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We first consider the theory of Jordan homomorphisms and Jordan ideals in Banach algebras. If B is a B^* -algebra or a semi-simple annihilator algebra, any closed Jordan ideal in B is a two-sided ideal. Any Jordan homomorphism of a Banach algebra onto B is automatically continuous. That Jordan homomorphisms are continuous and Jordan ideals are ideals is shown to hold in a number of other situations. We also study the Lie ideals in a semi-simple Banach algebra A . If the center of A is zero and proper closed Lie ideals do not contain their Lie annihilators, then A is direct topological sum of its minimal closed ideals. An H^* -algebra with zero center is an example of such an algebra.

The utility of the study of Jordan isomorphisms in Banach algebra was noted by Kadison [8] in the study of isometrics of B^* -algebras. The Jordan and Lie structures of simple associative rings has been investigated by Herstein in a series of paper (see [3], [4], [5]). Essential use is made of these results in the present work.

2. Pure algebra. Let R be an associative ring. As is well-known [3] we can make R into a Jordan (Lie) ring by introducing the Jordan (Lie) multiplication $x \cdot y = xy + yx$ ($[x, y] = xy - yx$). For a subset S of R we consider the sets

$$\begin{aligned} S^J &= \{x \in R \mid x \cdot u = 0 \text{ for all } u \in S\}, \\ S^L &= \{x \in R \mid [x, u] = 0 \text{ for all } u \in S\}, \\ \mathfrak{R}(S) &= \{x \in R \mid ux = 0 \text{ for all } u \in S\} \text{ and} \\ \mathfrak{L}(S) &= \{x \in R \mid xu = 0 \text{ for all } u \in S\}. \end{aligned}$$

By an *ideal* in R we mean, unless otherwise specified, a two-sided ideal.

2.1. LEMMA. *Let U be a Lie ideal in R . Then U^J and U^L are Lie ideals.*

Let $x \in U^J$, $u \in U$ and $b \in R$. Since $xu = -ux$, an easy computation shows that $[x, b] \cdot u = [b, u] \cdot x = 0$. Let $y \in U^L$. Since $yu = uy$, we obtain by straightforward calculation that $[[y, b], u] = [y, [b, u]] = 0$.

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2.2. LEMMA. *Let I be an ideal in R . Then*

$$[I^\perp, R] \subset \mathfrak{L}(I) \cap \mathfrak{R}(I) \subset I^\perp.$$

Let $y \in I, x \in I^\perp$ and $z \in R$. We get successively the relations $(xz)y = x(zy) = (zy)x = z(yx) = z(xy) = (zx)y$ or $[I^\perp, R] \subset \mathfrak{L}(I)$. Likewise $[I^\perp, R] \subset \mathfrak{R}(I)$.

For the remainder of this section, A shall denote a semi-prime algebra over a field of characteristic $\neq 2$. (A semi-prime algebra is one without nilpotent right or left ideals $\neq (0)$). It follows from the hypothesis that A is semi-prime, that $\mathfrak{L}(I) = \mathfrak{R}(I)$ for any ideal I in A [10] p. 99].

2.3. LEMMA. *Let U be a Lie ideal in A . Then $U^J = \mathfrak{L}(U) = \mathfrak{R}(U)$ is an ideal in A and $U \cap U^J = (0)$.*

Take $x \in U \cap U^J$. Clearly $x^2 = 0$. Also, by Lemma 2.1, $[x, b] \in U \cap U^J$ for each $b \in A$. Then

$$0 = [x, b]^2 = xbx b - xb^2x + bxbx.$$

Multiplying on the left by x and on the right by b , we obtain $(xb)^3 = 0$. Thus xA is a nilpotent right ideal so that $xA = (0)$ and hence $x = 0$. Therefore $U \cap U^J = (0)$.

Next let $u \in U$ and $x \in U^J$. The above shows that $[u, x] = 0$ and $u \cdot x = 0$. Thus $ux = xu = 0$. Hence

$$(1) \quad U^J \subset \mathfrak{L}(U) \cap \mathfrak{R}(U).$$

Next we show that

$$(2) \quad \mathfrak{L}(U) = \mathfrak{R}(U).$$

For take $x \in \mathfrak{L}(U)$, $u \in U$ and $b \in A$. We have $0 = x[b, u] = xbu$. Hence $(uxb)^2 = 0$. It follows that $ux = 0$ or $\mathfrak{L}(U) \subset \mathfrak{R}(U)$. Likewise $\mathfrak{R}(U) \subset \mathfrak{L}(U)$. A combination of (1) and (2) gives the desired result. The corresponding proposition for Jordan ideals is harder to prove (see Theorem 2.5).

Let U be a Jordan ideal in A . Let K denote the algebraic sum of the ideals in A contained in U . Clearly K is maximal in the set of ideals in A contained in U .

2.4. LEMMA. *The ideal K contains $a \cdot b$ and aAa for each a, b of the Jordan ideal U in A . If $U \neq (0)$ then $K \neq (0)$.*

By a lemma of Herstein [3, Lemma 1], $(a \cdot b)x - x(a \cdot b) \in U$ for each $x \in A$. Since $(a \cdot b) \cdot x \in U$ we see that $(a \cdot b)x$ and $x(a \cdot b)$ lie in U .

If also $y \in A$ then $(a \cdot b)xy + y(a \cdot b)x \in U$. From this we see that the ideal generated by $a \cdot b$ is contained in U . Therefore $a \cdot b \in K$. Also $(a \cdot x) \cdot a \in K$. Using $a^2 \in K$ we obtain $axa \in K$. If $K = (0)$, then $(aA)^2 = (0)$ and $a = 0$ for all $a \in U$.

2.5. THEOREM. *Let U be a Jordan ideal in A . Then $U^J = \mathfrak{L}(U) = \mathfrak{R}(U)$ is an ideal in A and $U \cap U^J = (0)$.*

Let $u \in U, x \in A$ and $z \in \mathfrak{L}(U)$. We have $0 = z(u \cdot x) = (zx)u$. Therefore $\mathfrak{L}(U)$ is an ideal in A . Since $[u\mathfrak{L}(U)]^2 = (0)$ we can conclude that $u\mathfrak{L}(U) = (0)$ or $\mathfrak{L}(U) \subset \mathfrak{R}(U)$. Likewise $\mathfrak{R}(U) \subset \mathfrak{L}(U)$. Thus $\mathfrak{L}(U) \subset U^J$.

Suppose that either a or b lies in U . Then

$$(a \cdot b) \cdot b = a \cdot (b^2) + 2bab \in U.$$

In either case it follows that $bab \in U$. Next let $u \in U, x \in A$. Then $(u \cdot x) \cdot (xu) = ux^2u + xu^2x + 2(xu)^2 \in U$. By the preceding remark we see that $(xu)^2 \in U$. Likewise $(ux)^2 \in U$. Consider now an ideal I in A such that $I \cap U = (0)$. If $z \in I$ then $(uz)^2 \in I \cap U = (0)$. Therefore $uI = (0)$. We then have $(0) = UI = IU$.

Let K be the ideal in A maximal in the set of ideals in A contained in U (see Lemma 2.4). In particular $a^2 \in K$ for $a \in U$. Now $\mathfrak{L}(K)$ is an ideal in A . We show that $\mathfrak{L}(K) \cap U = (0)$. For let $b \in \mathfrak{L}(K) \cap U, x \in U$. Then $(bx)^2 \in U \cap \mathfrak{L}(K)$. This makes $(bx)^4$ an element of $K \cap \mathfrak{L}(K) = (0)$. Therefore $(bA)^4 = (0)$ and $b = 0$.

It follows from the above that $U\mathfrak{L}(K) = \mathfrak{L}(K)U = (0)$. Consequently $\mathfrak{L}(K) \subset \mathfrak{L}(U)$. But, inasmuch as $K \subset U$, we have also that $\mathfrak{L}(K) = \mathfrak{L}(U)$. From Lemma 2.3 we get $K^J = \mathfrak{L}(K) = \mathfrak{L}(U) \subset U^J$. But as $K \subset U$ we see that $K^J = U^J$. Whereas $U \cap \mathfrak{L}(K) = (0)$, we obtain $U \cap U^J = (0)$.

2.6. LEMMA. *If A satisfies the descending chain condition on right ideals, every Jordan ideal in A is an ideal.*

By the proof of Theorem 2.5, bab lies in the Jordan ideal U if a or b lies in U . It follows from [9, Lemma 5] that U is an ideal.

As we shall see below, Jordan ideals are automatically ideals under favorable conditions. If an algebra B (even a Banach algebra) is not semi-simple, this situation does not prevail and even Lemma 2.6 can fail there, as easy examples show.

For S, T subsets of A , by $S \cdot T$ we mean the collection of all finite sums of elements $x \cdot y, x \in S, y \in T$.

2.7. LEMMA. *Let I be an ideal in A . Then $I \cdot A$ is an ideal.*

Also $A \cdot A$ contains A^3 .

The relation $a \cdot (b \cdot c) - (a \cdot b) \cdot c = [[a \cdot c], b]$ is readily verified. Suppose at least one of a, b and c is in I . Then $[[a, c], b] \in I \cdot A$. Since I is an ideal, $[a, c] \cdot b \in I \cdot A$. Hence $[a, c]b \in I \cdot A$ and $b[a, c] \in I \cdot A$.

Suppose that $b \in I$. Then $(ac) \cdot b - a[c, b] = (a \cdot b)c \in I \cdot A$. Thus $(a \cdot b)c \in I \cdot A$ and $[b, a]c \in I \cdot A$. Taking the sum and difference of these items, we see that abc and $bac \in I \cdot A$ for all $a, c \in A$. Therefore $(a \cdot b)c$ and likewise $c(a \cdot b) \in I \cdot A$ and $I \cdot A$ is an ideal. The proof shows that $A \cdot A \supset A^3$.

