proof makes use of the estimate

\[ \sum_{x=0}^{p-1} \phi(f(x)) = O(p^{1/2}) , \]

where \( \phi(a) \) is the Legendre symbol and \( f(x) \) is a polynomial of the fourth degree.

If we put

\[ K(a, b) = \sum_{x=1}^{p-1} e(ax + bx') , \]

where \( ab \not\equiv 0 \pmod{p} \), it is known that

\[ (2) \quad |K(a, b)| \leq 2p^{1/2} . \]

For proof of (2) see [1], [4].

Since

\[ S = \sum_{x=1}^{p-1} e(ax) \sum_{y=1}^{p-1} e(by + cx'y') \]
\[ = \sum_{x=1}^{p-1} e(ax) K(b, cx') , \]

it follows that

\[ |S| \leq \sum_{x=1}^{p-1} |K(b, cx')| \leq 2(p - 1)p^{1/2} \]

by (2). Thus, assuming (2), we get

\[ (3) \quad S = O(p^{3/2}) . \]

However it is not difficult to prove (3) directly without making use of (2). Put

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(4) \[ S(c) = \sum_{x,y=1}^{p-1} e(x + y + cx'y') . \]

There is evidently no loss in generality in taking \( a = b = 1 \). Then we have

\[ \sum_{c=0}^{p-1} |S(c)|^2 = \sum_{c=0}^{p-1} \sum_{x,y=1}^{p-1} \sum_{u,v=1}^{p-1} e\{x + y - u v + c(x'y' - u'v')\} \]
\[ = p \sum_{x,y=1}^{p-1} e(x + y - u - v) . \]

But

\[ \sum_{x,y=1}^{p-1} e(x + y - u - v) = \sum_{x,y,u=1}^{p-1} e(x + y - u - xyu') \]
\[ = \sum_{y,u=1}^{p-1} e(y - u) \sum_{x=1}^{p-1} e\{x(1 - yu')\} \]
\[ = - \sum_{y,u=1}^{p-1} e(y - u) + \sum_{y,u=1}^{p-1} e(y - u) \sum_{x=0}^{p-1} e\{x(1 - yu')\} \]
\[ = - 1 + p \sum_{y=1}^{p-1} 1 = p^2 - p - 1 , \]

so that

(5) \[ \sum_{c=0}^{p-1} |S(c)|^2 = p^2 - p^2 - p . \]

It follows at once from (5) that

(6) \[ |S(c)| < p^{3/2} , \]

so that we have proved (3).

2. Generalizing (4) we define

(7) \[ S_n(c) = \sum_{x_1,\ldots,x_n=1}^{p-1} e(x_1 + \cdots + x_n + cx_1'\cdots x_n') . \]

We shall show that

(8) \[ S_n(c) = O(p^{1/2(n+1)}) . \]

Exactly as above we have

(9) \[ \sum_c |S_n(c)|^2 = p \sum_{x_1,\ldots,x_n} \sum_{y_1,\ldots,y_n} e(x_1 + \cdots + x_n - y_1 - \cdots - y_n) , \]

where the summation is over all \( x_j, y_j \) such that

\[ x_1x_2\cdots x_n \equiv y_1y_2\cdots y_n , \quad x_j \equiv 0 , \quad y_j \equiv 0 \pmod{p} . \]
Let $T_n$ denote the sum on the right of (9). Then we have
\[
T_n = \sum e(x_1 + \cdots + x_n - y_1 - \cdots - y_{n-1})
\]
\[
= \sum_{x_1, \ldots, x_n - 1, y_1, \ldots, y_{n-1}} e(x_1 + \cdots + x_{n-1} - y_1 - \cdots - y_{n-1})
\]
\[
\cdot \sum_x e[(1 - x_1 \cdots x_{n-1} y_1 \cdots y'_{n-1})x] .
\]

