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We first consider the theory of Jordan homomorphisms and Jordan ideals in Banach algebras. If B is a B*-algebra or a semi-simple annihilator algebra, any closed Jordan ideal in B is a two-sided ideal. Any Jordan homomorphism of a Banach algebra onto B is automatically continuous. That Jordan homomorphisms are continuous and Jordan ideals are ideals is shown to hold in a number of other situations. We also study the Lie ideals in a semi-simple Banach algebra A. If the center of A is zero and proper closed Lie ideals do not contain their Lie annihilators, then A is direct topological sum of its minimal closed ideals. An H^*-algebra with zero center is an example of such an algebra.

The utility of the study of Jordan isomorphisms in Banach algebra was noted by Kadison [8] in the study of isometrics of B^*-algebras. The Jordan and Lie structures of simple associative rings has been investigated by Herstein in a series of paper (see [3], [4], [5]). Essential use is made of these results in the present work.

2. **Pure algebra.** Let R be an associative ring. As is well-known [3] we can make R into a Jordan (Lie) ring by introducing the Jordan (Lie) multiplication $x \cdot y = xy + yx([x, y] = xy - yx)$. For a subset S of R we consider the sets

$$S^J = \{x \in R \mid x \cdot u = 0 \text{ for all } u \in S\} ,$$
$$S^L = \{x \in R \mid [x, u] = 0 \text{ for all } u \in S\} ,$$
$$\mathfrak{R}(S) = \{x \in R \mid ux = 0 \text{ for all } u \in S\} \text{ and}$$
$$\mathfrak{L}(S) = \{x \in R \mid xu = 0 \text{ for all } u \in S\} .$$

By an *ideal* in R we mean, unless otherwise specified, a two-sided ideal.

2.1. LEMMA. *Let U be a Lie ideal in R. Then U^J and U^L are Lie ideals.*

Let $x \in U^J$, $u \in U$ and $b \in R$. Since $xu = - ux$, an easy computation shows that $[x, b]\cdot u = [b, u]\cdot x = 0$. Let $y \in U^L$. Since $yu = uy$, we obtain by straightforward calculation that $[[y, b], u]=[y, [b, u]] = 0$.

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2.2. LEMMA. *Let I be an ideal in R. Then*

$$[I^L, R] \subset \mathfrak{L}(I) \cap \mathfrak{R}(I) \subset I^L .$$

Let $y \in I, x \in I^L$ and $z \in R$. We get successively the relations $(xz)y = x(zy) = (zy)x = z(yx) = z(xy) = (zx)y$ or $[I^L, R] \subset \mathfrak{L}(I)$. Likewise $[I^L, R] \subset \mathfrak{R}(I)$.

For the remainder of this section, A shall denote a semi-prime alegbra over a field of characteristic $\neq 2$. (A semi-prime algebra is one without nilpotent right or left ideals $\neq (0)$). It follows from the hypothesis that A is semi-prime, that $\mathfrak{L}(I) = \mathfrak{R}(I)$ for any ideal I in A [10] p. 99].

2.3. LEMMA. *Let U be a Lie ideal in A. Then $U^J = \mathfrak{L}(U) = \mathfrak{R}(U)$ is an ideal in A and $U \cap U^J = (0)$.*

Take $x \in U \cap U^J$. Clearly $x^2 = 0$. Also, by Lemma 2.1, $[x, b] \in U \cap U^J$ for each $b \in A$. Then

$$0 = [x, b]^2 = xbxb - xb^2x + bxbx .$$

Multiplying on the left by x and on the right by b, we obtain $(xb)^3 = 0$. Thus xA is a nilpotent right ideal so that $xA = (0)$ and hence $x = 0$. Therefore $U \cap U^J = 0$.

Next let $u \in U$ and $x \in U^J$. The above shows that $[u, x] = 0$ and $u \cdot x = 0$. Thus $ux = xu = 0$. Hence

$$U^J \subset \mathfrak{L}(U) \cap \mathfrak{R}(U) . \tag{1}$$

Next we show that

$$\mathfrak{L}(U) = \mathfrak{R}(U) . \tag{2}$$

For take $x \in \mathfrak{L}(U)$, $u \in U$ and $b \in A$. We have $0 = x[b, u] = xbu$. Hence $(uxb)^2 = 0$. It follows that $ux = 0$ or $\mathfrak{L}(U) \subset \mathfrak{R}(U)$. Likewise $\mathfrak{R}(U) \subset \mathfrak{L}(U)$. A combination of (1) and (2) gives the desired result. The corresponding proposition for Jordan ideals in harder to prove (see Theorem 2.5).

Let U be a Jordan ideal in A. Let K denote the algebraic sum of the ideals in A contained in U. Clearly K is maximal in the set of ideals in A contained in U.

2.4. LEMMA. *The ideal K contains $a \cdot b$ and aAa for each a, b of the Jordan ideal U in A. If $U \neq (0)$ then $K \neq (0)$.*

By a lemma of Herstein [3, Lemma 1], $(a \cdot b)x - x(a \cdot b) \in U$ for each $x \in A$. Since $(a \cdot b) \cdot x \in U$ we see that $(a \cdot b)x$ and $x(a \cdot b)$ lie in U.

If also $y \in A$ then $(a \cdot b)xy + y(a \cdot b)x \in U$. From this we see that the ideal generated by $a \cdot b$ is contained in U. Therefore $a \cdot b \in K$. Also $(a \cdot x) \cdot a \in K$. Using $a^2 \in K$ we obtain $axa \in K$. If $K = (0)$, then $(aA)^2 = (0)$ and $a = 0$ for all $a \in U$.

2.5. THEOREM. *Let U be a Jordan ideal in A. Then $U^J = \mathfrak{L}(U) = \mathfrak{R}(U)$ is an ideal in A and $U \cap U^J = (0)$.*

Let $u \in U, x \in A$ and $z \in \mathfrak{L}(U)$. We have $0 = z(u \cdot x) = (zx)u$. Therefore $\mathfrak{L}(U)$ is an ideal in A. Since $[u\mathfrak{L}(U)]^2 = (0)$ we can conclude that $u\mathfrak{L}(U) = (0)$ or $\mathfrak{L}(U) \subset \mathfrak{R}(U)$. Likewise $\mathfrak{R}(U) \subset \mathfrak{L}(U)$. Thus $\mathfrak{L}(U) \subset U^J$.

Suppose that either a or b lies in U. Then

$$(a \cdot b) \cdot b = a \cdot (b^2) + 2bab \in U .$$

In either case it follows that $bab \in U$. Next let $u \in U$, $x \in A$. Then $(u \cdot x) \cdot (xu) = ux^2u + xu^2x + 2(xu)^2 \in U$. By the preceeding remark we see that $(xu)^2 \in U$. Likewise $(ux)^2 \in U$. Consider now an ideal I in A such that $I \cap U = (0)$. If $z \in I$ then $(uz)^2 \in I \cap U = (0)$. Therefore $uI = (0)$. We then have $(0) = UI = IU$.

Let K be the ideal in A maximal in the set of ideals in A contained in U (see Lemma 2.4). In particular $a^2 \in K$ for $a \in U$. Now $\mathfrak{L}(K)$ is an ideal in A. We show that $\mathfrak{L}(K) \cap U = (0)$. For let $b \in \mathfrak{L}(K) \cap U, x \in U$. Then $(bx)^2 \in U \cap \mathfrak{L}(K)$. This makes $(bx)^4$ an element of $K \cap \mathfrak{L}(K) = (0)$. Therefore $(bA)^4 = (0)$ and $b = 0$.

It follows from the above that $U\mathfrak{L}(K) = \mathfrak{L}(K)U = (0)$. Consequently $\mathfrak{L}(K) \subset \mathfrak{L}(U)$. But, inasmuch as $K \subset U$, we have also that $\mathfrak{L}(K) = \mathfrak{L}(U)$. From Lemma 2.3 we get $K^J = \mathfrak{L}(K) = \mathfrak{L}(U) \subset U^J$. But as $K \subset U$ we see that $K^J = U^J$. Whereas $U \cap \mathfrak{L}(K) = (0)$, we obtain $U \cap U^J = (0)$.

2.6. LEMMA. *If A satisfies the descending chain condition on right ideals, every Jordan ideal in A is an ideal.*

By the proof of Theorem 2.5, bab lies in the Jordan ideal U if a or b lies in U. It follows from [9, Lemma 5] that U is an ideal.

As we shall see below, Jordan ideals are automatically ideals under favorable conditions. If an algebra B (even a Banach algebra) is not semi-simple, this situation does not prevail and even Lemma 2.6 can fail there, as easy examples show.

For S, T subsets of A, by $S \cdot T$ we mean the collection of all finite sums of elements $x \cdot y, x \in S, y \in T$.

2.7. LEMMA. *Let I be an ideal in A. Then $I \cdot A$ is an ideal.*

Also $A \cdot A$ contains A^3.

The relation $a \cdot (b \cdot c) - (a \cdot b) \cdot c = [[a \cdot c], b]$ is readily verified. Suppose at least one of a, b and c is in I. Then $[[a, c], b] \in I \cdot A$. Since I is an ideal, $[a, c] \cdot b \in I \cdot A$. Hence $[a, c]b \in I \cdot A$ and $b[a, c] \in I \cdot A$.

Suppose that $b \in I$. Then $(ac) \cdot b - a[c, b] = (a \cdot b)c \in I \cdot A$. Thus $(a \cdot b)c \in I \cdot A$ and $[b, a]c \in I \cdot A$. Taking the sum and difference of these items, we see that abc and $bac \in I \cdot A$ for all $a, c \in A$. Therefore $(a \cdot b)c$ and likewise $c(a \cdot b) \in I \cdot A$ and $I \cdot A$ is an ideal. The proof shows that $A \cdot A \supset A^3$.