2.8. LEMMA. *A maximal ideal M in A is a maximal Jordan ideal.*

Let π be the natural homomorphism of A onto A/M . If A/M is not a zero algebra, then, by [3, Theorem 1], A/M has only itself and (0) as Jordan ideals. Therefore M is a maximal Jordan ideal. Suppose that A/M is a zero algebra. Since its only ideals are trivial, A/M is one-dimensional. This makes M a maximal linear subspace in A and hence a maximal Jordan ideal.

2.9. THEOREM. *If $A \cdot A = A$, then every maximal Jordan ideal M in A is an ideal.*

Let K be the largest ideal in A contained in M . If K is a maximal ideal in A , then $K = M$ by Lemma 2.8. We show that K is always a maximal ideal. For suppose otherwise. There exists an ideal I in A , $A \neq I$, $K \subset I$, $K \neq I$. If $I \supset M$ we are through and $I \subset M$ is impossible. Therefore $A = I + M$. Then $A \cdot M = I \cdot M + M \cdot M$. Lemma 2.4 gives $M \cdot M \subset K \subset I$ so that $A \cdot M \subset I$. Also $A \cdot A = A \cdot I + A \cdot M \subset I$. By hypothesis, this is impossible. In particular, by Lemma 2.7, the conclusion holds if $A^3 = A$.

2.10. THEOREM. *Suppose that $A \cdot A = A$ and each ideal in A is the intersection of the maximal ideals containing it. Then every Jordan ideal in A is an ideal.*

These conditions are satisfied, for example, if A is biregular in the sense of Arens and Kaplansky [1]. Let U be a Jordan ideal and let K be the largest ideal in A contained in U . If K is a maximal ideal, $K = U$ by Lemma 2.8. Suppose that K is not a maximal ideal and let M be any maximal ideal containing K . We show that $U \subset M$. For suppose otherwise. Clearly $M \not\subset U$ and $A = M + U$. Reasoning as in the proof of Theorem 2.9, we get $A \cdot A \subset M$, which is impossible. Hence

U is contained in every maximal ideal $M, M \supset K$. Therefore $U = K$.

3. On topological rings. Now let A be a semi-prime topological algebra over a field of characteristic $\neq 2$.

3.1. THEOREM. *Suppose that $\mathfrak{L}(I) \neq (0)$ for every closed ideal $I \neq A$ in A . Then any closed Jordan ideal U in A is an ideal in A .*

As in the preceding section we consider the ideal K in A maximal in the set of ideals contained in U . Clearly K is closed. Let $W = K \oplus \mathfrak{L}(K)$. Clearly $x^2 = 0$ for each $x \in \mathfrak{L}(W) = \mathfrak{R}(W)$. Therefore $\mathfrak{L}(W) = (0)$ and W is dense in A . Let $a \in U, u \in U$. There exists a net a_α in W such that $a_\alpha \rightarrow a$. For each $a_\alpha, a_\alpha = b_\alpha + c_\alpha$, where $b_\alpha \in K$ and $c_\alpha \in \mathfrak{L}(K)$. From the proof of Theorem 2.5, $\mathfrak{L}(K) = \mathfrak{L}(U)$. Then we see that $a_\alpha u = b_\alpha u \in K$ for each α . Consequently $au \in K$. Likewise $ua \in K$, and U is an ideal in A .

The condition $\mathfrak{L}(I) \neq (0)$ for proper closed ideals in A gives the natural two-sided analogue of the annihilator algebras of Bonsall and Goldie [2].

3.2. THEOREM. *Let B be a semi-simple topological ring. Suppose that*

- (a) *the primitive ideals of B are closed;*
- (b) *$\mathfrak{L}(I) \neq (0)$ for each closed ideal $I \neq A$.*

Then B is the direct topological sum of its minimal closed ideals.

Note that (a) is automatically satisfied if B is a Banach algebra. Suppose first that (0) is a primitive ideal, i.e., B is primitive. Then (b) and $\mathfrak{L}(I)I = (0)$ for a closed ideal I show that all ideals in B other than (0) must be dense. The desired conclusion is then readily seen. We assume then that (0) is not a primitive ideal.

We call an ideal K in B *dual* if $\mathfrak{L}\mathfrak{R}(K) = K$. Our first step is to show that each primitive ideal P is dual. There exists a modular maximal right ideal M in B such that $P = \{a \in B \mid xa \in M \text{ for all } x \in B\}$ and P is the largest ideal in B contained in M . Let j be a left identity for B modulo M , let $b \in \mathfrak{L}\mathfrak{R}(P)$ and suppose that $b \notin P$. Then there exists an element $z \in B$ such that $zb \notin M$. We can write $j = u + kzb + zbx$ where $u \in M, x \in B$ and k is an integer. Multiplying on the right by an element $w \in \mathfrak{R}(P) = \mathfrak{L}(P)$, we see that $jwt = uw \in M$. Since $jwt - w \in M$, we also get $w \in M$. Therefore $P \oplus \mathfrak{R}(P) \subset M$. This is impossible since $\mathfrak{R}(P) \neq (0)$ by hypothesis and $P \oplus \mathfrak{R}(P)$ is a larger ideal than P contained in M . Hence P is dual.

Next take any ideal I in B , where $I \supset P, I \neq P$, for a primitive

ideal P . We show that $\mathfrak{L}(I) = (0)$ so that I is dense. Since $I \supset P$, we have $\mathfrak{L}(I) \subset \mathfrak{L}(P)$. Also $(0) = \mathfrak{L}(I)I \subset P$ and $I \not\subset P$. The theory of primitive ideals shows that $\mathfrak{L}(I) \subset P$. Therefore $\mathfrak{L}(I) \subset P \cap \mathfrak{L}(P) = (0)$.

Now let I be any dual ideal in B . We show that I is the intersection of the primitive ideals in B which contain I . Let π be the natural homomorphism of B onto B/I . Since $I \cap \mathfrak{R}(I) = (0)$, π is one-to-one on $\mathfrak{R}(I)$ and $\pi(\mathfrak{R}(I))$ is semi-simple. Let W denote the radical of B/I . Then $W \cap \pi(\mathfrak{R}(I)) = (0) = W\pi(\mathfrak{R}(I))$ by [6, p. 10]. Therefore $\pi^{-1}(W)\mathfrak{R}(I) \subset I$ from which we see that $\pi^{-1}(W)\mathfrak{R}(I) = (0)$. Thus $\pi^{-1}(W) \subset \mathfrak{R}\mathfrak{R}(I) = I$ and $W = (0)$. Inasmuch as B/I is now seen to be semi-simple, I is the intersection of the primitive ideals which contain it.

Next consider a primitive ideals P_0 and let P be any primitive ideal, $P \neq P_0$. By the above and our hypothesis, $P \not\subset P_0$ and $P_0 \not\subset P$ and $\mathfrak{L}(P + P_0) = (0)$. Therefore $\mathfrak{L}(P)\mathfrak{L}(P_0) = (0) = \mathfrak{L}(P_0)\mathfrak{L}(P)$ so that $\mathfrak{L}(P_0) \subset \mathfrak{R}\mathfrak{L}(P) = P$. Also $\mathfrak{R}\mathfrak{L}\mathfrak{R}(P_0) = \mathfrak{R}(P_0) = \mathfrak{L}(P_0)$ so that $\mathfrak{L}(P_0)$ is dual and $\mathfrak{L}(P_0) \not\subset P_0$. Therefore, as $\mathfrak{L}(P_0)$ is the intersection of the primitive ideals containing it, $\mathfrak{L}(P_0) = \bigcap P$, for $P \neq P_0$, P primitive. It is now clear that the ideals $\mathfrak{L}(P)$, P primitive, are the minimal dual ideals of B .

View $\mathfrak{L}(P_0)$ as a ring. By [6, p. 206], every primitive ideal of $\mathfrak{L}(P_0)$ is the intersection of $\mathfrak{L}(P_0)$ with a primitive ideal of B . Thus $\mathfrak{L}(P_0)$ is a primitive ring. Let $I \neq (0)$ be any closed ideal in B , $I \subset \mathfrak{L}(P_0)$ and let J be the algebraic sum of the ideals $\mathfrak{L}(P)$, $P \neq P_0$, P primitive. We claim that $I + J$ is dense in B . For let $x \in \mathfrak{R}(I + J)$. Then $x \in \mathfrak{R}\mathfrak{L}(P) = P$ for all $P \neq P_0$. Thus $x \in \mathfrak{L}(P_0) \cap \mathfrak{L}(I) = K$, say. Since $IK = (0)$ and $\mathfrak{L}(P_0)$ is a primitive ring, we get $K = (0)$. By (b), $I + J$ is dense in B .

We continue with this notation and show next that $\mathfrak{L}(P_0)\mathfrak{L}(P_0) \subset I$. Let $u \in \mathfrak{L}(P_0)$, $u = \lim (v_\alpha + q_\alpha)$, $v_\alpha \in I$, $q_\alpha \in J$ and let $x \in \mathfrak{L}(P_0)$. Now each $q_\alpha \in P_0$ so that $q_\alpha x = 0$. Therefore $ux = \lim v_\alpha x \in I$. This enables us to see that $\mathfrak{L}(P_0)(P_0 + \mathfrak{L}(P_0)) \subset I$. Inasmuch as $P_0 + \mathfrak{L}(P_0)$ is dense in B , we get $\mathfrak{L}(P_0)B \subset I$. Likewise $B\mathfrak{L}(P_0) \subset I$. From semi-simplicity we see that $B\mathfrak{L}(P_0) \neq (0)$, $\mathfrak{L}(P_0)B \neq (0)$. It follows that $\overline{B\mathfrak{L}(P_0)} = \overline{\mathfrak{L}(P_0)B}$ is a minimal closed ideal in B .

