DEDEKIND DOMAINS: OVERRINGS AND SEMI-PRIME ELEMENTS

LUTHER ELIC CLABORN
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This paper develops two themes: (1) the relation of the class group of a Dedekind domain $A$ to that of an overring $B$ and (2) the question of finding a nonzero, nonunit element $x$ of a Dedekind domain $A$ such that $A/xA$ is regular. We obtain complete results in answer to the first question, giving a corollary concerning the realization of certain groups as class groups. We give various sufficient conditions in answer to the second question; some in terms of the class group, others concerning Dedekind domains which often arise in practice.

In § 1 of the present paper, we study the class group of an overring $B$ of a Dedekind domain $A$ and determine its class group in terms of that of $A$. We generalize and also strengthen the results of § 1 of an earlier article [1]. Combining several results, we obtain an interesting fact: if $G$ is the class group of a Dedekind domain and $G'$ is a homorphic image of $G$, then $G'$ is the class group of a suitable Dedekind domain.

Section 2 introduces the question of finding a nonunit $x$ in a Dedekind domain $A$ for which $A/xA$ is a direct sum of fields. Although we obtain no definitive result, various sufficient conditions are given. These require in part the developments of § 1. We also give examples Dedekind domains with “pathological” class groups.

1. We state two well known propositions which we will need by way of background.

PROPOSITION 1.1. Let $A$ be a Dedekind Domain with quotient field $F$. Let $B$ be a ring such that $A \subseteq B \subseteq F$. Then $B = \cap A_p$ over those prime ideals $P$ of $A$ for which $B \subseteq A_p$.

PROPOSITION 1.2. Let $A$ be a Dedekind domain with quotient field $F$. Let $B$ be a ring such that $A \subseteq B \subseteq F$. Then $B$ is a Dedekind domain.

PROPOSITION 1.3. Let $A$ be a Dedekind domain with quotient field $F$ and let $B$ be a ring such that $A \subseteq B \subseteq F$. The assignment $I \to IB$ is a homomorphism of the set of fractionary ideals of $A$ onto the set of fractionary ideals of $B$.
Proof. Let $Q$ be a prime ideal of $B$ and set $P = Q \cap A$. Then $PB = Q$. The inclusion $PB \subseteq Q$ is trivial, while $B_q = A_p$ implies that $PB_q = (PB)B_q = QB_q$. This yields $PB = Q$ if we know that $PB$ is not contained in another prime ideal $Q'$ of $B$. But then $Q'$ would also lie over $P$, which is not the case by Prop. 1.1.

If $I$ is a fractionary ideal of $A$, then there is a $d \neq 0$ in $A$ such that $dI \subseteq A$. But then clearly $d(IB) \subseteq B$, so $IB$ is a fractionary ideal of $B$. The mapping is clearly a homomorphism for multiplication. To see that the mapping is onto, let $Q$ be a prime ideal of $B$. We have seen above that if $P = Q \cap A$, then $PB = Q$. Thus the mapping is onto the prime ideals of $B$, and these generate the group of fractionary ideals of $B$.

**Corollary 1.4.** Let $A$ be a Dedekind domain with quotient field $F$ and let $B$ be a ring such that $A \subseteq B \subseteq F$. The assignment $I \mapsto IB$ of fractionary ideals of $A$ onto fractionary ideals of $B$ induces a homomorphism $\varphi : I \mapsto IB$ of the class group of $A$ onto that of $B$.

**Proof.** It is sufficient to note that if $I = xA$, then $IB = xB$.

**Proposition 1.5.** The kernel of $\varphi$ is generated by all $\bar{P}_a$, where $P_a$ ranges over all prime ideals such that $P_aB = B$.

**Proof.** Suppose $P_aB = B$, and let $I$ be a fractionary ideal such that $\bar{I} = \bar{P}_a$, i.e. $I = xP_a$ for $x \in F$. Then $IB = xP_aB = xB$, so $\bar{IB}$ is the identity.

Suppose now that $I$ is a fractionary ideal of $A$ such that $IB = yB$ for $y \in F$. Then $yI^{-1}B = B$, showing that $yI^{-1}$ is a product of primes $P_a$ of $A$ for which $P_aB = B$, and this establishes the assertion.

**Corollary 1.6.** Let $A$ be a Dedekind domain and $W = \{P_a\}$ be a collection of primes such that $\{\bar{P}_a\}$ does not generate the full class group of $A$. Then there are an infinite number of prime ideals of $A$ not in the set $\{P_a\}$.

**Proof.** Let $B = \bigcap_{P \in W} A_p$. By Proposition 1.5, $B$ is not a principal ideal domain. Therefore there are an infinite number of prime ideals of $B$, hence an infinite number of prime ideals of $A$ which are not in $W$.

**Corollary 1.7.** Let $A$ be a Dedekind domain with class group
Let $H$ be any subgroup of $G$. Then there is a Dedekind domain whose class group is $G/H$.