2.8. LEMMA. *A maximal ideal M in A is a maximal Jordan ideal.*

Let π be the natural homomorphism of A onto A/M. If A/M is not a zero algebra, then, by [3, Theorem 1], A/M has only itself and (0) as Jordan ideals. Therefore M is a maximal Jordan ideal. Suppose that A/M is a zero algebra. Since its only ideals are trivial, A/M is one-dimensional. This makes M a maximal linear subspace in A and hence a maximal Jordan ideal.

2.9. THEOREM. *If $A \cdot A = A$, then every maximal Jordan ideal M in A is an ideal.*

Let K be the largest ideal in A contained in M. If K is a maximal ideal in A, then $K = M$ by Lemma 2.8. We show that K is always a maximal ideal. For suppose otherswise. There exists an ideal I in A, $A \neq I$, $K \subset I$, $K \neq I$. If $I \supset M$ we are through and $I \subset M$ is impossible. Therefore $A = I + M$. Then $A \cdot M = I \cdot M + M \cdot M$. Lemma 2.4 gives $M \cdot M \subset K \subset I$ so that $A \cdot M \subset I$. Also $A \cdot A = A \cdot I + A \cdot M \subset I$. By hypothesis, this is impossible. In particular, by Lemma 2.7, the conclusion holds if $A^3 = A$.

2.10. THEOREM. *Suppose that $A \cdot A = A$ and each ideal in A is the intersection of the maximal ideals containing it. Then every Jordan ideal in A is an ideal.*

These conditions are satisfied, for example, if A is biregular in the sense of Arens and Kaplansky [1]. Let U be a Jordan ideal and let K be the largest ideal in A contained in U. If K is a maximal ideal, $K = U$ by Lemma 2.8. Suppose that K is not a maximal ideal and let M be any maximal ideal containing K. We show that $U \subset M$. For suppose otherwise. Clearly $M \not\subset U$ and $A = M + U$. Reasoning as in the proof of Theorem 2.9, we get $A \cdot A \subset M$, which is impossible. Hence

U is contained in every maximal ideal M, $M \supset K$. Therefore $U = K$.

3. On topological rings. Now let A be a semi-prime topological algebra over a field of characteristic $\neq 2$.

3.1. THEOREM. *Suppose that $\mathfrak{L}(I) \neq (0)$ for every closed ideal $I \neq A$ in A. Then any closed Jordan ideal U in A is an ideal in A.*

As in the preceding section we consider the ideal K in A maximal in the set of ideals contained in U. Clearly K is closed. Let $W = K \oplus \mathfrak{L}(K)$. Clearly $x^2 = 0$ for each $x \in \mathfrak{L}(W) = \mathfrak{R}(W)$. Therefore $\mathfrak{L}(W) = (0)$ and W is dense in A. Let $a \in U$, $u \in U$. There exists a net a_α in W such that $a_\alpha \to a$. For each a_α, $a_\alpha = b_\alpha + c_\alpha$, where $b_\alpha \in K$ and $c_\alpha \in \mathfrak{L}(K)$. From the proof of Theorem 2.5, $\mathfrak{L}(K) = \mathfrak{L}(U)$. Then we see that $a_\alpha u = b_\alpha u \in K$ for each α. Consequently $au \in K$. Likewise $ua \in K$, and U is an ideal in A.

The condition $\mathfrak{L}(I) \neq (0)$ for proper closed ideals in A gives the natural two-sided analogue of the annihilator algebras of Bonsall and Goldie [2].

3.2. THEOREM. *Let B be a semi-simple topological ring. Suppose that*

(a) *the primitive ideals of B are closed;*

(b) *$\mathfrak{L}(I) \neq (0)$ for each closed ideal $I \neq A$.*

Then B is the direct topological sum of its minimal closed ideals.

Note that (a) is automatically satisfied if B is a Banach algebra. Suppose first that (0) is a primitive ideal, i.e., B is primitive. Then (b) and $\mathfrak{L}(I)I = (0)$ for a closed ideal I show that all ideals in B other than (0) must be dense. The desired conclusion is then readily seen. We assume then that (0) is not a primitive ideal.

We call an ideal K in B *dual* if $\mathfrak{L}\mathfrak{R}(K) = K$. Our first step is to show that each primitive ideal P is dual. There exists a modular maximal right ideal M in B such that $P = \{a \in B \mid xa \in M \text{ for all } x \in B\}$ and P is the largest ideal in B contained in M. Let j be a left identity for B modulo M, let $b \in \mathfrak{L}\mathfrak{R}(P)$ and suppose that $b \notin P$. Then there exists an element $z \in B$ such that $zb \notin M$. We can write $j = u + kzb + zbx$ where $u \in M$, $x \in B$ and k is an integer. Multiplying on the right by an element $w \in \mathfrak{R}(P) = \mathfrak{L}(P)$, we see that $jw = uw \in M$. Since $jw - w \in M$, we also get $w \in M$. Therefore $P \oplus \mathfrak{R}(P) \subset M$. This is impossible since $\mathfrak{R}(P) \neq (0)$ by hypothesis and $P \oplus \mathfrak{R}(P)$ is a larger ideal than P contained in M. Hence P is dual.

Next take any ideal I in B, where $I \supset P$, $I \neq P$, for a primitive ideal P. We show that $\mathfrak{L}(I) = (0)$ so that I is dense. Since $I \supset P$, we have $\mathfrak{L}(I) \subset \mathfrak{L}(P)$. Also $(0) = \mathfrak{L}(I)I \subset P$ and $I \not\subset P$. The theory of primitive ideals shows that $\mathfrak{L}(I) \subset P$. Therefore $\mathfrak{L}(I) \subset P \cap \mathfrak{L}(P) = (0)$.

Now let I be any dual ideal in B. We show that I is the intersection of the primitive ideals in B which contain I. Let π be the natural homomorphism of B onto B/I. Since $I \cap \mathfrak{R}(I) = (0)$, π is one-to-one on $\mathfrak{R}(I)$ and $\pi(\mathfrak{R}(I))$ is semi-simple. Let W denote the radical of B/I. Then $W \cap \pi(\mathfrak{R}(I)) = (0) = W\pi(\mathfrak{R}(I))$ by [6, p. 10]. Therefore $\pi^{-1}(W)\mathfrak{R}(I) \subset I$ from which we see that $\pi^{-1}(W)\mathfrak{R}(I) = (0)$. Thus $\pi^{-1}(W) \subset \mathfrak{L}\mathfrak{R}(I) = I$ and $W = (0)$. Inasmuch as B/I is now seen to be semi-simple, I is the intersection of the primitive ideals which contain it.

Next consider a primitive ideals P_0 and let P be any primitive ideal, $P \neq P_0$. By the above and our hypothesis, $P \not\subset P_0$ and $P_0 \not\subset P$ and $\mathfrak{L}(P + P_0) = (0)$. Therefore $\mathfrak{L}(P)\mathfrak{L}(P_0) = (0) = \mathfrak{L}(P_0)\mathfrak{L}(P)$ so that $\mathfrak{L}(P_0) \subset \mathfrak{R}\mathfrak{L}(P) = P$. Also $\mathfrak{R}\mathfrak{L}\mathfrak{R}(P_0) = \mathfrak{R}(P_0) = \mathfrak{L}(P_0)$ so that $\mathfrak{L}(P_0)$ is dual and $\mathfrak{L}(P_0) \not\subset P_0$. Therefore, as $\mathfrak{L}(P_0)$ is the intersection of the primitive ideals containing it, $\mathfrak{L}(P_0) = \cap P$, for $P \neq P_0$, P primitive. It is now clear that the ideals $\mathfrak{L}(P)$, P primitive, are the minimal dual ideals of B.

View $\mathfrak{L}(P_0)$ as a ring. By [6, p. 206], every primitive ideal of $\mathfrak{L}(P_0)$ is the intersection of $\mathfrak{L}(P_0)$ with a primitive ideal of B. Thus $\mathfrak{L}(P_0)$ is a primitive ring. Let $I \neq (0)$ be any closed ideal in B, $I \subset \mathfrak{L}(P_0)$ and let J be the algebraic sum of the ideals $\mathfrak{L}(P)$, $P \neq P_0$, P primitive. We claim that $I + J$ is dense in B. For let $x \in \mathfrak{R}(I + J)$. Then $x \in \mathfrak{R}\mathfrak{L}(P) = P$ for all $P \neq P_0$. Thus $x \in \mathfrak{L}(P_0) \cap \mathfrak{L}(I) = K$, say. Since $IK = (0)$ and $\mathfrak{L}(P_0)$ is a primitive ring, we get $K = (0)$. By (*b*), $I + J$ is dense in B.

We continue with this notation and show next that $\mathfrak{L}(P_0)\mathfrak{L}(P_0) \subset I$. Let $u \in \mathfrak{L}(P_0)$, $u = \lim (v_\alpha + q_\alpha)$, $v_\alpha \in I$, $q_\alpha \in J$ and let $x \in \mathfrak{L}(P_0)$. Now each $q_\alpha \in P_0$ so that $q_\alpha x = 0$. Therefore $ux = \lim v_\alpha x \in I$. This enables us to see that $\mathfrak{L}(P_0)(P_0 + \mathfrak{L}(P_0)) \subset I$. Inasmuch as $P_0 + \mathfrak{L}(P_0)$ is dense in B, we get $\mathfrak{L}(P_0)B \subset I$. Likewise $B\mathfrak{L}(P_0) \subset I$. From semi-simplicity we see that $B\mathfrak{L}(P_0) \neq (0)$, $\mathfrak{L}(P_0)B \neq (0)$. It follows that $\overline{B\mathfrak{L}(P_0)} = \overline{\mathfrak{L}(P_0)B}$ is a minimal closed ideal in B.