For each primitive ideal P_α , let $Z_\alpha = \overline{B\mathfrak{L}(P_\alpha)}$. It is clear that if $P_\alpha \neq P_\beta$, then $Z_\alpha \cap Z_\beta = (0) = Z_\alpha Z_\beta$. Then the algebraic sum of the Z_α is a direct sum. Let Q be its closure, the direct topological sum of the Z_α . Note that $\mathfrak{R}(Z_\alpha) = \mathfrak{R}(B\mathfrak{L}(P_\alpha)) = \mathfrak{R}\mathfrak{L}(P_\alpha) = P_\alpha$. Therefore $Q = B$. If I is any minimal closed ideal, I must be Z_α , for some α , for otherwise I would annihilate every Z_α .

3.3. COROLLARY. *Under the conditions of Theorem 3.2. the primitive ideals of B are the maximal closed ideals of B and every*

ideal in B which is not dense is contained in a primitive ideal. Also B is the direct topological sum of its minimal dual ideals.

Let I be closed ideal in B , $I \neq B$. Then $\mathfrak{L}(I) \neq (0)$ and $\mathfrak{L}(I)I = (0)$. If I were contained in no primitive ideal we would have $\mathfrak{L}(I)$ contained in every primitive ideal, which is impossible.

The proof of Theorem 3.2 shows that a primitive ideal is a maximal closed ideal. Let M be a maximal closed ideal. As just seen, $M \subset P$ for some primitive ideal P , so $M = P$. The proof of Theorem 3.2 also demonstrates that $\mathfrak{L}(P_1)\mathfrak{L}(P_2) = (0)$ if P_1, P_2 are two distinct primitive ideals and that B is the direct topological sum of the $\mathfrak{L}(P)$ which are the minimal dual ideals.

4. Continuity of Jordan homomorphisms. We consider a Jordan homomorphism T defined on a Banach algebra A with range a dense subset of a semi-simple Banach algebra B . We seek to show that, under reasonable conditions, such a mapping T is automatically continuous. For an element x in A or B we let $\rho(x)$ denote its spectral radius [10, p. 30].

Two useful identities are noted by Kadison [8, p. 330] for the behavior of Jordan homomorphisms relative to Lie products. These are

$$T([[a, b], c]) = [[Ta, Tb], Tc]$$

and

$$T([a, b]^2) = [Ta, Tb]^2$$

for all $a, b, c \in A$. Hence, as noted in [8], $ab = ba$ makes $[Ta, Tb]$ a central nilpotent element of B . Thus, as B is semi-simple, $[Ta, Tb] = 0$. Then $T(ab) = T(ab + ba)/2 = T(a)T(b) = T(b)T(a)$.

4.1. LEMMA. *For each $x \in A$, $\rho(T(x)) \leq \rho(x)$.*

From the above discussion we see that if $a \circ b = b \circ a = 0$ that $T(a) \circ T(b) = T(b) \circ T(a) = 0$. From this the result is evident.

By the *separating set* for T we mean the set of $s \in B$ for which there exists a sequence $\{x_n\}$ in A with $\|x_n\| \rightarrow 0$ and $\|s - T(x_n)\| \rightarrow 0$. By the closed graph theorem, T is continuous if only if $S = (0)$.

Straightforward arguments show that S is a closed Jordan ideal in B .

4.2. LEMMA. *If B has a left or right identity j , then $j \in S$.*

Note that j need not lie in $T(A)$. Suppose that $j \in S$ where j is a left identity for B . Then there exists a sequence $\{x_n\}$ in A with

$\|x_n\| \rightarrow 0$ and $\|j - T(x_n)\| \rightarrow 0$. For each $z \in B$, $(zj)^n = z^n j$. Thus

$$\|(zj)^n\|^{1/n} \leq \|z^n\|^{1/n} \|j\|^{1/n}.$$

From this we see that $\rho(zj) \leq \rho(z)$. Clearly $\rho(j) = 1$. Now $j = (j - zj) + zj$ where the summands permute. Therefore

$$1 = \rho(j) \leq \rho(j - zj) + \rho(zj) \leq \rho(j - zj) + \rho(z).$$

Now replace z by $T(x_n)$. We see that

$$\rho(j - T(x_n)j) \leq \|j - T(x_n)\| \|j\| \rightarrow 0$$

and that, by Lemma 4.1, $\rho(T(x_n)) \leq \|x_n\| \rightarrow 0$. This yields a contradiction.

4.3. LEMMA. *Each element of S is a two-sided topological divisor of zero in B .*

This is a variation on a result of Rickart [10, p. 72]. Let $s \in S$. Arguments used there show that, if $1 - \lambda s$ is a two-sided topological divisor of zero for arbitrarily large λ , then s is a two-sided topological divisor of zero. We assume, then, that $1 - \lambda s$ is not a two-sided topological divisor of zero for $|\lambda| \geq k$. Then λs is quasi-regular, $|\lambda| \geq k$. Suppose that s is not a left topological divisor of zero in B . Then [10, p. 24], B has a left identity j and s is a regular element of the algebra jBj ; there exists $u \in jBj$ such that $us = su = j$. Then $(u \cdot s)/2 = j \in S$ which is contrary to Lemma 4.2. Consequently s is a left (and similarly right) topological divisor of zero.

4.4. LEMMA. *Let $e \neq 0$ be any idempotent in $T(A)$. Then $e \notin S$.*

Suppose that $e = T(x)$, $e \in S$. There exists a sequence $\{x_n\}$ in A with $\|x_n\| \rightarrow 0$ and $\|e - T(x_n)\| \rightarrow 0$. Then also $\|e - eT(x_n)e\| \rightarrow 0$. From the theory of Jordan homomorphisms (see, for examples [8, p. 329]), $T(xx_nx) = eT(x_n)e$. Now $e - eT(x_n)e$ and $eT(x_n)e$ permute so that, by Lemma 4.1,

$$\begin{aligned} 1 = \rho(e) &\leq \rho(e - eT(x_n)e) + \rho(eT(x_n)e) \\ &\leq \|e - eT(x_n)e\| + \rho(T(xx_nx)) \\ &\leq \|e - eT(x_n)e\| + \|xx_nx\| \rightarrow 0. \end{aligned}$$

Therefore $e \notin S$.

If it is not required that the idempotent e be in $T(A)$ the following weaker conclusion holds.

4.5. LEMMA. *No central idempotent $e \neq 0$ of B lies in S .*

We use the notation of the proof of Lemma 4.4. If $e \in S$, then $1 = \rho(e) \leq \rho(e - T(x_n)) + \rho(T(x_n)) \leq \|e - T(x_n)\| + \|x_n\| \rightarrow 0$.

As an application we take $B = L_2(G)$ where G is a compact group and the multiplication in $L_2(G)$ is convolution (see [10, p. 330]). Suppose $T(A)$ is dense in $L_2(G)$. Lemma 4.5 shows that S cannot contain a central idempotent $\neq 0$. But in $L_2(G)$, every nonzero ideal contains a central idempotent. Therefore $S = (0)$ and T is continuous.

4.6. THEOREM. *A Jordan homomorphism T of a Banach algebra A onto a dense subset of a strongly semi-simple Banach algebra B is continuous.*

Let M be a modular maximal ideal of B , and let π denote the natural homomorphism of A onto B/M . The mapping πT is a Jordan homomorphism of A onto a dense subset of the simple algebra B/M . The separating set S_1 for the mapping πT is a Jordan ideal of B/M which cannot contain the identity of B/M by Lemma 4.2. But by Herstein's result [3, Theorem 1], S_1 is an ideal of B/M . Therefore $S_1 = (0)$ and πT is continuous. Simple arguments now show that $S \subset M$. Hence $S = (0)$.

4.7. THEOREM. *Let T be a Jordan homomorphism of a Banach algebra A onto a semi-simple Banach algebra B where every nonzero right (left) ideal in B contains a minimal right (left) ideal in B . Then T is continuous.*

Examples of such B are the semi-simple annihilator Banach algebras of Bonsall and Goldie ([10, p. 96] and [2]).

Fix $a, b \in S$. By Lemma 2.4, the ideal generated by $a \cdot b$ lies in S . If the right ideal $\{(a \cdot b)x \mid x \in A\} \neq (0)$, it contains an idempotent $e \neq 0$ contrary to Lemma 4.4. Consequently $(a \cdot b)B = (0)$ and $a \cdot b = 0$ for all $a, b \in S$. For $a \in S, x \in B$ we have $(a \cdot x) \cdot a = 0$. Using $a^2 = 0$ we see that $(aB)^2 = (0)$. Hence a lies in the radical of B so that $S = (0)$.

4.8. THEOREM. *Let T be a Jordan homomorphism of a Banach algebra A onto a dense subset of a primitive Banach algebra B with minimal one-sided ideals. Suppose that $T(A) \cap I$ is dense for some minimal right (left) ideal I . Then T is continuous.*

For the case of an algebra homomorphism see [12, p. 378]. Suppose that I is a right ideal. Let $I_1 = T^{-1}(I)$. If $T(x^2) = 0$ for all $x \in I_1$, then $u^2 = 0$ for each u in a dense subset of I . This would

make $I^2 = (0)$ which is impossible. We can express $I = jB$ where $j^2 = j$. Set $R = \{y \in B \mid jy \in I \cap T(A)\}$. Clearly R is a linear manifold where jR is dense in jB . Thus jRj is dense in jBj . By the Gelfand-Mazur theorem, $jRj = jBj$. Select $x \in I_1$ where $T(x) = jw \neq 0$ and $T(x^2) = jwjw \neq 0$. We have $jwj \neq 0$. Since jBj is a division ring there exists $z \in R$ where $jwjzj = jzjwj = j$. Consequently $(jw) \cdot (jz) \neq 0$. There exists $x_1 \in I_1$ such that $T(x_1) = jz$. Simple computations give $T((x \cdot x_1)/2) = (jw) \cdot (jz)/2 = h$, say, and $T[(x \cdot x_1)^2/4] = h$. Hence h is an idempotent, $\neq 0$, in $T(A) \cap I$. By Lemma 4.4, $h \notin S$ where S is the separating set for T . Consequently $I \not\subset S$.

