TENSOR PRODUCTS OVER $H^*$-ALGEBRAS

LARRY CHARLES GROVE
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Larry C. Grove

Throughout, $A$, $B$, and $C$ denote (semi-simple) $H^*$-algebras whose respective decompositions into minimal closed ideals are $A = \Sigma A_a$, $B = \Sigma B_\beta$, and $C = \Sigma C_y$. It is assumed that $A$ is a right $C$-module and $B$ is a left $C$-module. We define a tensor product $A \otimes \delta B$ that is again an $H^*$-algebra, and show that it is isometric and isomorphic with an ideal in $A \otimes B \otimes C$. As a corollary, $A \otimes \delta B$ is strongly semi-simple if $A$, $B$, and $C$ are each strongly semi-simple. The converse to the corollary is shown to be false. When $A$, $B$, and $C$ are closed ideals in some $H^*$-algebra, with ordinary multiplication as the module action, then $A \otimes \delta B$ is shown to be isomorphic with the direct sum of all the one-dimensional ideals in $A \cap B \cap C$. When $A = L^k(G)$, $B = L^k(H)$, and $C = L^k(K)$, for suitable related compact groups $G$, $H$, and $K$, then the module actions are defined, and $A \otimes \delta B$ can be constructed. When $G = H = K$, it is shown that $A \otimes \delta B \cong L^k(G/N)$, where $N$ is the closure of the commutator subgroup of $G$. A conjecture is stated that would generalize this result to the case where $K$ is a closed subgroup of $G \cap H$.

Since $A \otimes \delta B$ will be represented in terms of ordinary tensor products $A \otimes B$ of $H^*$-algebras, the requisite facts concerning $A \otimes B$ are stated here (details may be found in [2]).

$A \otimes B$ is the Hilbert space completion of the space $A \otimes' B$ of all conjugate bilinear functionals $T$ on $A \times B$ of the form $T = \sum_i a_i \otimes b_i$, where $T(a, b) = \sum (a_i, a)(b_i, b)$ (see [3]). We define $(a \otimes b)(c \otimes d) = ac \otimes bd$, and extend by linearity and continuity to multiplication on $A \otimes B$. Then

I. $A \otimes B$ is an $H^*$-algebra and each $A_a \otimes B_\beta$ may be identified with a closed ideal in $A \otimes B$.

II. $A \otimes B = \Sigma (A_a \otimes B_\beta)$ is the decomposition of $A \otimes B$ into minimal closed ideals.

III. $A \otimes B$ is strongly semi-simple (see [5], p. 59) if and only if both $A$ and $B$ are strongly semi-simple.

1. Tensor products.

DEFINITION. $F_\delta(A, B)$ will denote the collection of all finite formal
sums of the form
\[ \sum_{i=1}^n c_i(a_i, b_i) \]
with \( a_i \in A, b_i \in B, \) and \( c_i \in C; \) i.e. \( F_0(A, B) \) is the free \( C \)-module generated by \( A \times B. \)

\( F_0(A, B) \) becomes an algebra and a pseudo-inner product space if the operations are defined by the formulas:

\[
\begin{align*}
(c(a, b)) \cdot (c'(a', b')) &= cc'(aa', bb'), \\
\lambda \sum c_i(a_i, b_i) &= \sum (\lambda c_i)(a_i, b_i), \quad \lambda \text{ complex, and} \\
(c(a, b), c'(a', b')) &= (c, c')((a, a') (b, b')
\end{align*}
\]

(the first and third must be extended by linearity). The positive semi-definiteness of the pseudo-inner product follows from the fact that

\[
(c(a, b), c'(a', b')) = (a \otimes b \otimes c, a' \otimes b' \otimes c');
\]

the other properties required of an inner product obviously hold.

Let \( I'_1 \) be the ideal in \( F_0(A, B) \) spanned by the set of all elements of the following forms:

\[ \begin{align*}
(1) & \quad c(a_1 + a_2, b) - c(a_1, b) - c(a_2, b), \\
(2) & \quad c(a, b_1 + b_2) - c(a, b_1) - c(a, b_2), \\
(3) & \quad (c_1 + c_2)(a, b) - c_1(a, b) - c_2(a, b), \\
(4) & \quad \lambda c(a, b) - c(\lambda a, b), \quad \text{and} \\
(5) & \quad \lambda c(a, b) - c(a, \lambda b)
\end{align*} \]

for arbitrary \( a, a_i \in A; \ b, b_i \in B; c, c_i \in C; \) and complex numbers \( \lambda. \)

Let \( I'^2 \) be the ideal in \( F_0(A, B) \) generated by the set of all elements of the forms:

\[ \begin{align*}
(6) & \quad c_i c_2(a, b) - c_i(ac_2, b), \quad \text{and} \\
(7) & \quad c_i c_2(a, b) - c_i(a, c_2 b)
\end{align*} \]

for arbitrary \( a \in A, b \in B, \) and \( c_i \in C. \) Then let \( I' = I'_1 \vee I'^2 = I'_1 + I'^2, \) the ideal generated by the set of all elements of the forms (1)–(7).