For each primitive ideal P_α, let $Z_\alpha = \overline{B\mathfrak{L}(P_\alpha)}$. It is clear that if $P_\alpha \neq P_\beta$, then $Z_\alpha \cap Z_\beta = (0) = Z_\alpha Z_\beta$. Then the algebraic sum of the Z_α is a direct sum. Let Q be its closure, the direct topological sum of the Z_α. Note that $\mathfrak{R}(Z_\alpha) = \mathfrak{R}(B\mathfrak{L}(P_\alpha)) = \mathfrak{R}\mathfrak{L}(P_\alpha) = P_\alpha$. Therefore $Q = B$. If I is any minimal closed ideal, I must be Z_α, for some α, for otherwise I would annihilate every Z_α.

3.3. Corollary. *Under the conditions of Theorem* 3.2. *the primitive ideals of B are the maximal closed ideals of B and every* *ideal in B which is not dense is contained in a primitive ideal. Also B is the direct topological sum of its minimal dual ideals.*

Let I be closed ideal in B, $I \neq B$. Then $\mathfrak{L}(I) \neq (0)$ and $\mathfrak{L}(I)I = (0)$. If I were contained in no primitive ideal we would have $\mathfrak{L}(I)$ contained in every primitive ideal, which is imposible.

The proof of Theorem 3.2 shows that a primitive ideal is a maximal closed ideal. Let M be a maximal closed ideal. As just seen, $M \subset P$ for some primitive ideal P, so $M = P$. The proof of Theorem 3.2 also demonstrates that $\mathfrak{L}(P_1)\mathfrak{L}(P_2) = (0)$ if P_1, P_2 are two distinct primitive ideals and that B is the direct topological sum of the $\mathfrak{L}(P)$ which are the minimal dual ideals.

4. Continuity of Jordan homomorphisms. We consider a Jordan homomorphism T defined on a Banach algebra A with range a *dense subset of a semi-simple Banach algebra B*. We seek to show that, under reasonable conditions, such a mapping T is automatically continuous. For an element x in A or B we let $\rho(x)$ denote its spectral radius [10, p. 30].

Two useful identities are noted by Kadison [8, p. 330] for the behavior of Jordan homomorphisms relative to Lie products. These are

$$T([[a, b], c]) = [[Ta, Tb], Tc]$$

and

$$T([a, b]^2) = [Ta, Tb]^2$$

for all $a, b, c \in A$. Hence, as noted in [8], $ab = ba$ makes $[Ta, Tb]$ a central nilpotent element of B. Thus, as B is semi-simple, $[Ta, Tb] = 0$. Then $T(ab) = T(ab + ba)/2 = T(a)T(b) = T(b)T(a)$.

4.1. LEMMA. *For each $x \in A$, $\rho(T(x)) \leqq \rho(x)$.*

From the above discussion we see that if $a \circ b = b \circ a = 0$ that $T(a) \circ T(b) = T(b) \circ T(a) = 0$. From this the result is evident.

By the *separating set* for T we mean the set of $s \in B$ for which there exists a sequence $\{x_n\}$ in A with $\| x_n \| \to 0$ and $\| s - T(x_n) \| \to 0$. By the closed graph theorem, T is continuous if only if $S = (0)$.

Sraightforward arguments show that S is a closed Jordan ideal in B.

4.2. LEMMA. *If B has a left or right identity j, then $j \notin S$.*

Note that j need not lie in $T(A)$. Suppose that $j \in S$ where j is a left identity for B. Then there exists a sequence $\{x_n\}$ in A with

$\| x_n \| \to 0$ and $\| j - T(x_n) \| \to 0$. For each $z \in B$, $(zj)^n = z^n j$. Thus

$$\| (zj)^n \|^{1/n} \leqq \| z^n \|^{1/n} \| j \|^{1/n} .$$

From this we see that $\rho(zj) \leqq \rho(z)$. Clearly $\rho(j) = 1$. Now $j = (j - zj) + zj$ where the summands permute. Therefore

$$1 = \rho(j) \leqq \rho(j - zj) + \rho(zj) \leqq \rho(j - zj) + \rho(z) .$$

Now replace z by $T(x_n)$. We see that

$$\rho(j - T(x_n)j) \leqq \| j - T(x_n) \| \| j \| \to 0$$

and that, by Lemma 4.1, $\rho(Tx_n) \leqq \| x_n \| \to 0$. This yields a contradiction.

4.3. LEMMA. *Each element of S is a two-sided topological divisor of zero in B.*

This is a variation on a result of Rickart [10, p. 72]. Let $s \in S$. Arguments used there show that, if $1 - \lambda s$ is a two-sided topological divisor of zero for arbitrarily large λ, then s is a two-sided topological divisor of zero. We assume, then, that $1 - \lambda s$ is not a two-sided topological divisor of zero for $|\lambda| \geqq k$. Then λs is quasi-regular, $|\lambda| \geqq k$. Suppose that s is not a left topological divisor of zero in B. Then [10, p. 24], B has a left identity j and s is a regular elemement of the algebra jBj; there exists $u \in jBj$ such that $us = su = j$. Then $(u \cdot s)/2 = j \in S$ which is contrary to Lemma 4.2. Consequently s is a left (and similarly right) topological divisor of zero.

4.4. LEMMA. *Let $e \neq 0$ be any idempotent in $T(A)$. Then $e \notin S$.*

Suppose that $e = T(x)$, $e \in S$. There exists a sequence $\{x_n\}$ in A with $\| x_n \| \to 0$ and $\| e - T(x_n) \| \to 0$. Then also $\| e - eT(x_n)e \| \to 0$. From the theory of Jordan homomorphisms (see, for examples [8, p. 329]), $T(xx_nx) = eT(x_n)e$. Now $e - eT(x_n)e$ and $eT(x_n)e$ permute so that, by Lemma 4.1,

$$\begin{aligned} 1 = \rho(e) &\leqq \rho(e - eT(x_n)e) + \rho(eT(x_n)e) \\ &\leqq \| e - eT(x_n)e \| + \rho(T(xx_nx)) \\ &\leqq \| e - eT(x_n)e \| + \| xx_nx \| \to 0 . \end{aligned}$$

Therefore $e \notin S$.

If it is not required that the idempotent e be in $T(A)$ the following weaker conclusion holds.

4.5. LEMMA. *No central idempotent $e \neq 0$ of B lies in S.*

We use the notation of the proof of Lemma 4.4. If $e \in S$, then $1 = \rho(e) \leqq \rho(e - T(x_n)) + \rho(T(x_n)) \leqq \| e - T(x_n) \| + \| x_n \| \to 0$.

As an application we take $B = L_2(G)$ where G is a compact group and the multiplication in $L_2(G)$ is convolution (see [10, p. 330]). Suppose $T(A)$ is dense in $L_2(G)$. Lemma 4.5 shows that S cannot contain a central idempotent $\neq 0$. But in $L_2(G)$, every nonzero ideal contains a central idempotent. Therefore $S = (0)$ and T is continuous.

4.6. THEOREM. *A Jordan homomorphism T of a Banach algebra A onto a dense subset of a strongly semi-simple Banach algebra B is continuous.*

Let M be a modular maximal ideal of B, and let π denote the natural homomorphism of A onto B/M. The mapping πT is a Jordan homomorphism of A onto a dense subset of the simple algebra B/M. The separating set S_1 for the mapping πT is a Jordan ideal of B/M which cannot contain the identity of B/M by Lemma 4.2. But by Herstein's result [3, Theorem 1], S_1 is an ideal of B/M. Therefore $S_1 = (0)$ and πT is continuous. Simple arguments now show that $S \subset M$. Hence $S = (0)$.

4.7. THEOREM. *Let T be a Jordan homomorphism of a Banach algebra A onto a semi-simple Banach algebra B where every nonzero right (left) ideal in B contains a minimal right (left) ideal in B. Then T is con-tinuous.*

Examples of such B are the semi-simple annihilator Banach algebras of Bonsall and Goldie ([10, p. 96] and [2]).

Fix $a, b \in S$. By Lemma 2.4, the ideal generated by $a \cdot b$ lies in S. If the right ideal $\{(a \cdot b)x \mid x \in A\} \neq (0)$, it contains an idempotent $e \neq 0$ contrary to Lemma 4.4. Consequently $(a \cdot b)B = (0)$ and $a \cdot b = 0$ for all $a, b \in S$. For $a \in S, x \in B$ we have $(a \cdot x) \cdot a = 0$. Using $a^2 = 0$ we see that $(aB)^2 = (0)$. Hence a lies in the radical of B so that $S = (0)$.

4.8. THEOREM. *Let T be a Jordan homomorphism of a Banach algebra A onto a dense subset of a primitive Banach algebra B with minimal one-sided ideals. Suppose that $T(A) \cap I$ is dense I for some minimal right (left) ideal I. Then T is continuous.*

For the case of an algebra homomorphism see [12, p. 378]. Suppose that I is a right ideal. Let $I_1 = T^{-1}(I)$. If $T(x^2) = 0$ for all $x \in I_1$, then $u^2 = 0$ for each u in a dense subset of I. This would make $I^2 = (0)$ which is impossible. We can express $I = jB$ where $j^2 = j$. Set $R = \{y \in B \mid jy \in I \cap T(A)\}$. Clearly R is a linear manifold where jR is dense in jB. Thus jRj is dense in jBj. By the Gelfand-Mazur theorem, $jRj = jBj$. Select $x \in I_1$ where $T(x) = jw \neq 0$ and $T(x^2) = jwjw \neq 0$. We have $jwj \neq 0$. Since jBj is a division ring there exists $z \in R$ where $jwjzj = jzjwj = j$. Consequently $(jw)\cdot(jz) \neq 0$. There exists $x_1 \in I_1$ such that $T(x_1) = jz$. Simple computations give $T((x\cdot x_1)/2) = (jw)\cdot(jz)/2 = h$, say, and $T[(x\cdot x_1)^2/4] = h$. Hence h is an idempotent, $\neq 0$, in $T(A) \cap I$. By Lemma 4.4, $h \notin S$ where S is the separating set for T. Consequently $I \not\subset S$.