The inner sum is equal to
\[
\begin{cases}
p - 1 & (x_1 \cdots x_{n-1} \equiv y_1 \cdots y_{n-1}) \\
-1 & (x_1 \cdots x_{n-1} \not\equiv y_1 \cdots y_{n-1}) ,
\end{cases}
\]
so that
\[
T_n = p T_{n-1} - \sum_{x_1, \ldots, x_n - 1, y_1, \ldots, y_{n-1}} e(x_1 + \cdots + x_{n-1} - y_1 - \cdots - y_{n-1}) .
\]

Hence
\[
T_n = p T_{n-1} - 1 .
\]

Now
\[
T_1 = \sum_{x, y} e(x - y) = p - 1 ,
\]
\[
T_2 = p(p - 1) - 1 = p^2 - p - 1
\]
and generally
\[
T_n = p^n - p^{n-1} - \cdots - 1 .
\]

Thus (9) becomes
\[
(12) \sum |S_n(c)|^2 = p^{n+1} - p^n - \cdots - p
\]
and (8) follows at once.

It follows from (12) that
\[
S_n(c) = o(p^{n/2})
\]
cannot hold for all $c$.

3. Returning to (4) we shall now show that
\[
(13) S(c) = O(p^{11/8}) .
\]

It is convenient to put
\[
S(a, b, c) = \sum_{x, y} e(ax + by + cx'y') .
\]
Then

\[ \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} |S(a, b, c)|^4 = p^3 \],

where \( N \) denotes the number of solutions of the system

\[
\begin{align*}
    x_1 + x_2 &\equiv x_3 + x_4 \\
    y_1 + y_2 &\equiv y_3 + y_4 \\
    x_1' y_1' + x_2' y_2' &\equiv x_3' y_3' + x_4' y_4' \\
    x_1 x_2 x_3 x_4 y_1 y_2 y_3 y_4 &\not\equiv 0.
\end{align*}
\]

Eliminating \( x_4, y_4 \) it follows that \( N \) is the number of solutions of

\[ (x_1 y_1 + x_2 y_2) x_3 y_3 (x_1 + x_2 - x_3)(y_1 + y_2 - y_3) \equiv x_1 y_1 x_2 y_2 [(x_1 + x_2 - x_3)(y_1 + y_2 - y_3) + x_3 y_3] \]

such that

\[ x_1 x_2 x_3 y_1 y_2 y_3 (x_1 + x_2 - x_3)(y_1 + y_2 - y_3) \not\equiv 0. \]

Now by a theorem of Lang and Weil [2] we have

\[ N = p^5 + O(p^{5-1/2}), \]

so that (14) becomes

\[ \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} |S(a, b, c)|^4 = p^3 + O(p^{15/2}) . \]

On the other hand

\[
\begin{align*}
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} |S(a, b, c)|^4 &\equiv |S(0, 0, 0)|^4 + 3 \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |S(a, b, 0)|^4 \\
&\quad + 3 \sum_{a=1}^{p-1} |S(a, 0, 0)|^4 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} |S(a, b, c)|^4 \\
&\equiv (p - 1)^6 + (p - 1)^5 + 3(p - 1)^6 + (p - 1)^5 \sum_{c=1}^{p-1} |S(c)|^4 ,
\end{align*}
\]

so that (17) reduces to

\[ \sum_{c=1}^{p-1} |S(c)|^4 = O(p^{11/2}) . \]

Clearly (18) implies (13).

4. If an exact formula for

\[ \sum_{c=0}^{p-1} |S(c)|^4 \]
were available we should presumably be able to prove

\[(19) \quad S(c) = O(p^{5/4}).\]

In this connection it may be of interest to remark that the sum

\[(20) \quad \sum_{c=0}^{p-1} S^2(c)\]

can be evaluated. Indeed if we put

\[S(a, b, c) = \sum_{x, y} e(ax + by + cx'y')\]

then

\[(21) \quad \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} (S(a, b, c))^2 = p^3 N,\]

where \(N\) denotes the number of solutions of the system

\[(22) \quad \begin{cases} x_1 + x_2 + x_3 = 0 \\ y_1 + y_2 + y_3 = 0 \\ x'_1y'_1 + x'_2y'_2 + x'_3y'_3 = 0 \\ x_1x_2x_3y_1y_2y_3 \neq 0. \end{cases}\]