Denote the socle of B by Σ and let $a, b \in S$. By the arguments of Theorem 4.7, the two-sided ideal Q generated by $a \cdot b$ lies in S . As B is primitive, $\Sigma \subset Q \subset S$ unless $a \cdot b = 0$. Thus, if $a \cdot b \neq (0)$, $h \in S$ which is impossible. Consequently we can reason as in Theorem 4.7 to see that $S = (0)$.

5. Jordan ideals and homomorphisms in B^* -algebras. Throughout this section U will denote a closed Jordan ideal in a B^* -algebra A . For an element $x \in A$ which is quasi-regular [10, p. 16] we denote its quasi-inverse by x' . Since $x \cdot x' = (x + x')/2$, we see that $x' \in U$ if $x \in U$.

5.1. LEMMA. *If h is self-adjoint and $h^3 \in U$, then $h \in U$.*

For all $\alpha > 0$, $(-\alpha h^2)'$ exists and lies in U . Thus $h \cdot (-\alpha h^2)' \in U$. An argument of Rickart [10, Theorem 4.9.2] shows that for any $x \in A$,

$$(5.1) \quad x = \lim_{\alpha \rightarrow \infty} x(-\alpha x^*x)' = \lim_{\alpha \rightarrow \infty} (-\alpha x x^*)'x .$$

Applying this to our case, we obtain $2h = \lim h \cdot (-\alpha h^2)' \in U$.

5.2. LEMMA. $U^* = U$.

Let $a \in U$. By Lemma 2.4, the ideal generated by a^2 lies in U . In particular, $[-\alpha(a^2)(a^2)^*]' \in U$ for all $\alpha > 0$. By the proof of Theorem 2.5, $(a^2)^*[-\alpha a^2(a^2)^*]'(a^2)^* \in U$. Formula (5.1) implies that $(a^4)^* \in U$. Since $a^4 \in U$ it follows that $[a^2 + (a^2)^*]^2 = a^4 + a^2 \cdot (a^2)^* + (a^4)^* \in U$, and so, by Lemma 5.1, $a^2 + (a^2)^* \in U$. Therefore $(a^2)^* \in U$, so that

$$(a + a^*)^2 = a^2 + a \cdot a^* + (a^2)^* \in U.$$

Another application of Lemma 5.1 shows that $a + a^*$, and thus a^* , lies in U .

5.3. THEOREM. *Any closed Jordan ideal in a B^* -algebras A is*

an ideal of A .

Let U be a closed Jordan ideal of A and let $a \in U$. Suppose $a = h + ik$, where h and k are self-adjoint. Since $a^* \in U$, it follows from Lemma 5.2 that $h \in U$ and $k \in U$. It thus suffices to see that if h and w are self-adjoint and $h \in U$, then hw and wh are in U . Now $(wh)^*(wh) = hw^2h \in U$, since $h \in U$. Also $(wh)(wh)^* = wh^2w \in U$, since $h \in U$. Consequently for $\alpha > 0$, $[-\alpha(wh)^*(wh)]' \in U$, and so $\lim_{\alpha \rightarrow \infty} (wh)[- \alpha(wh)^*(wh)]'(wh) \in U$. By formula (5.1), $(wh)^2 \in U$, and so by Lemma 5.2 $(hw)^2 \in U$. Since U is a Jordan ideal, it contains $wh + hw$. But also $[i(wh - hw)]^2 = -\{(wh)^2 - wh^2w - hw^2h + (hw)^2\} \in U$, so again by Lemma 5.1 $wh - hw \in U$, and thus wh and $hw \in U$, so U is an ideal.

Automatic continuity occurs for Jordan $*$ -homomorphisms.

5.4. THEOREM. *Let T be a Jordan $*$ -homomorphism of a B^* -algebra A onto a dense subset of a B^* -algebra B . Then*

- (a) T is continuous and $\|T\| = 1$,
- (b) the range of T is B , and
- (c) the adjoint mapping T' is an isometry.

Before proceeding with the proof we should remark that these are instances of positive linear maps on operator algebras, which have been extensively studied. See [15].

Let A_0 be the closed $*$ -subalgebra generated by a self-adjoint element $h \in A$. Then T is a $*$ -homomorphism of the commutative B^* -algebra A_0 into B . Therefore $\|T(h)\| \leq \rho(T(h)) \leq \rho(h) \leq \|h\|$. Consequently if h and k are any two self-adjoint elements of A ,

$$\begin{aligned} \|T(h + ik)\| &\leq \|T(h)\| + \|T(k)\| \leq \|h\| + \|k\| \\ &\leq \|h + ik\| + \|h - ik\| = 2\|h + ik\|. \end{aligned}$$

Thus T is continuous.

Hence the kernel $T^{-1}(0)$ of the mapping T is a closed Jordan ideal in A . By Theorem 5.3, $T^{-1}(0)$ is an ideal and [10, p. 249] $A/T^{-1}(0)$

is a B^* -algebra in the quotient space norm. Let T_0 be the mapping of $A/T^{-1}(0)$ onto B defined by $T_0(x + T^{-1}(0)) = T(x)$ and let π be the natural homomorphism of A onto $A/T^{-1}(0)$. The mapping T_0 restricted to the closed $*$ -subalgebra of $A/T^{-1}(0)$ generated by a normal element is an isometry by [10, p. 241]. That T_0 is an isometry on normal elements allows us to use the arguments of [8, Theorem 5] to assert that T_0 is an isometry on $A/T^{-1}(0)$. The range of T_0 and therefore that of T is then all of B , and $\|T(x)\| = \|\pi(x)\|$ for all $x \in A$. Since a natural homomorphism has norm one, $\|T\| = 1$. Now $T = T_0\pi$, so

$T' = \pi' T'_0$. As each of T'_0 and π' is an isometry by the theory of normed linear spaces, so is T' .

5.5. THEOREM. *Let A, B be B^* -algebras with identities e_A and e_B , respectively. The Jordan $*$ -homomorphisms of A onto B are precisely the continuous linear mappings T of A onto B such that (a) the adjoint T' is isometric, (b) the kernel of T is an ideal of A , and (c) $T(e_A) = e_B$.*

By Theorem 5.4, any Jordan $*$ -homomorphism of A onto B has properties (a), (b) and (c).

Suppose T has the properties (a), (b), and (c). Let π be the natural homomorphism of A onto $A/T^{-1}(0)$. Let T_0 be as in the prior theorem. Then $T = T_0\pi$, and $T' = \pi' T'_0$. Since T' is an isometry, so is T'_0 . As T_0 is a one-to-one mapping of $A/T^{-1}(0)$ onto B , T'_0 is an isometry of the conjugate space of B onto the conjugate space of $A/T^{-1}(0)$. Thus we conclude that T_0 is an isometry of $A/T^{-1}(0)$ onto B . Since $T_0(e_A + T^{-1}(0)) = e_B$, a result of Kadison [8, p. 330] shows that T_0 is a Jordan $*$ -isomorphism and thus that $T = T_0\pi$ is a Jordan $*$ -homomorphism.

In [8, p. 329] Kadison shows that if T is a Jordan $*$ -isomorphism of a B^* -algebra A onto a B^* -algebra B , then T is an isometry. The conventions of that paper require a unit for the algebra. However, a check of the argument involved in the proposition quoted above shows that the identity plays no role whatsoever. The next proposition is in the nature of a converse statement.

5.6. THEOREM. *Let A be a B^* -algebra, and B a Banach algebra with an involution. Let T be a Jordan $*$ -isomorphism of A onto B which is an isometry. Then B is a B^* -algebra.*

Let $x \in A$. Then $\|(Tx)^*\| = \|Tx^*\| = \|x^*\|$, so $\rho(Tx) = \rho(x)$ for all $x \in A$, where ρ designates the spectral radius in either algebra.

Suppose $Tx = y \in \text{Rad } B$, the radical of B . Then $y^* \in \text{Rad } B$ and consequently $\rho(y \pm y^*) = 0 = \rho(x \pm x^*)$. But $x + x^*$ and $x - x^*$ are self-adjoint and skew elements in a B^* -algebra, so $x \pm x^* = 0$. Consequently $x = 0 = y$, and thus B is semi-simple.

It then follows (see § 4) that the spectrum of $T(x)$ is the same as that of x . Consequently, if $u \in B$ and u is self-adjoint, then its spectrum is real and $\rho(u) = \|u\|$. A result of Yood [13, p. 148] now asserts that there is a bicontinuous $*$ -isomorphism σ of B onto a B^* -algebra B_1 . The mapping σT of A onto B_1 is a Jordan $*$ -isomorphism.

By the quoted result of Kadison, σT is an isometry. Since T is an isometry, so is σ and, therefore, B is a B^* -algebra.

We use the terminology *Jordan involution* for a conjugate linear mapping $x \rightarrow x^*$ of period two on a semi-simple complex Banach algebra A where $(x \cdot y)^* = x^* \cdot y^*$, $x, y \in A$. We write $H = \{x \in A \mid x = x^*\}$. Clearly $A = H \oplus iH$ and $\rho(x^*) = \rho(x)$, $x \in A$.