Denote the socle of B by Σ and let $a, b \in S$. By the arguments of Theorem 4.7, the two-sided ideal Q generated by $a\cdot b$ lies in S. As B is primitive, $\Sigma \subset Q \subset S$ unless $a\cdot b = 0$. Thus, if $a\cdot b \neq (0)$, $h \in S$ which is impossible. Consequently we can reason as in Theorem 4.7 to see that $S = (0)$.

5. Jordan ideals and homomorphisms in B*-algebras.

Throughout this section U will denote a closed Jordan ideal in a B^*-algebra A. For an element $x \in A$ which is quasi-regular [10, p. 16] we denote its quasi-inverse by x'. Since $x\cdot x' = (x + x')/2$, we see that $x' \in U$ if $x \in U$.

5.1. LEMMA. *If h is self-adjoint and $h^2 \in U$, then $h \in U$.*

For all $\alpha > 0$, $(-\alpha h^2)'$ exists and lies in U. Thus $h\cdot(-\alpha h^2)' \in U$. An argument of Rickart [10, Theorem 4.9.2] shows that for any $x \in A$,

$$x = \lim_{\alpha=\infty} x(-\alpha x^* x)' = \lim_{\alpha=\infty} (-\alpha x x^*)' x \,. \tag{5.1}$$

Applying this to our case, we obtain $2h = \lim h\cdot(-\alpha h^2)' \in U$.

5.2. LEMMA. $U^* = U$.

Let $a \in U$. By Lemma 2.4, the ideal generated by a^2 lies in U. In particular, $[-\alpha(a^2)(a^2)^*]' \in U$ for all $\alpha > 0$. By the proof of Theorem 2.5, $(a^2)^*[-\alpha a^2(a^2)^*]'(a^2)^* \in U$. Formula (5.1) implies that $(a^4)^* \in U$. Since $a^4 \in U$ it follows that $[a^2 + (a^2)^*]^2 = a^4 + a^2\cdot(a^2)^* + (a^4)^* \in U$, and so, by Lemma 5.1, $a^2 + (a^2)^* \in U$. Therefore $(a^2)^* \in U$, so that

$$(a + a^*)^2 = a^2 + a\cdot a^* + (a^2)^* \in U.$$

Another application of Lemma 5.1 shows that $a + a^*$, and thus a^*, lies in U.

5.3. THEOREM. *Any closed Jordan ideal in a B^*-algebras A is* *an ideal of A.*

Let U be a closed Jordan ideal of A and let $a \in U$. Suppose $a = h + ik$, where h and k are self-adjoint. Since $a^* \in U$, it follows from Lemma 5.2 that $h \in U$ and $k \in U$. It thus suffices to see that if h and w are self-adjoint and $h \in U$, then hw and wh are in U. Now $(wh)^*(wh) = hw^2h \in U$, since $h \in U$. Also $(wh)(wh)^* = wh^2w \in U$, since $h \in U$. Consequently for $\alpha > 0, [-\alpha(wh)^*(wh)]' \in U$, and so $\lim_{\alpha\to\infty} (wh)[-\alpha(wh)^*(wh)]'(wh) \in U$. By formula (5.1), $(wh)^2 \in U$, and so by Lemma 5.2 $(hw)^2 \in U$. Since U is a Jordan ideal, it contains $wh + hw$. But also $[i(wh - hw)]^2 = -\{(wh)^2 - wh^2w - hw^2h + (hw)^2\} \in U$, so again by Lemma 5.1 $wh - hw \in U$, and thus wh and $hw \in U$, so U is an ideal.

Automatic continuity occurs for Jordan $*$-homomorphisms.

5.4. THEOREM. *Let T be a Jordan $*$-homomorphism of a B^*-algebra A onto a dense subset of a B^*-algebra B. Then*

(a) *T is continuous and $\| T \| = 1$,*

(b) *the range of T is B, and*

(c) *the adjoint mapping T' is an isometry.*

Before proceeding with the proof we should remark that these are instances of positive linear maps on operator algebras, which have been extensively studied. See [15].

Let A_0 be the closed $*$-subalgebra generated by a self-adjoint element $h \in A$. Then T is a $*$-homomorphism of the commutative B^*-algebra A_0 into B. Therefore $\| T(h) \| \leqq \rho(T(h)) \leqq \rho(h) \leqq \| h \|$. Consequently if h and k are any two self-adjoint elements of A,

$$\begin{aligned}\| T(h + ik) \| &\leqq \| T(h) \| + \| T(k) \| \leqq \| h \| + \| k \| \\ &\leqq \| h + ik \| + \| h - ik \| = 2 \| h + ik \| .\end{aligned}$$

Thus T is continuous.

Hence the kernel $T^{-1}(0)$ of the mapping T is a closed Jordan ideal in A. By Theorem 5.3, $T^{-1}(0)$ is an ideal and [10, p. 249] $A/T^{-1}(0)$

is a B^*-algebra in the quotient space norm. Let T_0 be the mapping of $A/T^{-1}(0)$ onto B defined by $T_0(x + T^{-1}(0)) = T(x)$ and let π be the natural homomorphism of A onto $A/T^{-1}(0)$. The mapping T_0 restricted to the closed $*$-subalgebra of $A/T^{-1}(0)$ generated by a normal element is an isometry by [10, p. 241]. That T_0 is an isometry on normal elements allows us to use the arguments of [8, Theorem 5] to assert that T_0 is an isometry on $A/T^{-1}(0)$. The range of T_0 and therefore that of T is then all of B, and $\| T(x) \| = \| \pi(x) \|$ for all $x \in A$. Since a natural homomorphism has norm one, $\| T \| = 1$. Now $T = T_0\pi$, so

$T' = \pi' T_0'$. As each of T_0' and π' is an isometry by the theory of normed linear spaces, so is T'.

5.5. THEOREM. *Let A, B be B^*-algebras with identities e_A and e_B, respectively. The Jordan $*$-homomorphisms of A onto B are precisely the continuous linear mappings T of A onto B such that (a) the adjoint T' is isometric, (b) the kernel of T is an ideal of A, and (c) $T(e_A) = c_B$.*

By Theorem 5.4, any Jordan $*$-homomorphism of A onto B has properties (a), (b) and (c).

Suppose T has the properties (a), (b), and (c). Let π be the natural homomorphism of A onto $A/T^{-1}(0)$. Let T_0 be as in the prior theorem. Then $T = T_0\pi$, and $T' = \pi' T_0'$. Since T' is an isometry, so is T_0'. As T_0 is a one-to-one mapping of $A/T^{-1}(0)$ onto B, T_0' is an isometry of the conjugate space of B onto the conjugate space of $A/T^{-1}(0)$. Thus we conclude that T_0 is an isometry of $A/T^{-1}(0)$ onto B. Since $T_0(e_A + T^{-1}(0)) = e_B$, a result of Kadison [8, p. 330] shows that T_0 is a Jordan $*$-isomorphism and thus that $T = T_0\pi$ is a Jordan $*$-homomorphism.

In [8, p. 329] Kadison shows that if T is a Jordan $*$-isomorphism of a B^*-algebra A onto a B^*-algebra B, then T is an isometry. The conventions of that paper require a unit for the algebra. However, a check of the argument involved in the proposition quoted above shows that the identity plays no role whatsoever. The next proposition is in the nature of a converse statement.

5.6. THEOREM. *Let A be a B^*-algebra, and B a Banach algebra with an involution. Let T be a Jordan $*$-isomorphism of A onto B which is an isometry. Then B is a B^*-algebra.*

Let $x \in A$. Then $\| (Tx)^n \| = \| Tx^n \| = \| x^n \|$, so $\rho(Tx) = \rho(x)$ for all $x \in A$, where ρ designates the spectral radius in either algebra.

Suppose $Tx = y \in \mathrm{Rad}\, B$, the radical of B. Then $y^* \in \mathrm{Rad} B$ and consequently $\rho(y \pm y^*) = 0 = \rho(x \pm x^*)$. But $x + x^*$ and $x - x^*$ are self-adjoint and skew elements in a B^*-algebra, so $x \pm x^* = 0$. Consequently $x = 0 = y$, and thus B is semi-simple.

It then follows (see § 4) that the spectrum of $T(x)$ if the same as that of x. Consequently, if $u \in B$ and u is self-adjoint, then its spectrum is real and $\rho(u) = \| u \|$. A result of Yood [13, p. 148] now asserts that there is a bicontinuous $*$-isomorphism σ of B onto a B^*-algebra B_1. The mapping σT of A onto B_1 is a Jordan $*$-isomorphism.

By the quoted result of Kadison, σT is an isometry. Since T is an isometry, so is σ and, therefore, B is a B^*-algebra.

We use the terminology *Jordan involution* for a conjugate linear mapping $x \longrightarrow x^*$ of period two on a semi-simple complex Banach algebra A where $(x \cdot y)^* = x^* \cdot y^*$, $x, y \in A$. We write $H = \{x \in A \mid x = x^*\}$. Clearly $A = H \oplus iH$ and $\rho(x^*) = \rho(x)$, $x \in A$.