Eliminating \(x_3, y_3\), we find that (22) reduces to

\[(23) \quad x_1(x_1 + x_2)y_1^2 + (x_1^2 + 3x_1x_2 + x_2^2)y_1y_2 + x_2(x_1 + x_2)y_2^2 = 0\]

together with

\[(24) \quad x_1x_2y_1y_2(x_1 + x_2)(y_1 + y_2) \neq 0.\]

We may replace (23) by

\[(25) \quad [(x_1 + x_2)y_1 + x_2y_2][x_1y_1 + (x_1 + x_2)y_2] = 0.\]

If \(x_1x_2(x_1 + x_2)y_1 \neq 0\), it is clear from (25) that \(y_2 \neq 0\) and \(y_1 - y_2 \neq 0\). The two factors in (25) may vanish simultaneously. This will happen when

\[(26) \quad x_1^2 + x_1x_2 + x_2^2 = 0,\]

that is when \(-3\) is a quadratic residue of \(p\); moreover if \(x_1, x_2\) satisfy (26) with \(x_1x_2 \neq 0\) then \(x_1 + x_2 \neq 0\). Thus the number of solutions of (26) is equal to

\[\left\{1 + \left(\frac{-3}{p}\right)\right\} \frac{p-1}{2}.\]
If $-3$ is a nonresidue we find that
\begin{equation}
N = 2(p - 1)^t (p - 2),
\end{equation}
while, if $-3$ is a residue,
\begin{equation}
N = 2(p - 1)^t (p - 2) - (p - 1)^2.
\end{equation}

For $p = 3$ we have
\begin{equation}
N = 4,
\end{equation}
for it is evident from (22) that $x_1 = x_2 = x_3$, $y_1 = y_2 = y_3$.

Combining (27) and (28) we have
\begin{equation}
N = 2(p - 1)^t (p - 2) - \left\{1 + \left(\frac{-3}{p}\right)\right\} \left(\frac{p - 1}{2}\right) \quad (p > 3).
\end{equation}

On the other hand, since
\begin{align*}
S(0, 0, 0) &= (p - 1)^t S(a, 0, 0) = - (p - 1) \quad (a \neq 0), \\
S(a, b, 0) &= 1 \quad (ab \neq 0),
\end{align*}
we have
\begin{align*}
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} (S(a, b, c))^3 &= (p - 1)^t - 3(p - 1)^t + 3(p - 1)^2 \\
+ & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} (S(a, b, c))^3 \\
= (p - 1)^t - 3(p - 1)^t + 3(p - 1)^2 + (p - 1)^2 \sum_{c=1}^{p-1} (S(c))^3.
\end{align*}

Therefore, using (21) and (30), we get
\begin{equation}
\sum_{c=1}^{p-1} (S(c))^3 = 2p^3(p - 2) - (p - 1)^t \\
+ 3(p - 1)^t - 3 - \frac{1}{2} \left\{1 + \left(\frac{-3}{p}\right)\right\}.
\end{equation}

5. We shall now show that
\begin{equation}
S(c) = O(p^{5/4}).
\end{equation}

With the notation of § 3 we have, as above,
\begin{equation}
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \left| S(a, b, c) \right|^t = p^3 N,
\end{equation}
where $N$ is the number of solutions of the system
Note that we have replaced each $x_j, y_j$ by its reciprocal (mod $p$).