5.7. LEMMA. *Suppose that there is a real normed linear space norm $|x|$ on H such that $|x| \leq \rho(x)$, $x \in H$. Then the Jordan involution $x \rightarrow x^*$ is continuous.*

Let H' be the derived set of H . Note that $y^2 \in H$ if $y \in H$. The arguments of [11, Lemma 3.3] show $y^2 = 0$ if $y \in (iH) \cap H'$. Thus $\rho(y) = \rho(iy) = 0$ and $y = 0$ for such y . It follows [11, p. 157] that H is closed. If $x \in A$ and $x = u + iv$, $u, v \in H$ the norm $\|x\|_1 = \|u\| + \|v\|$ is then a complete linear space norm for A topologically equivalent to the given norm so that the Jordan involution is continuous.

We are now able to establish the continuity of a Jordan homomorphism of a complex Banach algebra A onto a B^* -algebra. There need be no involution on A .

5.8. THEOREM. *Let T be a Jordan homomorphism of a complex Banach algebra A onto a B^* -algebra B . Then T is continuous.*

Let P_α be the set of primitive ideals of B , and let π_α be the canonical homomorphism of B onto B/P_α . Then $\pi_\alpha T$ is a Jordan homomorphism of A onto a primitive algebra. By a result of Herstein [4, p. 340], $\pi_\alpha T$ is either a homomorphism or an anti-homomorphism. Since B/P_α is a B^* -algebra, in either case we see that $\pi_\alpha T$ is continuous [10, Theorem 4.1.20]. This shows that $(\pi_\alpha T)^{-1}(0)$ is a closed ideal in A . But

$$T^{-1}(0) = \bigcap_\alpha (\pi_\alpha T)^{-1}(0).$$

Therefore $T^{-1}(0)$ is a closed ideal in A . It is not difficult to show that $A/T^{-1}(0)$ is semi-simple.

Let π denote the natural homomorphism of A onto $A/T^{-1}(0)$ and let T_0 be the Jordan isomorphism $T_0(\pi(x)) = T(x)$ of $A/T^{-1}(0)$ onto B . By § 4 we see that $\rho[T(x)] = \rho[T_0(\pi(x))] = \rho[\pi(x)]$ since B is semi-simple.

The natural involution $x \rightarrow x^*$ on B induces a Jordan involution on $A/T^{-1}(0)$ by the rule $[T_0^{-1}(w)]^* = T_0^{-1}(w^*)$. Suppose that $T_0^{-1}(h)$ is self-adjoint under this Jordan involution. Then $T_0^{-1}(h) = [T_0^{-1}(h)]^* = T_0^{-1}(h^*)$ so that $h = h^*$ and h is self-adjoint in B . Thus $\rho(T_0^{-1}(h)) = \rho(h) = \|h\|$ for all h self-adjoint in B and $|T_0^{-1}(h)| = \|h\|$ serves as the auxiliary norm needed for Lemma 5.7. By that result, the Jordan involution

is continuous on $A/T^{-1}(0)$.

To show that T is continuous it is enough to show that T_0 is continuous. For the continuity of T_0 it suffices to show that $\|\pi(x_n)\| \rightarrow 0$ and $\|T_0(\pi(x_n)) - T_0(\pi(x_0))\| \rightarrow 0$ imply that $T_0(\pi(x_0)) = T(x_0) = 0$. We can write $\pi(x_n) = \pi(u_n) + i\pi(v_n)$, $n = 0, 1, 2, \dots$, where each $\pi(u_n)$ and $\pi(v_n)$ is self-adjoint in terms of the Jordan involution of $A/T^{-1}(0)$. Since that involution is continuous, $\|\pi(u_n)\| \rightarrow 0$ and $\|\pi(v_n)\| \rightarrow 0$. Each $T_0(\pi(u_n)) = T(u_n)$, and each $T(v_n)$, is self-adjoint in the B^* -algebra B . Thus $\|T(u_n) - T(u_0)\| \rightarrow 0$ and $\|T(v_n) - T(v_0)\| \rightarrow 0$. Also

$$\begin{aligned} \rho(T(u_0)) &= \|T(u_0)\| \leq \|T(u_n) - T(u_0)\| + \|T(u_n)\| \\ &= \|T(u_n) - T(u_0)\| + \rho(T(u_n)) \\ &= \|T(u_n) - T(u_0)\| + \rho(\pi(u_n)) \rightarrow 0. \end{aligned}$$

Therefore $T(u_0) = 0$ so that $\pi(u_0) = 0$. Likewise $\pi(v_0) = 0$ and thus $\pi(x_0) = 0$.

6. Lie ideals in Banach algebras. Throughout let B be a complex Banach algebra with center \mathfrak{Z} . Suppose that B is semi-simple. If $e^2 = e$ and $eB(Be)$ is a minimal right (left) ideal of B , we call e a *minimal idempotent* of B . By the Gelfand-Mazur Theorem, $eBe = Ke$, where K is the set of complex numbers. If B is primitive, it is known [10, p. 61] that either $\mathfrak{Z} = (0)$ or B has an identity u and $\mathfrak{Z} = Ku$.

Consider the Lie multiplication $[x, y] = xy - yx$ in B . In this topological algebraic setting we modify standard notation [7] as follows. Let $[S, T]$ be the closed linear span of the elements $[s, t]$ where $s \in S$, $t \in T$ and S, T are subsets of B . If \mathfrak{L} is a Lie ideal of B we define $D^k\mathfrak{L}$ inductively by $D^0\mathfrak{L} = \bar{\mathfrak{L}}$ and $D^{k+1}\mathfrak{L} = [D^k\mathfrak{L}, D^k\mathfrak{L}]$. A Lie ideal \mathfrak{L} is called *solvable* if for some positive integer k , $D^k\mathfrak{L} = (0)$. The *Lie radical* of B is defined to be the closed linear span of all the solvable Lie ideals of B .

As in [3] we define $S(\mathfrak{L}) = \{x \in B \mid [x, y] \in \mathfrak{L} \text{ for all } y \in B\}$.

6.1. THEOREM. *Let B be a primitive Banach algebra with socle $S \neq (0)$. Let e be a minimal idempotent and $\mathfrak{L} \neq (0)$ be a Lie ideal of B . Then*

- (a) $e\mathfrak{L}e = Ke$.
- (b) If $[\mathfrak{L}, \mathfrak{L}] = (0)$, then $\mathfrak{L} = \mathfrak{Z}$.
- (c) If $e \in \mathfrak{L}$, then $\mathfrak{L} \supset S$.
- (d) If $\mathfrak{L} \neq \mathfrak{Z}$, then \mathfrak{L} contains $[x, y]$ for all $x \in S$, $y \in B$.
- (a) Suppose that $e\mathfrak{L}e \neq Ke$. Then $e\mathfrak{L}e = (0)$. Let $a \in \mathfrak{L}$ and $x \in B$.

We have $e(ax - xa)e = 0$ or $eaxe = exae$. Replacing x by xe , we see that $eaxe = 0$. This gives $(eax)^2 = 0$ for all $x \in B$, so that $ea = 0$.

Then $e(ax - xa) = 0 = exa$. From this we get $(BeB)(BaB) = (0)$. As B is primitive it follows that $a = 0$. This makes $\mathfrak{L} = (0)$.

(b) For any $x \in \mathfrak{L}$, $y \in B$ we have $[x, y^2] = [x, y] \cdot y \in \mathfrak{L}$. Since $[[x, y], y] \in \mathfrak{L}$, it follows that $(xy - yx)y$ and $y(xy - yx) \in \mathfrak{L}$. As $[\mathfrak{L}, \mathfrak{L}] = (0)$, x permutes with these elements and with $xy - yx$. Therefore $(xy - yx)yx = (xy - yx)xy$ or $(xy - yx)^2 = 0$. Since $e(xe - ex)^2e = 0$, we obtain $ex^2e = (exe)^2$, $x \in \mathfrak{L}$. Replacing x by $xy - yx$ we see that $[e(xy - yx)e]^2 = 0$. Since $eBe = Ke$, we have $e[\mathfrak{L}, B]e = (0)$. But $[\mathfrak{L}, B]$ is a Lie ideal so that, by (a), $\mathfrak{L} \subset \mathfrak{Z}$. But \mathfrak{Z} is one-dimensional or (0) , whence $\mathfrak{L} = \mathfrak{Z}$.

(c) Given $e \in \mathfrak{L}$, we have $[e, [e, x]] \in \mathfrak{L}$ for all $x \in B$. Inasmuch as exe is a scalar multiple of e , we see that $e \cdot x \in \mathfrak{L}$. As $[e, x] \in \mathfrak{L}$, we get $ex \in \mathfrak{L}$ and $[y, ex] \in \mathfrak{L}$ for all $x, y \in B$. But $[y, ex] = yex - exy$ so that $BeB \subset \mathfrak{L}$. By [6, p. 75] we see that $\mathfrak{L} \supset S$.

(d) By [3, p. 282], $S(\mathfrak{L})$ is both a Lie ideal and a subalgebra of B . Consequently, by [3, p. 281], either $\mathfrak{L} \subset S(\mathfrak{L}) \subset \mathfrak{Z}$ (so that $\mathfrak{L} = \mathfrak{Z}$) or $S(\mathfrak{L})$ contains a nonzero ideal of B . In the latter case, $S(\mathfrak{L}) \supset S$ since B is primitive. This yields (d).