5.7. LEMMA. *Suppose that there is a real normed linear space norm $|x|$ on H such that $|x| \leqq \rho(x)$, $x \in H$. Then the Jordan involution $x \longrightarrow x^*$ is continuous.*

Let H' be the derived set of H. Note that $y^2 \in H$ if $y \in H$. The arguments of [11, Lemma 3.3] show $y^2 = 0$ if $y \in (iH) \cap H'$. Thus $\rho(y) = \rho(iy) = 0$ and $y = 0$ for such y. It follows [11, p. 157] that H is closed. If $x \in A$ and $x = u + iv$, $u, v \in H$ the norm $\|x\|_1 = \|u\| + \|v\|$ is then a complete linear space norm for A topologically equivalent to the given norm so that the Jordan involution is continuous.

We are now able to establish the continuity of a Jordan homomorphism of a complex Banach algebra A onto a B^*-algebra. There need be no involution on A.

5.8. THEOREM. *Let T be a Jordan homomorphism of a complex Banach algebra A onto a B^*-algebra B. Then T is continuous.*

Let P_α be the set of primitive ideals of B, and let π_α be the canonical homomorphism of B onto B/P_α. Then $\pi_\alpha T$ is a Jordan homomorphism of A onto a primitive algebra. By a result of Herstein [4, p. 340], $\pi_\alpha T$ is either a homomorphism or an anti-homomorphism. Since B/P_α is a B^*-algebra, in either case we see that $\pi_\alpha T$ is continuous [10, Theorem 4.1.20]. This shows that $(\pi_\alpha T)^{-1}(0)$ is a closed ideal in A. But

$$T^{-1}(0) = \cap_\alpha (\pi_\alpha T)^{-1}(0) .$$

Therefore $T^{-1}(0)$ is a closed ideal in A. It is not difficult to show that $A/T^{-1}(0)$ is semi-simple.

Let π denote the natural homomorphism of A onto $A/T^{-1}(0)$ and let T_0 be the Jordan isomorphism $T_0(\pi(x)) = T(x)$ of $A/T^{-1}(0)$ onto B. By § 4 we see that $\rho[T(x)] = \rho[T_0(\pi(x))] = \rho[\pi(x)]$ since B is semi-simple.

The natural involution $x \longrightarrow x^*$ on B induces a Jordan involution on $A/T^{-1}(0)$ by the rule $[T_0^{-1}(w)]^* = T_0^{-1}(w^*)$. Suppose that $T_0^{-1}(h)$ is self-adjoint under this Jordan involution. Then $T_0^{-1}(h) = [T_0^{-1}(h)]^* = T_0^{-1}(h^*)$ so that $h = h^*$ and h is self-adjoint in B. Thus $\rho(T_0^{-1}(h)) = \rho(h) = \|h\|$ for all h self-adjoint in B and $|T_0^{-1}(h)| = \|h\|$ serves as the auxiliary norm needed for Lemma 5.7. By that result, the Jordan involution is continuous on $A/T^{-1}(0)$.

To show that T is continuous it is enough to show that T_0 is continuous. For the continuity of T_0 it suffices to show that $\|\pi(x_n)\| \to 0$ and $\| T_0(\pi(x_n)) - T_0(\pi(x_0)) \| \to 0$ imply that $T_0(\pi(x_0)) = T(x_0) = 0$. We can write $\pi(x_n) = \pi(u_n) + i\pi(v_n)$, $n = 0, 1, 2, \cdots$, where each $\pi(u_n)$ and $\pi(v_n)$ is self-adjoint in terms of the Jordan involution of $A/T^{-1}(0)$. Since that involution is continuous, $\| \pi(u_n) \| \to 0$ and $\| \pi(v_n) \| \to 0$. Each $T_0(\pi(u_n)) = T(u_n)$, and each $T(v_n)$, is self-adjoint in the B^*-algebra B. Thus $\| T(u_n) - T(u_0) \| \to 0$ and $\| T(v_n) - T(v_0) \| \to 0$. Also

$$\begin{aligned}\rho(T(u_0)) &= \| T(u_0) \| \leqq \| T(u_n) - T(u_0) \| + \| T(u_n) \| \\ &= \| T(u_n) - T(u_0) \| + \rho(T(u_n)) \\ &= \| T(u_n) - T(u_0) \| + \rho(\pi(u_n)) \to 0 \ .\end{aligned}$$

Therefore $T(u_0) = 0$ so that $\pi(u_0) = 0$. Likewise $\pi(v_0) = 0$ and thus $\pi(x_0) = 0$.

6. Lie ideals in Banach algebras.

Throughout let B be a complex Banach algebra with center $\mathfrak{Z}$. Suppose that B is semi-simple. If $e^2 = e$ and $eB(Be)$ is a minimal right (left) ideal of B, we call e a *minimal idempotent* of B. By the Gelfand-Mazur Theorem, $eBe = Ke$, where K is the set of complex numbers. If B is primitive, it is known [10, p. 61] that either $\mathfrak{Z} = (0)$ or B has an identity u and $\mathfrak{Z} = Ku$.

Consider the Lie multiplication $[x, y] = xy - yx$ in B. In this topological algebraic setting we modify standard notation [7] as follows. Let $[S, T]$ be the closed linear span of the elements $[s, t]$ where $s \in S$, $t \in T$ and S, T are subsets of B. If $\mathfrak{L}$ is a Lie ideal of B we define $D^k\mathfrak{L}$ inductively by $D^\circ\mathfrak{L} = \bar{\mathfrak{L}}$ and $D^{k+1}\mathfrak{L} = [D^k\mathfrak{L}, D^k\mathfrak{L}]$. A Lie ideal $\mathfrak{L}$ is called *solvable* if for some positive integer k, $D^k\mathfrak{L} = (0)$. The *Lie radical* of B is defined to be the closed linear span of all the solvable Lie ideals of B.

As in [3] we define $S(\mathfrak{L}) = \{x \in B \mid [x, y] \in \mathfrak{L} \text{ for all } y \in B\}$.

6.1. THEOREM. *Let B be a primitive Banach algebra with socle $S \neq (0)$. Let e be a minimal idempotent and $\mathfrak{L} \neq (0)$ be a Lie ideal of B. Then*

(a) *$e\mathfrak{L}e = Ke$.*
(b) *If $[\mathfrak{L}, \mathfrak{L}] = (0)$, then $\mathfrak{L} = \mathfrak{Z}$.*
(c) *If $e \in \mathfrak{L}$, then $\mathfrak{L} \supset S$.*
(d) *If $\mathfrak{L} \neq \mathfrak{Z}$, then $\mathfrak{L}$ contains $[x, y]$ for all $x \in S$, $y \in B$.*

(a) Suppose that $e\mathfrak{L}e \neq Ke$. Then $e\mathfrak{L}e = (0)$. Let $a \in \mathfrak{L}$ and $x \in B$. We have $e(ax - xa)e = 0$ or $eaxe = exae$. Replacing x by xe, we see that $eaxe = 0$. This gives $(eax)^2 = 0$ for all $x \in B$, so that $ea = 0$.

Then $e(ax - xa) = 0 = exa$. From this we get $(BeB)(BaB) = (0)$. As B is primitive it follows that $a = 0$. This makes $\mathfrak{L} = (0)$.

(b) For any $x \in \mathfrak{L}$, $y \in B$ we have $[x, y^2] = [x, y]\cdot y \in \mathfrak{L}$. Since $[[x, y], y] \in \mathfrak{L}$. it follows that $(xy - yx)y$ and $y(xy - yx) \in \mathfrak{L}$. As $[\mathfrak{L}, \mathfrak{L}] = (0)$, x permutes with these elements and with $xy - yx$. Therefore $(xy - yx)yx = (xy - yx)xy$ or $(xy - yx)^2 = 0$. Since $e(xe - ex)^2e = 0$, we obtain $ex^2e = (exe)^2, x \in \mathfrak{L}$. Replacing x by $xy - yx$ we see that $[e(xy - yx)e]^2 = 0$. Since $eBe = Ke$, we have $e[\mathfrak{L}, B]e = (0)$. But $[\mathfrak{L}, B]$ is a Lie ideal so that, by (a), $\mathfrak{L} \subset \mathfrak{Z}$. But $\mathfrak{Z}$ is one-dimensional or (0). whence $\mathfrak{L} = \mathfrak{Z}$.

(c) Given $e \in \mathfrak{L}$, we have $[e, [e, x]] \in \mathfrak{L}$ for all $x \in B$. Inasmuch as exe is a scalar multiple of e, we see that $e\cdot x \in \mathfrak{L}$. As $[e, x] \in \mathfrak{L}$, we get $ex \in \mathfrak{L}$ and $[y, ex] \in \mathfrak{L}$ for all $x, y \in B$. But $[y, ex] = yex - exy$ so that $BeB \subset \mathfrak{L}$. By [6, p. 75] we see that $\mathfrak{L} \supset S$.

(d) By [3, p. 282], $S(\mathfrak{L})$ is both a Lie ideal and a subalgebra of B. Consequently, by [3, p. 281], either $\mathfrak{L} \subset S(\mathfrak{L}) \subset \mathfrak{Z}$ (so that $\mathfrak{L} = \mathfrak{Z}$) or $S(\mathfrak{L})$ contains a nonzero ideal of B. In the latter case, $S(\mathfrak{L}) \supset S$ since B is primitive. This yields (d).