Proof. In [1], we constructed the Dedekind domain $A' = A[X]_S$, where $S$ denotes the set of all monic polynomials of $A[X]$. We showed that $A$ has the same class group as $A$ [1, Prop. 2.3] and also that $A$ has a prime ideal in every class of the class group [1, Cor. 2-5]. Identify $G$ and $H$ at the class group and a subgroup of the class group of $A'$. For each class of $H$, choose a prime $P'$ of $A'$ in the given class. Let $W$ denote the set \{P'\} so chosen. Then $B = \bigcap_{q \in W} A_q$ has class group $G/H$ by Proposition 1.5.

2. Definition 2.1. Let $A$ be a Dedekind domain. An element $x$ of $A$ which is not zero and not a unit will be said to be semi-prime if $A/xA$ is a regular ring.

Remark 2.2. This condition is equivalent to (1) $A/xA$ is a direct sum of fields, or (2) $xA$ is not contained in the square of any prime ideal of $A$.

In what follows, sufficient conditions will be given for $A$ to contain semi-prime elements. If $A$ has only a finite number of prime ideals, then $A$ is a principal domain and obviously $A$ contains semi-prime elements. This case ($A$ has only a finite number of prime ideals) will be excluded from the developments which follow.

Proposition 2.3. If $A$ has a finite class group, then there are semi-prime elements in $A$.

Proof. Since we are assuming that $A$ has infinitely many prime ideals, there must be at least one class of the class group containing an infinite set $\{P_i\}$ of the prime ideals. If $n$ is the class number of $A$, then $P_1 \cdots P_n$ must be principal, say $xA = P_1 \cdots P_n$. $x$ is then a semi-prime element.

Proposition 2.4. Let $A$ be a Dedekind domain, and suppose that every class of the class group (except possibly the principal class) contains a prime ideal. Then $A$ contains a semi-prime element.

Proof. If $A$ is a principal ideal domain, then there is nothing to prove. Otherwise let $P$ be a nonprincipal prime ideal and let $Q$ be a prime in the class of $P^{-1}$. Then $PQ$ is principal, say $PQ = xA$, and $x$ will be semi-prime unless $P = Q$. We are therefore done unless every class has exponent 2 and there is only one prime in each class.
Choose $P$ to represent one nonprincipal class and $Q$ to represent a different nonprincipal class. Choose a prime ideal $R$ in the class of $PQ$. Obviously $R \neq P$, $R \neq Q$, while $PQR$ is principal. This gives a semi-prime element in $A$.

We can actually prove a little more.

**Proposition 2.5.** Let $A$ be a Dedekind domain, and suppose that for every prime ideal $P$ there is a prime of $A$ in the class of $P^{-1}$. Then $A$ contains a semi-prime element.

**Proof.** As in the proof Proposition 2.4 we may assume that every class has exponent 2. The class group of $A$ may therefore be regarded as a vector space over the field with 2 elements. Since the prime ideals of $A$ generate the class group, we may choose a basis $\{\bar{P}_a\}$ for the class group consisting of classes of prime ideals. Let $P$ be any prime ideal of $A$ and let $P$ be its class. Let $\bar{P} = \bar{P}_a_1 \cdots \bar{P}_a_k$ be its representation in terms of the given basis. Thus $PP_{a_1} \cdots P_{a_k}$ is principal and we get a semi-prime element unless $P$ is in the set $\{P_a\}$. We may assume then that the set $\{P_a\}$ contains all prime ideals of $A$. But this contradicts Corollary 1.6, and the proposition is established.

Before giving an example violating the hypothesis of Proposition 2.5, we present a lemma which will be useful in constructing such an example and in a later proof.

**Lemma 2.6.** Let $F$ be a field of characteristic $p$ such that $[F^{1/p} : F] = p$. Let $K$ be a separable extension of $F$; then $[K^{1/p} : K] = p$.

**Proof.** Since $K$ is a separable extension of $F$, we have $K = F(K^n)$ [3, Thm. 8, p. 69]. Thus $K^{1/p} = F^{1/p}(K)$. But $F^{1/p}$ and $K$ are linearly disjoint [3, Thm. 35, p. 111], so we get $[K^{1/p} : K] = [F^{1/p}(K) : K] = [F^{1/p} : F] = p$.