If we put

\[ x_3 = x_1 u_1, \quad x_4 = x_2 u_2, \quad y_3 = y_1 v_1, \quad y_4 = y_2 v_2, \]

(34) becomes

\[ \begin{cases} (x_1 + x_2) u_1 u_2 & = x_1 u_1 + x_2 u_2 \\ (y_1 + y_2) v_1 v_2 & = y_1 v_1 + y_2 v_2 \\ x_1 y_1 + x_2 y_2 & = x_1 y_1 u_1 v_1 + x_2 y_2 u_2 v_2 \\ x_1 x_2 y_1 y_2 u_1 u_2 v_1 v_2 & = 0 \end{cases} \]

(35)

Now put $x_2 = x_1 x$, $y_2 = y_1 y$ and (35) reduces to

\[ \begin{cases} (1 + x) u_1 u_2 & = u_1 + xu_2 \\ (1 + y) v_1 v_2 & = v_1 + yv_2 \\ 1 + xy & = u_1 v_1 + xyu_2 v_2 \\ xyx_1 y_1 u_1 v_1 u_2 v_2 & = 0 \end{cases} \]

(36)

Finally, eliminating $x$, $y$ we get the single equation

\[ \frac{(1 - u_1)(1 - v_1)(1 - u_2 v_1)}{u_1 v_1} + \frac{(1 - u_2)(1 - v_2)(1 - u_2 v_2)}{u_2 v_2} = 0 \]

subject to

\[ x_1 y_1 u_1 v_1 u_2 v_2 \neq 0 \]

(37)

It should be noted that for fixed $u_1$, $v_1$, $u_2$, $v_2$ satisfying (37), $x$, $y$ are uniquely determined by (36) unless $u_1 \equiv u_2 v_1 \equiv v_1 \equiv v_2 \equiv 1$; also we find that the forbidden cases $xy = 0$ or $xy$ “infinite” contribute $O(p^2)$.

Let $N'(k)$ denote the number of solutions $u$, $v \neq 0$ of

\[ (1 - u)(1 - v)(1 - uv) = kwv \]

(39)

and let $N(k)$ denote the total number of solutions of (39), so that

\[ N(k) = N'(k) + O(1) \]

Then clearly the number of nonzero solutions of (37) is equal to

\[ \sum_{k=0}^{p-1} N(k)N(-k) + O(p^2) \]
Let \( \psi(a) \) denote the Legendre symbol \((a/p)\). Then for fixed \( u \) and \( k \), the number of solutions of (39) is equal to

\[
1 + \psi\{(1 + ku - u^2)^2 - 4u(1 - u)^2\},
\]

so that

\[
N(k) = p + \sum_{u=0}^{p-1} \psi(f(k, u)),
\]

where

\[
f(k, u) = (1 + ku - u^2)^2 - 4u(1 - u)^2.
\]

Thus (40) becomes

\[
p^3 + 2p \sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \psi(f(k, u)) + \sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \psi(f(k, u))\psi(-k, v) + O(p^2).
\]

Since \( f(k, u) \) is a quadratic in \( k \) we have

\[
\sum_{k=0}^{p-1} \psi(f(k, u)) = -1
\]

unless \( u(1 - u) = 0 \). It follows that

\[
\sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \psi(f(k, u)) = O(p^c).
\]

Consider next the sum

\[
\sum_{u=0}^{p-1} \psi(f(k, u)).
\]

It is easily seen from (41) that for fixed \( k \), \( f(k, u) \) is the square of a polynomial in \( u \) only when \( k = 0 \). We therefore have the estimate

\[
\sum_{u=0}^{p-1} \psi(f(k, u)) = O(p^{1/2}),
\]

so that

\[
\sum_{k=0}^{p-1} \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \psi(f(k, u))\psi(-k, v) = O(p^c).
\]

Substituting from (43) and (45) in (42) we see that the number of nonzero solutions (37) is

\[
p^3 + O(p^c).
\]

Therefore \( N \), the number of solutions of (34) is
and (33) becomes
\[ \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} |S(a, b, c)|^4 = p^8 + O(p^7); \]
since \( S(0, 0, 0) = p^2 \),
\[ S(a, b, c) = S(1, 1, abc) \quad (abc \neq 0) \]
and there are \((p - 1)^2\) terms \(S(a, b, c)\) in the sum that give the same \(S(1, 1, c)\), (32) now follows immediately.

Note that, except for (44), the proof is elementary.

**References**

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