6.2. THEOREM. *Let B be a primitive Banach algebra with minimal one-sided ideals. Then the Lie radical of B coincides with $\mathfrak{Z}$.*

It is sufficient to show that any solvable Lie ideal $\mathfrak{L}$ of B is contained in $\mathfrak{Z}$. Suppose $D^0\mathfrak{L} = \bar{\mathfrak{L}}$ and $D^{p+1}\mathfrak{L} = [D^p\mathfrak{L}, D^p\mathfrak{L}], p = 0, 1, \cdots$, where k is the smallest integer such that $D^k\mathfrak{L} = (0)$. We may suppose $k \geqq 1$ and then $[D^{k-1}\mathfrak{L}, D^{k-1}\mathfrak{L}] = (0)$. By Theorem 6.1, we see that $D^{k-1}\mathfrak{L} \subset \mathfrak{Z}$. If $k = 1$ we are through. Otherwise $[D^{k-2}\mathfrak{L}, D^{k-2}\mathfrak{L}] \subset \mathfrak{Z}$. If $\mathfrak{Z} \neq (0)$ then $\mathfrak{Z} = Ku$ where u is the identity of B. But for $x, y \in D^{k-2}\mathfrak{L}, xy - yx = \alpha u, \alpha \neq 0$ is impossible. For if this relation persists then $sp(xy) = \alpha + sp(yx)$ whereas $sp(xy)$ and $sp(yx)$ agree up to the single value $\{0\}$ and are compact sets. Therefore $[D^{k-2}\mathfrak{L}, D^{k-2}\mathfrak{L}] = D^{k-1}\mathfrak{L} = (0)$ and $D^{k-2}\mathfrak{L} \subset \mathfrak{Z}$. This argument, repeated a finite number of times, leads to $\mathfrak{L} \subset \mathfrak{Z}$.

6.3. COROLLARY. *Let B be a semi-simple Banach algebra with socle $S \neq (0)$ where S is contained in no primitive ideal of B. Then the Lie radical of B is contained in $\mathfrak{Z}$.*

Let $\mathfrak{L}$ be a solvable Lie ideal of B, P be a primitive ideal and π be the natural homomorphism of B onto B/P. By hypothesis there exists a minimal idempotent e of B, $e \notin P$. Now $eBe = Ke$ so that $\pi(e)\pi(B)\pi(e) = K\pi(e)$ from which it follows that $\pi(e)$ lies in the socle of B/P. Now $\pi(\mathfrak{L})$ is a solvable Lie ideal of B/P so that, by Theorem 6.2, either $\pi(\mathfrak{L}) = (0)$ or $\pi(\mathfrak{L}) = K\pi(u)$ where $\pi(u)$ is an identity for B/P. Hence, for $x \in \mathfrak{L}$, either $x \in P$ or we can write $x = \alpha u + y$, $\alpha \in K, y \in P$. In either case we get $[x, w] \in P$ for all $w \in B$. Therefore $[\mathfrak{L}, B]$ is in every primitive ideal or $\mathfrak{L} \subset \mathfrak{Z}$ by semi-simplicity.

There is an additional fact relating to primitive ideals which will be useful in the sequel.

6.4. LEMMA. *Let P be a primitive ideal of a Banach algebra B. Then either $S(P) = P$ or $S(P) = Ku + P$ where u is an identity for B modulo P.*

Let π be the natural homomorphism of B onto B/P. Let $x \in S(P)$ and $y \in B$. Then clearly $\pi(x)\pi(y) = \pi(y)\pi(x)$ or $\pi(x)$ lies in the center $\mathfrak{Z}_1$ of B/P. Now B/P is primitive so either $\mathfrak{Z}_1 = (0)$ or B/P has an identity $\pi(u)$ and $\mathfrak{Z}_1 = K\pi(u)$. If $\mathfrak{Z}_1 = (0)$ clearly $S(P) = P$. If $\mathfrak{Z}_1 = K\pi(u)$ then $P \subset S(P) \subset Ku + P$ so that either $P = S(P)$ or $Ku + P = S(P)$.

7. Annihilation and Lie annihilation.

In Theorem 3.2 and Corollary 3.3 we obtained structure theorems for a class of algebras where, for each proper closed ideal $I, \mathfrak{L}(I) \neq (0)$. In this section we consider the consequences of a weaker hypothesis on the proper closed ideals for Banach algebras.

Before we investigate the weakened hypothesis, let us note the implication of the statement $\mathfrak{L}(I) = (0)$. In Lemma 2.2, we saw that if I is an ideal in a ring A, then $[I^L, A] \subset \mathfrak{L}(I) \cap \mathfrak{R}(I) \subset I^L$. Thus if $\mathfrak{L}(I) = (0)$, $I^L = \mathfrak{Z}$, the center of A. Suppose that A is semi-prime (so that $\mathfrak{L}(I) = \mathfrak{R}(I)$) and that $\mathfrak{Z}$ contains no nonzero ideal of A. Then if $I^L = \mathfrak{Z}, \mathfrak{L}(I) = (0)$. Hence for many of the algebras under consideration in this paper, $I^L = \mathfrak{Z}$ if and only if $\mathfrak{L}(I) = (0)$.

Throughout this section, B denotes a semi-simple Banach algebra, with center $\mathfrak{Z}$, satisfying the condition:

(7.1) *If $\mathfrak{L}(I) = (0)$, for an ideal I, then $I + \mathfrak{Z}$ is dense in B.*

Since $\mathfrak{L}(I + \mathfrak{R}(I)) = (0)$, an immediate consequence of (7.1) is that for every ideal $I, I + \mathfrak{R}(I) + \mathfrak{Z}$ is dense in B. Also, since $I + \mathfrak{Z} \subset S(I)$, it is immediate that $\mathfrak{R}(I) + S(I)$ is dense in B.

7.1. LEMMA. *Let I be a closed ideal of B. Then $\mathfrak{L}\mathfrak{R}(I) \subset S(I)$.*

Let $x \in \mathfrak{L}\mathfrak{R}(I), y \in B$. We can write $y = \lim (u_n + v_n + z_n)$ where $u_n \in I, v_n \in \mathfrak{R}(I)$ and $z_n \in \mathfrak{Z}$. Then $xy - yx = \lim (xu_n - u_nx) \in I$.

Therefore $x \in S(I)$.

7.2. LEMMA. *A primitive ideal P for which $\mathfrak{L}(P) \neq (0)$ is a ideal. Also $\mathfrak{L}(P)$ is a minimal dual ideal.*

The first statement is shown by the proof of Theorem 3.2. Let $I \neq (0)$, $I \subset \mathfrak{L}(P)$, where I is dual. Since $I \not\subset P$ and P is a primitive ideal we see that $\mathfrak{L}(I) \subset P$. Thus $I = \mathfrak{R}\mathfrak{L}(I) \supset \mathfrak{L}(P)$ which forces $I = \mathfrak{L}(P)$.

7.3. LEMMA. *Let P be a primitive ideal of B for which $\mathfrak{L}(P) =$ (0). Then P is a modular maximal ideal of B of deficiency one, and there is an identity for B modulo P which is central.*

Proof. By (7.1), $P + \mathfrak{Z}$ is dense. Let π be the natural homomorphism of B onto B/P. Then $\pi(P + \mathfrak{Z}) = \pi(\mathfrak{Z})$ is dense in B/P. Therefore B/P is a commutative primitive Banach algebra, i.e., a field. Thus there exists $u \in \mathfrak{Z}$ such that $B = P + Ku$. Clearly u can be selected to be an identity for B modulo P.

7.4. THEOREM. *Let B be a semi-simple Banach algebra satisfying* (7.1). *Let I be the smallest dual ideal of B which contains all the minimal dual ideals of B, then B/I is commutative.*

Let J be the algebraic sum of all the ideals $\mathfrak{R}(P) \neq (0)$ for P primitive. By Lemma 7.2, $I \supset J$. Also clearly $\mathfrak{R}(J) = \cap P$, for the primitive ideals with $\mathfrak{R}(P) \neq (0)$.

Let $W = \cap M$, where M is a maximal ideal of deficiency one with a relative identity in $\mathfrak{Z}$. For any maximal modular ideal of the stated form $B = M + Ku$, with $u \in \mathfrak{Z}$, so $[B, B] \subset M$. Thus $[B, B] \subset W$. By (7.1), Lemma 7.3, and semi-simplicity, $W \cap \mathfrak{R}(J) = (0)$, so $W \subset \mathfrak{L}\mathfrak{R}(J) \subset I$. Thus $[B, B] \subset I$, and consequently B/I is commutative.

We next consider a hypothesis on the Lie ideals which will enable us to obtain results on the Lie structure of certain Banach algebras.

(7.2) *If $\mathfrak{L}$ is a closed Lie ideal of B and $\mathfrak{L} \supset \mathfrak{L}^L$, then $\mathfrak{L} = B$.*

7.5. THEOREM. *Let B be a semi-simple Banach algebra which satisfies* (7.2). *Then B satisfies the conclusion of Theorem* 7.4.

We show that if (7.2) is satisfied so is (7.1). In view of Lemma 2.2, if $\mathfrak{L}(I) = 0$ for an ideal I then $[I^L, B] = (0)$ or $I^L \subset \mathfrak{Z}$. Thus if $\mathfrak{I} = I + \mathfrak{Z}$, $\mathfrak{I}$ is a Lie ideal and $\mathfrak{I} \supset \mathfrak{Z} \supset I^L = \mathfrak{I}^L$. Consequently (7.2) shows that (7.1) is satisfied.

If the semi-simple Banach algebra B satisfies (7.2) and also

(7.3) *B has zero center*

then (7.1) is satisfied with $\mathfrak{Z} = (0)$, so we are with in the framework of § 3, and in particular Corollary 3.3 holds. Thus a structure holds in terms of minimal associative objects. We next turn our attention to obtaining a structure theorem under these hypotheses in terms of certain Lie ideals.