6.2. THEOREM. *Let B be a primitive Banach algebra with minimal one-sided ideals. Then the Lie radical of B coincides with \mathfrak{Z} .*

It is sufficient to show that any solvable Lie ideal \mathfrak{L} of B is contained in \mathfrak{Z} . Suppose $D^\circ \mathfrak{L} = \bar{\mathfrak{L}}$ and $D^{p+1} \mathfrak{L} = [D^p \mathfrak{L}, D^p \mathfrak{L}]$, $p = 0, 1, \dots$, where k is the smallest integer such that $D^k \mathfrak{L} = (0)$. We may suppose $k \geq 1$ and then $[D^{k-1} \mathfrak{L}, D^{k-1} \mathfrak{L}] = (0)$. By Theorem 6.1, we see that $D^{k-1} \mathfrak{L} \subset \mathfrak{Z}$. If $k = 1$ we are through. Otherwise $[D^{k-2} \mathfrak{L}, D^{k-2} \mathfrak{L}] \subset \mathfrak{Z}$. If $\mathfrak{Z} \neq (0)$ then $\mathfrak{Z} = Ku$ where u is the identity of B . But for $x, y \in D^{k-2} \mathfrak{L}$, $xy - yx = \alpha u$, $\alpha \neq 0$ is impossible. For if this relation persists then $sp(xy) = \alpha + sp(yx)$ whereas $sp(xy)$ and $sp(yx)$ agree up to the single value $\{0\}$ and are compact sets. Therefore $[D^{k-2} \mathfrak{L}, D^{k-2} \mathfrak{L}] = D^{k-1} \mathfrak{L} = (0)$ and $D^{k-2} \mathfrak{L} \subset \mathfrak{Z}$. This argument, repeated a finite number of times, leads to $\mathfrak{L} \subset \mathfrak{Z}$.

6.3. COROLLARY. *Let B be a semi-simple Banach algebra with socle $S \neq (0)$ where S is contained in no primitive ideal of B . Then the Lie radical of B is contained in \mathfrak{Z} .*

Let \mathfrak{L} be a solvable Lie ideal of B , P be a primitive ideal and π be the natural homomorphism of B onto B/P . By hypothesis there exists a minimal idempotent e of B , $e \notin P$. Now $eBe = Ke$ so that $\pi(e)\pi(B)\pi(e) = K\pi(e)$ from which it follows that $\pi(e)$ lies in the socle

of B/P . Now $\pi(\mathfrak{L})$ is a solvable Lie ideal of B/P so that, by Theorem 6.2, either $\pi(\mathfrak{L}) = (0)$ or $\pi(\mathfrak{L}) = K\pi(u)$ where $\pi(u)$ is an identity for B/P . Hence, for $x \in \mathfrak{L}$, either $x \in P$ or we can write $x = \alpha u + y$, $\alpha \in K, y \in P$. In either case we get $[x, w] \in P$ for all $w \in B$. Therefore $[\mathfrak{L}, B]$ is in every primitive ideal or $\mathfrak{L} \subset \mathfrak{Z}$ by semi-simplicity.

There is an additional fact relating to primitive ideals which will be useful in the sequel.

6.4. LEMMA. *Let P be a primitive ideal of a Banach algebra B . Then either $S(P) = P$ or $S(P) = Ku + P$ where u is an identity for B modulo P .*

Let π be the natural homomorphism of B onto B/P . Let $x \in S(P)$ and $y \in B$. Then clearly $\pi(x)\pi(y) = \pi(y)\pi(x)$ or $\pi(x)$ lies in the center \mathfrak{Z}_1 of B/P . Now B/P is primitive so either $\mathfrak{Z}_1 = (0)$ or B/P has an identity $\pi(u)$ and $\mathfrak{Z}_1 = K\pi(u)$. If $\mathfrak{Z}_1 = (0)$ clearly $S(P) = P$. If $\mathfrak{Z}_1 = K\pi(u)$ then $P \subset S(P) \subset Ku + P$ so that either $P = S(P)$ or $Ku + P = S(P)$.

7. Annihilation and Lie annihilation. In Theorem 3.2 and Corollary 3.3 we obtained structure theorems for a class of algebras where, for each proper closed ideal $I, \mathfrak{L}(I) \neq (0)$. In this section we consider the consequences of a weaker hypothesis on the proper closed ideals for Banach algebras.

Before we investigate the weakened hypothesis, let us note the implication of the statement $\mathfrak{L}(I) = (0)$. In Lemma 2.2, we saw that if I is an ideal in a ring A , then $[I^\perp, A] \subset \mathfrak{L}(I) \cap \mathfrak{R}(I) \subset I^\perp$. Thus if $\mathfrak{L}(I) = (0)$, $I^\perp = \mathfrak{Z}$, the center of A . Suppose that A is semi-prime (so that $\mathfrak{L}(I) = \mathfrak{R}(I)$) and that \mathfrak{Z} contains no nonzero ideal of A . Then if $I^\perp = \mathfrak{Z}$, $\mathfrak{L}(I) = (0)$. Hence for many of the algebras under consideration in this paper, $I^\perp = \mathfrak{Z}$ if and only if $\mathfrak{L}(I) = (0)$.

Throughout this section, B denotes a semi-simple Banach algebra, with center \mathfrak{Z} , satisfying the condition:

(7.1) *If $\mathfrak{L}(I) = (0)$, for an ideal I , then $I + \mathfrak{Z}$ is dense in B .*

Since $\mathfrak{L}(I + \mathfrak{R}(I)) = (0)$, an immediate consequence of (7.1) is that for every ideal $I, I + \mathfrak{R}(I) + \mathfrak{Z}$ is dense in B . Also, since $I + \mathfrak{Z} \subset S(I)$, it is immediate that $\mathfrak{R}(I) + S(I)$ is dense in B .

7.1. LEMMA. *Let I be a closed ideal of B . Then $\mathfrak{LR}(I) \subset S(I)$.*

Let $x \in \mathfrak{LR}(I), y \in B$. We can write $y = \lim (u_n + v_n + z_n)$ where $u_n \in I, v_n \in \mathfrak{R}(I)$ and $z_n \in \mathfrak{Z}$. Then $xy - yx = \lim (xu_n - u_nx) \in I$.

Therefore $x \in S(I)$.

7.2. LEMMA. *A primitive ideal P for which $\mathfrak{L}(P) \neq (0)$ is a ideal. Also $\mathfrak{L}(P)$ is a minimal dual ideal.*

The first statement is shown by the proof of Theorem 3.2. Let $I \neq (0)$, $I \subset \mathfrak{L}(P)$, where I is dual. Since $I \not\subset P$ and P is a primitive ideal we see that $\mathfrak{L}(I) \subset P$. Thus $I = \mathfrak{R}\mathfrak{L}(I) \supset \mathfrak{L}(P)$ which forces $I = \mathfrak{L}(P)$.

7.3. LEMMA. *Let P be a primitive ideal of B for which $\mathfrak{L}(P) = (0)$. Then P is a modular maximal ideal of B of deficiency one, and there is an identity for B modulo P which is central.*

Proof. By (7.1), $P + \mathfrak{J}$ is dense. Let π be the natural homomorphism of B onto B/P . Then $\pi(P + \mathfrak{J}) = \pi(\mathfrak{J})$ is dense in B/P . Therefore B/P is a commutative primitive Banach algebra, i.e., a field. Thus there exists $u \in \mathfrak{J}$ such that $B = P + Ku$. Clearly u can be selected to be an identity for B modulo P .

7.4. THEOREM. *Let B be a semi-simple Banach algebra satisfying (7.1). Let I be the smallest dual ideal of B which contains all the minimal dual ideals of B , then B/I is commutative.*

Let J be the algebraic sum of all the ideals $\mathfrak{R}(P) \neq (0)$ for P primitive. By Lemma 7.2, $I \supset J$. Also clearly $\mathfrak{R}(J) = \bigcap P$, for the primitive ideals with $\mathfrak{R}(P) \neq (0)$.

Let $W = \bigcap M$, where M is a maximal ideal of deficiency one with a relative identity in \mathfrak{J} . For any maximal modular ideal of the stated form $B = M + Ku$, with $u \in \mathfrak{J}$, so $[B, B] \subset M$. Thus $[B, B] \subset W$. By (7.1), Lemma 7.3, and semi-simplicity, $W \cap \mathfrak{R}(J) = (0)$, so $W \subset \mathfrak{L}\mathfrak{R}(J) \subset I$. Thus $[B, B] \subset I$, and consequently B/I is commutative.

We next consider a hypothesis on the Lie ideals which will enable us to obtain results on the Lie structure of certain Banach algebras.

(7.2) *If \mathfrak{L} is a closed Lie ideal of B and $\mathfrak{L} \supset \mathfrak{L}^2$, then $\mathfrak{L} = B$.*

7.5. THEOREM. *Let B be a semi-simple Banach algebra which satisfies (7.2). Then B satisfies the conclusion of Theorem 7.4.*

We show that if (7.2) is satisfied so is (7.1). In view of Lemma 2.2, if $\mathfrak{L}(I) = 0$ for an ideal I then $[I^2, B] = (0)$ or $I^2 \subset \mathfrak{J}$. Thus if $\mathfrak{S} = I + \mathfrak{J}$, \mathfrak{S} is a Lie ideal and $\mathfrak{S} \supset \mathfrak{J} \supset I^2 = \mathfrak{S}^2$. Consequently (7.2) shows that (7.1) is satisfied.

If the semi-simple Banach algebra B satisfies (7.2) and also

(7.3) B has zero center

then (7.1) is satisfied with $\mathfrak{J} = (0)$, so we are with in the framework of § 3, and in particular Corollary 3.3 holds. Thus a structure holds in terms of minimal associative objects. We next turn our attention to obtaining a structure theorem under these hypotheses in terms of certain Lie ideals.

Any H^* -algebra [10, pp. 272–276] B with zero center satisfies (7.2) and (7.3). Let \mathfrak{L} be a closed Lie ideal of B , and suppose that $\mathfrak{L} \supset \mathfrak{L}^\perp$. Let I be any minimal closed ideal of B . Direct calculation with the “matrix” representation of I , shows that the only closed Lie ideals in I are (0) and I . Hence $[\mathfrak{L}, I] = (0)$ or $[\mathfrak{L}, I] = I$. In the first instance $I \subset \mathfrak{L}^\perp \subset \mathfrak{L}$, and in the second $I \subset \mathfrak{L}$ directly. Thus for all minimal closed ideals of B , $\mathfrak{L} \supset I$, so $\mathfrak{L} = B$ and (7.2) is satisfied.