**Example 2.7.** Let $F' = Z/3Z(a)$ where $Z$ denotes the integers and $a$ is indeterminant. Let $F'$ be the separable closure of $F'$ in its algebraic closure. By Lemma 2.6, $[F^{1/3} : F] = 3$. Consider the integral closure $A$ of $F[X]$ in the field $F(X, \delta)$, where $\delta = aX^a + X$. It is not difficult to show by a direct computation that $A = F[X, Y]$, but it is easier to notice that since the matrix of partial derivatives of the equation $Y^a - aX^a - X$ has always rank 1, $F[X, Y]$ is regular [2, Thm. 1, p. 201]. Over each prime ideal of $F[X]$ there lies only one prime ideal of $A$ and for the relative degree $f$ of the residue field and the ramification index $e$ we have $e = 3$, $f = 1$ or $e = 1$, $f = 3$ [3,
Thm. 22, p. 289. We show first that for all nonlinear prime elements of $F[X]$, we get $e = 1$, $f = 3$, so these remain principal. Let $Q$ be a prime ideal of $F[X]$ generated by a nonlinear element; $Q = X^q - t$, where $q$ is a power of 3 and $t \in F$. The residue field $F[X]/(X^q - t)$ is $F[t^{1/q}]$, while the residue field relative to $A$ will be $F[t^{1/q}, w]$, where $w^3 = at^{1/q} + t^{1/q}$. Since $[F^{1/3} : F] = 3$, we have $F^{1/3} \subseteq F[t^{1/q}]$, hence $\alpha^3 \in F[t^{1/q}]$. Thus $at^{1/q}$ is a cube in $F[t^{1/q}]$. But $t^{1/q}$ is not a cube in $F[t^{1/q}]$, so $[F[t^{1/q}, w] : F[t^{1/q}]] = 3$. That is, $f = 3$, $e = 1$; thus we see that nonlinear prime elements of $F[X]$ remain prime in $A$.

For the linear primes $X - t$, $t \in F$, we get $e = 1$, $f = 3$ if $at^3 + t$ is not a cube in $F$, while $e = 3$, $f = 1$ if $at^3 + t$ is a cube in $F$. Certainly we have the latter case at least for $t = 0$. Let $P$ be a prime ideal of $A$ lying over a linear prime ideal of $F[X]$ for which $e = 3$, $f = 1$. Then $P$ is not principal. For if $P$ were principal, say $P = (c_0(X) + c_1(X)Y + c_2(X)Y^2)$ we would get

$$P^3 = (c_0(X) + c_1(X)(aX^3 + X) + c_2(X)(a^2X^6 + 2aX^4 + X^2)).$$

But $P^3 = (x - t)$ for some $t \in F$. Comparing degrees and using the fact that 1, $a$, $a^2$ are independent over $F$, we get a contradiction. Again let $P$ be such a prime and suppose that the class of $P^3$ (which is the class of $P^{-1}$) contains a prime $Q$. $Q$ is certainly not principal; therefore $Q$ lies over a linear prime ideal of $F[X]$ and $e = 3$, $f = 1$ for $Q$. We also get that $P^3Q^2$ is principal, say

$$P^3Q^2 = (d_0(X) + d_1(X)Y + d_2(X)Y^2).$$

Cubing, we get

$$(P^3Q^2)^3 = (d_0^3(X) + d_1^3(X)(aX^3 + X) + d_2^3(X)(a^2X^6 + 2aX^4 + X^2)).$$

On the left side of this equation we have a polynomial of degree 4, while on the right we have a polynomial whose degree is divisible by 3, a contradiction.

**Proposition 2.8.** Let $A$ be a principal ideal domain and let $K$ be a finite separable extension of the quotient field $F$ of $A$. Let $B$ be the integral closure of $A$ in $K$ and let $C$ be a ring such that $B \subseteq C \subseteq K$. Then $C$ contains a semi-prime element.\(^1\)

**Proof.** There are only a finite number of prime ideals $Q_1, \ldots, Q_k$ of $B$ whose reduced ramification index is greater than 1 [3, Thm. 28, p. 302]. Let $P = \pi A$ be a prime ideal of $A$ not lying under any $Q_1, \ldots, Q_k$. Then $\pi B$ is a product of distinct primes and is a semi-

\(^1\) The referee has kindly pointed out that this Proposition (and thus the following) hold when $B$ is not necessarily integrally closed.
prime element in $B$. $\pi$ will also be a semi-prime element in $C$ unless all prime ideals of $B$ dividing $\pi$ generate $C$. The result now follows by Proposition 1.5 and Corollary 1.6.

**Proposition 2.9.** Let $A$ be the coordinate ring of an algebraic curve over a perfect ground field $F$. If $A$ is a Dedekind domain, then $A$ contains a semi-prime element.

*Proof.* $A = F[x_1, \cdots, x_n]$. Choose $X$ in $A$ such that $A$ is integral over $F[X]$; this is possible by [2, Thm. 1, p. 22]. Since $A$ is integrally closed in $K = F(x_1, \cdots, x_n)$, $A$ is the integral closure of $F[X]$ in $K$. Let $K'$ be the separable closure of $F(X)$ in $K$, and let $A'$ be the integral closure of $F[X]$ in $K$. The conclusion holds for $A'$ by Proposition 2.8.

Since $[F(X)^{1/p} : F(X)] = p$, we have $[K'^{1/p} : K'] = p$ by Lemma 2.6. $K$ is a purely inseparable extension of $K'$, so we may break the extension from $K'$ to $K$ into a chain of extensions each of which is pure inseparable of exponent $p$. This chain can only be

$$K' = K_0 \subset K_0^{1/p} \subset K_0^{1/p^2} \subset \cdots \subset K_0^{1/p^m} = K.$$  

But then we have an isomorphism of $K$ onto $K'$ given by $x \rightarrow x^{p^m}$ which induces an isomorphism of $A$ onto $A'$. Since $A'$ contained semi-prime elements, so does $A$.

**References**


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