Any H^*-algebra [10, pp. 272–276] B with zero center satisfies (7.2) and (7.3). Let $\mathfrak{L}$ be a closed Lie ideal of B, and suppose that $\mathfrak{L} \supset \mathfrak{L}^L$. Let I be any minimal closed ideal of B. Direct calculation with the "matrix" representation of I, shows that the only closed Lie ideals in I are (0) and I. Hence $[\mathfrak{L}, I] = (0)$ or $[\mathfrak{L}, I] = I$. In the first instance $I \subset \mathfrak{L}^L \subset \mathfrak{L}$, and in the second $I \subset \mathfrak{L}$ directly. Thus for all minimal closed ideals of B, $\mathfrak{L} \supset I$, so $\mathfrak{L} = B$ and (7.2) is satisfied.

Until otherwise is stated we assume B is a semi-simple Banach algebra satisfying (7.2) and (7.3).

7.6. LEMMA. *B is Lie semi-simple.*

It is sufficient to show that $[\mathfrak{L}, \mathfrak{L}] = (0)$ for a Lie ideal $\mathfrak{L}$ of B implies $\mathfrak{L} = (0)$. Now $[\mathfrak{L}, \mathfrak{L}] = 0$ implies $\mathfrak{L} \subset \mathfrak{L}^L$, and thus that $\mathfrak{L}^{LL} \subset \mathfrak{L}^L$. Condition (7.2) then shows that $\mathfrak{L}^L = B$, whence $\mathfrak{L} \subset \mathfrak{Z} = (0)$ by (7.3).

The Lie semi-simplicity will frequently be used in the form that $\mathfrak{L} \cap \mathfrak{L}^L = (0)$ for any Lie ideal $\mathfrak{L}$ of B.

7.7. COROLLARY. *B contains no modular two-sided ideals.*

If M were a maximal modular ideal of B, then (7.1) implies $B = M \oplus \mathfrak{L}(M)$. Let e be an identity for B modulo M, with $e \in \mathfrak{L}(M)$. Clearly $e \in (\mathfrak{L}(M))^L \cap \mathfrak{L}(M) = (0)$ by the prior lemma. Thus we have a contradiction and there can be no modular ideals in B.

7.8. LEMMA. *If $\mathfrak{L}$ is a Lie ideal of B, then $S(\mathfrak{L}^L) = \mathfrak{L}^L$ and $S(L^-) = \mathfrak{L}^{LL}$.*

Clearly $L^L \subset S(\mathfrak{L}^L)$. Let $x \in S(\mathfrak{L}^L)$ and $y \in \mathfrak{L}$. Then $[x, y] \in \mathfrak{L}^L \cap \mathfrak{L} = (0)$, so $S(\mathfrak{L}^L) \subset \mathfrak{L}^L$.

Since $\mathfrak{L} \subset \mathfrak{L}^{LL}$ and the latter is closed, $S(\mathfrak{L}^-) \subset S(\mathfrak{L}^{LL}) = \mathfrak{L}^{LL}$ by the above. Now (7.2) implies that $\mathfrak{L} + \mathfrak{L}^L$ is dense in B. Let $x \in \mathfrak{L}^{LL}$. Then if $y \in B$, $y = \lim (y_j + z_j)$ with $y_j \in \mathfrak{L}$ and $z_j \in \mathfrak{L}^L$. Consequently, $[x,y] = \lim [x, y_j + z_j] = \lim [x, y_j] \in \mathfrak{L}^-$, and so $\mathfrak{L}^{LL} \subset S(\mathfrak{L}^-)$.

7.9. COROLLARY. *If P is a primitive ideal of B, then $p = p^{LL}$.*

It follows from Lemma 6.4 that $S(P) = P$ unless P is a modular ideal. However, by Corollary 7.7 there are no modular ideals, so Lemma 7.8 yields the desired result.

7.10. LEMMA. *If I is a closed ideal in B, then $(\mathfrak{R}(I))^L = S(I)$.*

Since $\mathfrak{R}(I) \subset I^L$, it follows from Lemma 7.8 that $S(I) = I^{LL} \subset (\mathfrak{R}(I))^L$. Let $x \in (\mathfrak{R}(I))^L$ and $y \in B$. Since the ideal $I + \mathfrak{R}(I)$ is dense in B, $y = \lim (p_n + q_n)$, with $p_n \in I$ and $q_n \in \mathfrak{R}(I)$. Then

$$[x, y] = \lim [x, p_n + {}_{qn}] \lim [x, p_n] \in I ,$$

and consequently $(\mathfrak{R}(I))^L \subset S(I)$.

7.11. LEMMA. *Let P be a primitive ideal of B. Then $[\mathfrak{R}(P), \mathfrak{R}(P)]$ is contained in every nonzero closed Lie ideal in $\mathfrak{R}(P)$.*

Let $\mathfrak{J} \neq (0)$ be a closed Lie ideal of B with $\mathfrak{J} \subset \mathfrak{R}(P)$. By the use of two results of Herstein, we see that $S(\mathfrak{J})$ is a ring as well as a Lie ideal [3, p. 282] and therefore [3, p. 281] that either $S(\mathfrak{J})$ is in the center of B or contains a nonzero ideal I of B. The first possibility cannot pertain here since (7.3) holds. Thus we have a nonzero ideal $I \subset S(\mathfrak{J}) = \mathfrak{J}^{LL} \subset (\mathfrak{R}(P))^{LL} = P^L$ by use of Lemmas 7.8, 7.10 and Corollary 7.9. This implies that $\mathfrak{L}(\mathfrak{R}(I)) \cdot \mathfrak{R}(P) \neq 0$, for otherwise $I \subset \mathfrak{L}(\mathfrak{R}(I)) \subset P = \mathfrak{L}(\mathfrak{R}(P))$, and thus $I \subset P \cap P^L = (0)$, which is a contradiction. Consequently $\mathfrak{L}(\mathfrak{R}(I)) \cap \mathfrak{R}(P)$ is a dual ideal $\neq (0)$ contained in the minimal dual ideal $\mathfrak{R}(P)$. Therefore $\mathfrak{R}(P) \subset \mathfrak{L}\mathfrak{R}(I)$ and $\mathfrak{R}(I) \subset P$. Also $S(I) = I^{LL} \subset \mathfrak{J}^{LLLL} = \mathfrak{J}^{LL} = S(\mathfrak{J}) \subset P^L$. Another application of Lemma 7.10 yields $P^L \subset (\mathfrak{R}(I))^L = S(I)$. However, from $I \supset P^L$, we deduce $S(I) = I^{LL} \subset P^L$, so $P^L = S(I) = S(\mathfrak{J})$. Let D be the largest ideal contained in P^L. The Theorem 1 in [14, p. 156] together with its proof shows that $\mathfrak{J} \supset [D, B]$. Since $D \supset \mathfrak{R}(P)$, $\mathfrak{J} \supset [\mathfrak{R}(P), \mathfrak{R}(P)]$.

7.12. LEMMA. *For any primitive ideal P of B, $[\mathfrak{R}(P), \mathfrak{R}(P)]^L = P$.*

The inclusion $P \subset [\mathfrak{R}(P), \mathfrak{R}(P)]^L$ is immediate. Let $x \in P^L = (\mathfrak{R}(P))^{LL}$ by Lemmas 7.9 and 7.10. For $y \in B$, $y = \lim (p_n + r_n)$ with $p_n \in P$ and $r_n \in \mathfrak{R}(P)$, and $x = \lim (s_m + t_m)$ with $s_m \in P$ and $t_m \in \mathfrak{R}(P)$. Thus

$$[x, y] = \lim_n [x, p_n + r_n] = \lim_n [x, r_n] = \lim_n \lim_m [s_m + t_m, r_n]$$
$$= \lim_n \lim_m [t_m, r_n] \in [\mathfrak{R}(P), \mathfrak{R}(P)] .$$

Consequently $P^L \subset S([P), \mathfrak{R}(P)]) = [\mathfrak{R}(P), \mathfrak{R}(P)]^{LL}$, and so $[\mathfrak{R}(P), \mathfrak{R}(P)]^L \subset P^{LL} = P$. We thus have the asserted equality.

7.13. LEMMA. *Each primitive ideal P of B is a maximal proper closed Lie ideal of B.*

Let $\mathfrak{J}$ be a proper closed Lie ideal of B such that $\mathfrak{J} \supset P$. The hypothesis (7.2) implies that $\mathfrak{J}^L \neq (0)$. Corollary 3.3 and its proof imply that each minimal dual ideal of B is of the form $\mathfrak{R}(Q)$ for Q a primitive ideal of B. If $\mathfrak{J}^L \cap \mathfrak{R}(Q) = (0)$ for all primitive ideals Q, then $[\mathfrak{J}^L, \mathfrak{R}(Q)] \subset \mathfrak{J}^L \cap \mathfrak{R}(Q) = (0)$, so by Corollary 3.3, $[\mathfrak{J}^L, B] = (0)$ and $\mathfrak{J}^L \subset Z = (0)$ which is a contradiction. Thus there exists a primitive ideal Q_0 such that $\mathfrak{J}^L \cap \mathfrak{R}(Q_0) \neq (0)$. Lemma 7.11 then yields $[\mathfrak{R}(Q_0), \mathfrak{R}(Q_0)] \subset \mathfrak{J}^L$. Lemma 7.12 then implies that $\mathfrak{J} \subset \mathfrak{J}^{LL} \subset [\mathfrak{R}(Q_0), \mathfrak{R}(Q_0)]^L = Q_0$. As noted in the proof of Theorem 3.2, the resulting inequality $Q_0 \subset P$ with Q_0 and P primitive ideals implies that $Q_0 = P$ and consequently $\mathfrak{J} = P$.