Until otherwise is stated we assume B is a semi-simple Banach algebra satisfying (7.2) and (7.3).

7.6. LEMMA. B is Lie semi-simple.

It is sufficient to show that $[\mathfrak{L}, \mathfrak{L}] = (0)$ for a Lie ideal \mathfrak{L} of B implies $\mathfrak{L} = (0)$. Now $[\mathfrak{L}, \mathfrak{L}] = 0$ implies $\mathfrak{L} \subset \mathfrak{L}^\perp$, and thus that $\mathfrak{L}^{\perp\perp} \subset \mathfrak{L}^\perp$. Condition (7.2) then shows that $\mathfrak{L}^\perp = B$, whence $\mathfrak{L} \subset \mathfrak{J} = (0)$ by (7.3).

The Lie semi-simplicity will frequently be used in the form that $\mathfrak{L} \cap \mathfrak{L}^\perp = (0)$ for any Lie ideal \mathfrak{L} of B .

7.7. COROLLARY. B contains no modular two-sided ideals.

If M were a maximal modular ideal of B , then (7.1) implies $B = M \oplus \mathfrak{L}(M)$. Let e be an identity for B modulo M , with $e \in \mathfrak{L}(M)$. Clearly $e \in (\mathfrak{L}(M))^\perp \cap \mathfrak{L}(M) = (0)$ by the prior lemma. Thus we have a contradiction and there can be no modular ideals in B .

7.8. LEMMA. If \mathfrak{L} is a Lie ideal of B , then $S(\mathfrak{L}^\perp) = \mathfrak{L}^\perp$ and $S(\mathfrak{L}^-) = \mathfrak{L}^{\perp\perp}$.

Clearly $\mathfrak{L}^\perp \subset S(\mathfrak{L}^\perp)$. Let $x \in S(\mathfrak{L}^\perp)$ and $y \in \mathfrak{L}$. Then $[x, y] \in \mathfrak{L}^\perp \cap \mathfrak{L} = (0)$, so $S(\mathfrak{L}^\perp) \subset \mathfrak{L}^\perp$.

Since $\mathfrak{L} \subset \mathfrak{L}^{\perp\perp}$ and the latter is closed, $S(\mathfrak{L}^-) \subset S(\mathfrak{L}^{\perp\perp}) = \mathfrak{L}^{\perp\perp}$ by the above. Now (7.2) implies that $\mathfrak{L} + \mathfrak{L}^\perp$ is dense in B . Let $x \in \mathfrak{L}^{\perp\perp}$. Then if $y \in B$, $y = \lim (y_j + z_j)$ with $y_j \in \mathfrak{L}$ and $z_j \in \mathfrak{L}^\perp$. Consequently, $[x, y] = \lim [x, y_j + z_j] = \lim [x, y_j] \in \mathfrak{L}^-$, and so $\mathfrak{L}^{\perp\perp} \subset S(\mathfrak{L}^-)$.

7.9. COROLLARY. *If P is a primitive ideal of B , then $p = p^{LL}$.*

It follows from Lemma 6.4 that $S(P) = P$ unless P is a modular ideal. However, by Corollary 7.7 there are no modular ideals, so Lemma 7.8 yields the desired result.

7.10. LEMMA. *If I is a closed ideal in B , then $(\mathfrak{R}(I))^L = S(I)$.*

Since $\mathfrak{R}(I) \subset I^L$, it follows from Lemma 7.8 that $S(I) = I^{LL} \subset (\mathfrak{R}(I))^L$. Let $x \in (\mathfrak{R}(I))^L$ and $y \in B$. Since the ideal $I + \mathfrak{R}(I)$ is dense in B , $y = \lim (p_n + q_n)$, with $p_n \in I$ and $q_n \in \mathfrak{R}(I)$. Then

$$[x, y] = \lim [x, p_n + q_n] \lim [x, p_n] \in I,$$

and consequently $(\mathfrak{R}(I))^L \subset S(I)$.

7.11. LEMMA. *Let P be a primitive ideal of B . Then $[\mathfrak{R}(P), \mathfrak{R}(P)]$ is contained in every nonzero closed Lie ideal in $\mathfrak{R}(P)$.*

Let $\mathfrak{S} \neq (0)$ be a closed Lie ideal of B with $\mathfrak{S} \subset \mathfrak{R}(P)$. By the use of two results of Herstein, we see that $S(\mathfrak{S})$ is a ring as well as a Lie ideal [3, p. 282] and therefore [3, p. 281] that either $S(\mathfrak{S})$ is in the center of B or contains a nonzero ideal I of B . The first possibility cannot pertain here since (7.3) holds. Thus we have a nonzero ideal $I \subset S(\mathfrak{S}) = \mathfrak{S}^{LL} \subset (\mathfrak{R}(P))^{LL} = P^L$ by use of Lemmas 7.8, 7.10 and Corollary 7.9. This implies that $\mathfrak{S}(\mathfrak{R}(I)) \cdot \mathfrak{R}(P) \neq 0$, for otherwise $I \subset \mathfrak{S}(\mathfrak{R}(I)) \subset P = \mathfrak{S}(\mathfrak{R}(P))$, and thus $I \subset P \cap P^L = (0)$, which is a contradiction. Consequently $\mathfrak{S}(\mathfrak{R}(I)) \cap \mathfrak{R}(P)$ is a dual ideal $\neq (0)$ contained in the minimal dual ideal $\mathfrak{R}(P)$. Therefore $\mathfrak{R}(P) \subset \mathfrak{S}(\mathfrak{R}(I))$ and $\mathfrak{R}(I) \subset P$. Also $S(I) = I^{LL} \subset \mathfrak{S}^{LLLL} = \mathfrak{S}^{LL} = S(\mathfrak{S}) \subset P^L$. Another application of Lemma 7.10 yields $P^L \subset (\mathfrak{R}(I))^L = S(I)$. However, from $I \supset P^L$, we deduce $S(I) = I^{LL} \subset P^L$, so $P^L = S(I) = S(\mathfrak{S})$. Let D be the largest ideal contained in P^L . The Theorem 1 in [14, p. 156] together with its proof shows that $\mathfrak{S} \supset [D, B]$. Since $D \supset \mathfrak{R}(P)$, $\mathfrak{S} \supset [\mathfrak{R}(P), \mathfrak{R}(P)]$.

7.12. LEMMA. *For any primitive ideal P of B , $[\mathfrak{R}(P), \mathfrak{R}(P)]^L = P$.*

The inclusion $P \subset [\mathfrak{R}(P), \mathfrak{R}(P)]^L$ is immediate. Let $x \in P^L = (\mathfrak{R}(P))^{LL}$ by Lemmas 7.9 and 7.10. For $y \in B$, $y = \lim (p_n + r_n)$ with $p_n \in P$ and $r_n \in \mathfrak{R}(P)$, and $x = \lim (s_m + t_m)$ with $s_m \in P$ and $t_m \in \mathfrak{R}(P)$. Thus

$$\begin{aligned} [x, y] &= \lim_n [x, p_n + r_n] = \lim_n [x, r_n] = \lim_n \lim_m [s_m + t_m, r_n] \\ &= \lim_n \lim_m [t_m, r_n] \in [\mathfrak{R}(P), \mathfrak{R}(P)]. \end{aligned}$$

Consequently $P^L \subset S([P], \mathfrak{R}(P)) = [\mathfrak{R}(P), \mathfrak{R}(P)]^L$, and so $[\mathfrak{R}(P), \mathfrak{R}(P)]^L \subset P^{LL} = P$. We thus have the asserted equality.

7.13. LEMMA. *Each primitive ideal P of B is a maximal proper closed Lie ideal of B .*

Let \mathfrak{S} be a proper closed Lie ideal of B such that $\mathfrak{S} \supset P$. The hypothesis (7.2) implies that $\mathfrak{S}^L \neq (0)$. Corollary 3.3 and its proof imply that each minimal dual ideal of B is of the form $\mathfrak{R}(Q)$ for Q a primitive ideal of B . If $\mathfrak{S}^L \cap \mathfrak{R}(Q) = (0)$ for all primitive ideals Q , then $[\mathfrak{S}^L, \mathfrak{R}(Q)] \subset \mathfrak{S}^L \cap \mathfrak{R}(Q) = (0)$, so by Corollary 3.3, $[\mathfrak{S}^L, B] = (0)$ and $\mathfrak{S}^L \subset Z = (0)$ which is a contradiction. Thus there exists a primitive ideal Q_0 such that $\mathfrak{S}^L \cap \mathfrak{R}(Q_0) \neq (0)$. Lemma 7.11 then yields $[\mathfrak{R}(Q_0), \mathfrak{R}(Q_0)] \subset \mathfrak{S}^L$. Lemma 7.12 then implies that $\mathfrak{S} \subset \mathfrak{S}^{LL} \subset [\mathfrak{R}(Q_0), \mathfrak{R}(Q_0)]^L = Q_0$. As noted in the proof of Theorem 3.2, the resulting inequality $Q_0 \subset P$ with Q_0 and P primitive ideals implies that $Q_0 = P$ and consequently $\mathfrak{S} = P$.