7.14. LEMMA. *Each minimal closed Lie ideal of B is of the form $[\mathfrak{R}(Q), \mathfrak{R}(Q)]$ for some primitive ideal Q of B.*

Let $\mathfrak{D}$ be a minimal closed Lie ideal of B. As noted in the proof of Lemma 7.13, $\mathfrak{D} \cap \mathfrak{R}(Q) \neq (0)$ for some primitive ideal Q of B; and so $\mathfrak{D} \subset \mathfrak{R}(Q)$. Thus by Lemma 7.11, $[\mathfrak{R}(Q), \mathfrak{R}(Q)] = \mathfrak{D}$ by the minimal nature of $\mathfrak{D}$.

7.15. THEOREM. *If B is a semi-simple Banach algebra which satisfies* (7.2) *and* (7.3), *then B is the direct topological sum of its minimal closed Lie ideals.*

In view of Lemma 7.14, and the fact that any two ideals $\mathfrak{R}(Q)$, with Q primitive, have only zero in common, a sum of minimal closed Lie ideals is a direct sum. Let $\mathfrak{J}$ be the direct topological sum of the minimal closed Lie ideal in B. If $\mathfrak{J} \neq B$, then (7.2) implies $\mathfrak{J}^L \neq (0)$. The argument used in the proof of Lemmas 7.13 and 7.14 shows that there is a primitive ideal Q such that $\mathfrak{J}^L \cap \mathfrak{R}(Q) \neq (0)$. Lemma 7.11 then implies that $[\mathfrak{R}(Q), \mathfrak{R}(Q)] \subset \mathfrak{J}^L$. But then $[\mathfrak{R}(Q), \mathfrak{R}(Q)] \subset \mathfrak{J}^L \cap \mathfrak{J} = (0)$, which is a contradiction.

In the final portion of this section, B will denote a simple Banach algebra with an identity u, which satisfied (7.2) and

(7.4) *The center $\mathfrak{Z}$ of B consists of the complex multiples of the identity u of B.*

In view of the proof of Theorem 7.5, we note that (7.1) is automatically satisfied by B. Thus if $\mathfrak{L}(I) = (0)$, for an ideal I, either I is dense in B or I is a maximal ideal of deficiency one in B.

7.16. LEMMA. *If $\mathfrak{R}(M) \neq (0)$ for each maximal ideal M of B, then each primitive ideal of B is a maximal ideal. Furthermore, B is the direct topological sum of its minimal dual ideals.*

Since B has an identity, each primitive ideal P is contained in a maximal ideal M. If $M \neq P$, then by the argument used in the proof of Theorem 3.2, $\mathfrak{R}(M) = (0)$, which is a contradiction. Thus each primitive ideal is a maximal ideal. If B is a primitive algebra, then B is a simple algebra and thus is the only minimal dual ideal. Otherwise let J be the direct topological sum of the minimal dual ideals $\mathfrak{R}(P)$, where P is a primitive ideal. If J were proper, there would exist a maximal ideal N such that $N \supset J \supset \mathfrak{R}(N) \neq (0)$, which would contradict the semi-simplicity of B. Thus $J = B$.

7.17. LEMMA. *If M is a maximal ideal of B with $\mathfrak{R}(M) = (0)$, then $B = M \oplus Ku$, where K is the complex field. Moreover, M as an algebra satisfies condition* (7.2) *and* (7.3).

By Lemma 7.3 M has deficiency one and thus $B = M \oplus Ku$. Since M is a maximal ideal of B, M is a semi-simple Banach algebra and the center of M is contained in the center of B, so is zero. If $\mathfrak{U}$ is a proper closed Lie ideal of M, and $\mathfrak{U}$ contains $V = \{x \in M : [x, y] = 0, \forall y \in \mathfrak{U}\}$, then $\mathfrak{u} + Ku$ is a proper closed Lie ideal of B and $(\mathfrak{U} + Ku)^L = \mathfrak{U}^L = V + Ku$, so $(\mathfrak{U} + Ku) \supset (\mathfrak{U} + Ku)^L$. The hypothesis (7.2) yields $\mathfrak{U} + Ku = B$, which is a contradiction. Thus M as an algebra satisfies (7.2).

7.18. THEOREM. *Let B be a simple Banach algebra with satisfies* (7.2) *and* (7.4). *The only nonzero closed Lie ideals of B are either B and Ku or are B, Ku and $[B, B]$, in which case $B = [B, B] \oplus Ku$, and B is the direct sum of the minimal closed Lie ideals of B.*

Since B is simple, by a theorem of Herstein [3, p. 282], any closed Lie ideal is either contained in Ku or else contains $[B, B] = \mathfrak{L}$.

Suppose first that $Ku \subset \mathfrak{L}$. From Herstein's theorem applied to $\mathfrak{L}^L$, either (a) $\mathfrak{L}^L \subset Ku \subset \mathfrak{L}$ or (b) $\mathfrak{L} \subset \mathfrak{L}^L$. In case (a), the relation (7.2) yields $[B, B] = B$, so that the only closed Lie ideals are Ku and B. In case (b) $\mathfrak{L}^L \supset \mathfrak{L}^{LL}$ so that, by (7.2), $\mathfrak{L}^L = B$. Thus $\mathfrak{L} \subset \mathfrak{Z}$. In view of the reasoning for Theorem 6.2, $[B, B] = (0)$. Thus B is commutative and so $B = Ku$. Hence the case $Ku \subset \mathfrak{L}$ leads to the first possibility stated in the theorem.

If $Ku \not\subset [B, B]$, let $\mathfrak{T} = [B, B] \oplus Ku$. Suppose $\mathfrak{T} \neq B$. Another application of Herstein's theorem yields $\mathfrak{T}^L \supset [B, B]$ or $\mathfrak{T}^L \subset Ku \subset \mathfrak{T}$. The latter is in contradiction to (7.2), while the former implies $\mathfrak{T}^L \supset \mathfrak{T}$ so $\mathfrak{T}^{LL} \subset \mathfrak{T}^L$. Thus $\mathfrak{T}^L = B$, which as above shows that B is the complex field. Hence $B = [B, B] + Ku$, in which case B is the direct sum of the minimal closed Lie ideals of B.

7.19. THEOREM. *Let B be a semi-simple Banach algebra which satisfies* (7.2) *and* (7.4). *Suppose that B is not simple, and is not commutative. Then B is the direct topological sum of its minimal closed Lie ideals and is either the direct topological sum of its minimal dual ideals, or the minimal dual ideals and the center.*

Suppose that $\mathfrak{R}(M) \neq (0)$ for each maximal ideal M of B. Then by Lemma 7.16 each primitive ideal of B is a maximal ideal and B is the direct topological sum of its minimal dual ideals. Also, as noted in the proof of Lemma 7.16, each minimal dual ideal has the form $\mathfrak{R}(M)$ for a maximal ideal.

Let M be any maximal ideal of B, and let $\mathfrak{J} \neq (0)$ be a closed Lie ideal, $\mathfrak{J} \subset \mathfrak{R}(M)$. Then, by a result of Herstein [3, p. 281], either $S(\mathfrak{J})$ is in the center of B or there is a nonzero ideal I in $S(\mathfrak{J})$, in which case we may assume I is the largest such ideal. The first possibility cannot occur since it would imply $\mathfrak{J} \subset Ku$, while $\mathfrak{J} \neq (0)$, and $\mathfrak{J} \subset \mathfrak{R}(M)$. Thus the second possibility holds. If $a \in I \cap M$, then for all $x \in B$, $[a, x] \in \mathfrak{J} \subset \mathfrak{R}(M)$, while $[a, x] \in M$, so $[a, x] = 0$ for all $x \in B$. Thus a is central and since $u \notin M$, we conclude $a = 0$. Consequently $I \cap M = (0)$ and $B = M \oplus I$ so that $I = \mathfrak{R}(M)$. By a result of Zuev [14],

$$\mathfrak{J} \supset [I, B] \supset [\mathfrak{R}(M), \mathfrak{R}(M)] .$$

Consequently each Lie ideal $[R(M), R(M)]$ is a minimal closed Lie ideal of B. Let $\mathfrak{U}$ be the direct topological sum of its minimal closed Lie ideals of B. Since B is the direct topological sum of the minimal dual ideals and these have the form $\mathfrak{R}(M)$, we see that $\mathfrak{U} \supset [B, B] + Ku$. Another application of the result of Herstein used above shows that either $S(\mathfrak{U}^L) \subset Ku$ or these exists a nonzero ideal N such that $N \subset S(\mathfrak{U}^L)$. The first possibility implies that $\mathfrak{U}^L = Ku$, whence $\mathfrak{U} \supset \mathfrak{U}^L$ so by (7.2) $\mathfrak{U} = B$. In the second instance, let $a \neq 0, a \in N$. Then for all $x \in B$, $[a, x] \in \mathfrak{U}^L \cap \mathfrak{U} = \mathfrak{W}$, from which it is immediate that $\mathfrak{W}^{LL} \subset \mathfrak{W}^L$. Thus by (7.2) $\mathfrak{W}^L = B$, so $\mathfrak{W} \subset Ku$, and by the argument used in Theorem 6.2 we deduce that $a \in Ku$, which implies that $N = S(\mathfrak{U}^L) = B$. We therefore conclude that $[B, B] \subset \mathfrak{U} \cap \mathfrak{U}^L \subset Ku$, so as argued earlier B

is commutative, which is a contradiction.

In the event that $\mathfrak{R}(M) = (0)$ for some maximal ideal M of B, it follows from Lemma 7.17 that $B = M \oplus Ku$ where K is the complex field, and M as an algebra satisfies (7.2) and (7.3). Then by Theorem 7.15 M is the direct topological sum of its minimal closed Lie ideals and by Corollary 3.3 is the direct topological sum of the minimal dual ideals of M_0. Any minimal closed Lie ideal of M is one of B, and any minimal dual ideal is one of B. Since Ku is a minimal closed Lie ideal of B, the theorem follows.

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