7.14. LEMMA. *Each minimal closed Lie ideal of B is of the form $[\mathfrak{R}(Q), \mathfrak{R}(Q)]$ for some primitive ideal Q of B .*

Let \mathfrak{D} be a minimal closed Lie ideal of B . As noted in the proof of Lemma 7.13, $\mathfrak{D} \cap \mathfrak{R}(Q) \neq (0)$ for some primitive ideal Q of B ; and so $\mathfrak{D} \subset \mathfrak{R}(Q)$. Thus by Lemma 7.11, $[\mathfrak{R}(Q), \mathfrak{R}(Q)] = \mathfrak{D}$ by the minimal nature of \mathfrak{D} .

7.15. THEOREM. *If B is a semi-simple Banach algebra which satisfies (7.2) and (7.3), then B is the direct topological sum of its minimal closed Lie ideals.*

In view of Lemma 7.14, and the fact that any two ideals $\mathfrak{R}(Q)$, with Q primitive, have only zero in common, a sum of minimal closed Lie ideals is a direct sum. Let \mathfrak{S} be the direct topological sum of the minimal closed Lie ideal in B . If $\mathfrak{S} \neq B$, then (7.2) implies $\mathfrak{S}^L \neq (0)$. The argument used in the proof of Lemmas 7.13 and 7.14 shows that there is a primitive ideal Q such that $\mathfrak{S}^L \cap \mathfrak{R}(Q) \neq (0)$. Lemma 7.11 then implies that $[\mathfrak{R}(Q), \mathfrak{R}(Q)] \subset \mathfrak{S}^L$. But then $[\mathfrak{R}(Q), \mathfrak{R}(Q)] \subset \mathfrak{S}^L \cap \mathfrak{S} = (0)$, which is a contradiction.

In the final portion of this section, B will denote a simple Banach algebra with an identity u , which satisfied (7.2) and

(7.4) *The center \mathfrak{Z} of B consists of the complex multiples of the identity u of B .*

In view of the proof of Theorem 7.5, we note that (7.1) is automatically satisfied by B . Thus if $\mathfrak{L}(I) = (0)$, for an ideal I , either I is dense in B or I is a maximal ideal of deficiency one in B .

7.16. LEMMA. *If $\mathfrak{R}(M) \neq (0)$ for each maximal ideal M of B , then each primitive ideal of B is a maximal ideal. Furthermore, B is the direct topological sum of its minimal dual ideals.*

Since B has an identity, each primitive ideal P is contained in a maximal ideal M . If $M \neq P$, then by the argument used in the proof of Theorem 3.2, $\mathfrak{R}(M) = (0)$, which is a contradiction. Thus each primitive ideal is a maximal ideal. If B is a primitive algebra, then B is a simple algebra and thus is the only minimal dual ideal. Otherwise let J be the direct topological sum of the minimal dual ideals $\mathfrak{R}(P)$, where P is a primitive ideal. If J were proper, there would exist a maximal ideal N such that $N \supset J \supset \mathfrak{R}(N) \neq (0)$, which would contradict the semi-simplicity of B . Thus $J = B$.

7.17. LEMMA. *If M is a maximal ideal of B with $\mathfrak{R}(M) = (0)$, then $B = M \oplus Ku$, where K is the complex field. Moreover, M as an algebra satisfies condition (7.2) and (7.3).*

By Lemma 7.3 M has deficiency one and thus $B = M \oplus Ku$. Since M is a maximal ideal of B , M is a semi-simple Banach algebra and the center of M is contained in the center of B , so is zero. If \mathfrak{u} is a proper closed Lie ideal of M , and \mathfrak{u} contains $V = \{x \in M : [x, y] = 0, \forall y \in \mathfrak{u}\}$, then $\mathfrak{u} + Ku$ is a proper closed Lie ideal of B and $(\mathfrak{u} + Ku)^\perp = \mathfrak{u}^\perp = V + Ku$, so $(\mathfrak{u} + Ku) \supset (\mathfrak{u} + Ku)^\perp$. The hypothesis (7.2) yields $\mathfrak{u} + Ku = B$, which is a contradiction. Thus M as an algebra satisfies (7.2).

7.18. THEOREM. *Let B be a simple Banach algebra with satisfies (7.2) and (7.4). The only nonzero closed Lie ideals of B are either B and Ku or are B , Ku and $[B, B]$, in which case $B = [B, B] \oplus Ku$, and B is the direct sum of the minimal closed Lie ideals of B .*

Since B is simple, by a theorem of Herstein [3, p. 282], any closed Lie ideal is either contained in Ku or else contains $[B, B] = \mathfrak{L}$.

Suppose first that $Ku \subset \mathfrak{L}$. From Herstein's theorem applied to \mathfrak{L}^\perp , either (a) $\mathfrak{L}^\perp \subset Ku \subset \mathfrak{L}$ or (b) $\mathfrak{L} \subset \mathfrak{L}^\perp$. In case (a), the relation (7.2) yields $[B, B] = B$, so that the only closed Lie ideals are Ku and B . In case (b) $\mathfrak{L}^\perp \supset \mathfrak{L}^{\perp\perp}$ so that, by (7.2), $\mathfrak{L}^\perp = B$. Thus $\mathfrak{L} \subset \mathfrak{L}$. In view of the reasoning for Theorem 6.2, $[B, B] = (0)$. Thus B is commutative and

so $B = Ku$. Hence the case $Ku \subset \mathfrak{L}$ leads to the first possibility stated in the theorem.

If $Ku \not\subset [B, B]$, let $\mathfrak{X} = [B, B] \oplus Ku$. Suppose $\mathfrak{X} \neq B$. Another application of Herstein's theorem yields $\mathfrak{X}^\perp \supset [B, B]$ or $\mathfrak{X}^\perp \subset Ku \subset \mathfrak{X}$. The latter is in contradiction to (7.2), while the former implies $\mathfrak{X}^\perp \supset \mathfrak{X}$ so $\mathfrak{X}^{\perp\perp} \subset \mathfrak{X}^\perp$. Thus $\mathfrak{X}^\perp = B$, which as above shows that B is the complex field. Hence $B = [B, B] + Ku$, in which case B is the direct sum of the minimal closed Lie ideals of B .

7.19. THEOREM. *Let B be a semi-simple Banach algebra which satisfies (7.2) and (7.4). Suppose that B is not simple, and is not commutative. Then B is the direct topological sum of its minimal closed Lie ideals and is either the direct topological sum of its minimal dual ideals, or the minimal dual ideals and the center.*

Suppose that $\mathfrak{R}(M) \neq (0)$ for each maximal ideal M of B . Then by Lemma 7.16 each primitive ideal of B is a maximal ideal and B is the direct topological sum of its minimal dual ideals. Also, as noted in the proof of Lemma 7.16, each minimal dual ideal has the form $\mathfrak{R}(M)$ for a maximal ideal.

Let M be any maximal ideal of B , and let $\mathfrak{S} \neq (0)$ be a closed Lie ideal, $\mathfrak{S} \subset \mathfrak{R}(M)$. Then, by a result of Herstein [3, p. 281], either $S(\mathfrak{S})$ is in the center of B or there is a nonzero ideal I in $S(\mathfrak{S})$, in which case we may assume I is the largest such ideal. The first possibility cannot occur since it would imply $\mathfrak{S} \subset Ku$, while $\mathfrak{S} \neq (0)$, and $\mathfrak{S} \subset \mathfrak{R}(M)$. Thus the second possibility holds. If $a \in I \cap M$, then for all $x \in B$, $[a, x] \in \mathfrak{S} \subset \mathfrak{R}(M)$, while $[a, x] \in M$, so $[a, x] = 0$ for all $x \in B$. Thus a is central and since $u \notin M$, we conclude $a = 0$. Consequently $I \cap M = (0)$ and $B = M \oplus I$ so that $I = \mathfrak{R}(M)$. By a result of Zuev [14],

$$\mathfrak{S} \supset [I, B] \supset [\mathfrak{R}(M), \mathfrak{R}(M)].$$

Consequently each Lie ideal $[R(M), R(M)]$ is a minimal closed Lie ideal of B . Let \mathfrak{U} be the direct topological sum of its minimal closed Lie ideals of B . Since B is the direct topological sum of the minimal dual ideals and these have the form $\mathfrak{R}(M)$, we see that $\mathfrak{U} \supset [B, B] + Ku$. Another application of the result of Herstein used above shows that either $S(\mathfrak{U}^\perp) \subset Ku$ or there exists a nonzero ideal N such that $N \subset S(\mathfrak{U}^\perp)$. The first possibility implies that $\mathfrak{U}^\perp = Ku$, whence $\mathfrak{U} \supset \mathfrak{U}^\perp$ so by (7.2) $\mathfrak{U} = B$. In the second instance, let $a \neq 0, a \in N$. Then for all $x \in B$, $[a, x] \in \mathfrak{U}^\perp \cap \mathfrak{U} = \mathfrak{B}$, from which it is immediate that $\mathfrak{B}^{\perp\perp} \subset \mathfrak{B}^\perp$. Thus by (7.2) $\mathfrak{B}^\perp = B$, so $\mathfrak{B} \subset Ku$, and by the argument used in Theorem 6.2 we deduce that $a \in Ku$, which implies that $N = S(\mathfrak{U}^\perp) = B$. We therefore conclude that $[B, B] \subset \mathfrak{U} \cap \mathfrak{U}^\perp \subset Ku$, so as argued earlier B

is commutative, which is a contradiction.

In the event that $\mathfrak{R}(M) = (0)$ for some maximal ideal M of B , it follows from Lemma 7.17 that $B = M \oplus Ku$ where K is the complex field, and M as an algebra satisfies (7.2) and (7.3). Then by Theorem 7.15 M is the direct topological sum of its minimal closed Lie ideals and by Corollary 3.3 is the direct topological sum of the minimal dual ideals of M_0 . Any minimal closed Lie ideal of M is one of B , and any minimal dual ideal is one of B . Since Ku is a minimal closed Lie ideal of B , the theorem follows.

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