**Proposition 1.** \( I'_1 = \{X \in F_0(A, B): (X, X) = 0\}. \)

**Proof.** Straightforward computations show that \( (X, Y) = 0 \) if \( X \) is one of the forms (1)–(5) and \( Y = c'(a', b'). \) Extending by linearity we have immediately that \( (X, Y) = 0 \) for all \( X \in I'_1, \ Y \in F_0(A, B). \) Suppose then that \( X = \sum_{i=1}^n c_i(a_i, b_i) \) and that \( (X, X) = 0. \) It must be shown that \( X \in I'_1. \)

If \( \{c_i\}_{i=1}^n \) is not linearly independent, then we may assume that \( c_n = \sum_{i=1}^{n-1} \lambda_i c_i, \) and so
\[
X = \sum_{i=1}^{n-1} a_i(a_i, b_i) + \left( \sum_{i=1}^{n-1} \lambda_i c_i \right)(a_n, b_n) \\
= \sum_{i=1}^{n-1} c_i(a_i, b_i) + \sum_{i=1}^{n-1} c_i(\lambda_i a_n, b_n) \\
+ \left[ \left( \sum_{i=1}^{n-1} \lambda_i c_i \right)(a_n, b_n) - \sum_{i=1}^{n-1} c_i(\lambda_i a_n, b_n) \right].
\]

The expression in brackets is clearly an element of \( I_i \), call it \( \gamma_i \). Thus we have

\[
X = \sum_{j=1}^{2^p} \left( \sum_{i=1}^{n-1} c_i(a_{ij}, b_{ij}) \right) + \gamma_1,
\]

where \( a_{ij} = a_i, \ a_{i2} = \lambda_i a_n, \ b_{i1} = b_i, \ b_{i2} = b_n \). Repeating the process as many times as is necessary we obtain

\[
X = \sum_{j=1}^{2^p} \left( \sum_{i=1}^{n-1} c_i(a_{ij}, b_{ij}) \right) + \gamma_p,
\]

where \( \gamma_p \in I_i \) and \( \{c_i\}_{i=1}^{2^p} \) is linearly independent. Then, for each fixed index \( i \), by using an argument similar to the one above, we can write

\[
\sum_{j=1}^{2^p} c_i(a_{ij}, b_{ij}) = \sum_{k=1}^{2^p-q(i)} \left( \sum_{j=1}^{2^p-q(i)} c_i(a_{ij}, b_{ijk}) \right) + \gamma_{iq(i)},
\]

where \( \gamma_{iq(i)} \in I_i \) and \( \{a_{ij}: j = 1, \ldots, 2^p - q(i)\} \) is linearly independent.

As a result, we have

\[
X = \sum_{i=1}^{n-1} \sum_{j=1}^{2^p} \sum_{k=1}^{2^p-q(i)} c_i(a_{ij}, b_{ijk}) + \gamma,
\]

where \( \{c_i\} \) is linearly independent, \( \{a_{ij}\} \) is linearly independent for each fixed \( i \), and \( \gamma \in I_i' \).

Fix any pair \( < i, j > \) of indices. By the Hahn-Banach Theorem and the Riesz Theorem there exist \( \alpha \in A \) and \( \alpha' \in C \) such that

\[
\| \alpha' \| = \| \alpha \| = 1, \ (c_i, \alpha') = d_i > 0, \ (a_{ij}, \alpha') = d_{ij} > 0,
\]

\( (c', \alpha') = 0 \) if \( i' \neq i \), and \( (a_{ij'}, \alpha') = 0 \) if \( j' \neq j \). Since \( F_0(A, B) \) is a pseudo-inner product space, the Schwarz inequality holds. Thus if we let \( b' = \sum b_{ijh}: k = 1, \ldots, 2^p \), we have

\[
| (X, c'(\alpha', b')) | \leq (X, X)(c'(\alpha', b'), c'(\alpha', b')) = 0.
\]

On the other hand,

\[
(X, c'(\alpha', b')) = \sum_{m,n,h} (c_m, c')(a_{mn}, \alpha')(b_{mnh}, b') \\
= d_i d_{ij} \| b' \|^2 = 0,
\]

so that \( b' = 0 \). If we now write
\[ \sum_k c_i(a_{ij}, b_{ij}) = c_i(a_{ij}, \sum_k b_{ij}) + [\sum_k c_i(a_{ij}, b_{ijk})] = c_i(a_{ij}, 0) + \gamma_{ij}, \]

where \( \gamma_{ij} \) is the expression in brackets, which is clearly an element of \( I' \), then we have

\[ X = \sum_{i,j} c_i(a_{ij}, 0) + \gamma', \]

where \( \gamma' = \sum_{i,j} \gamma_{ij} \), and so \( X \in I' \).

\( F_c(A, B) \) is a pseudo-normed space, with \( \| X \|^2 = (X, X) \). Let us denote by \( \mathcal{F}_c(A, B) \) its pseudo-normed completion, i.e. the collection of all Cauchy sequences from \( F_c(A, B) \). Define a mapping

\[ \varphi: F_c(A, B) \to A \otimes B \otimes C \]

as follows:

\[ \varphi(\sum c_i(a_i, b_i)) = \sum a_i \otimes b_i \otimes c_i. \]

It is immediate that \( \varphi \) is a linear, homogeneous, multiplicative isometry, and that its range is dense. Thus \( \varphi \) can be extended to an isometric homomorphism on \( \mathcal{F}_c(A, B) \) onto \( A \otimes B \otimes C \). Note that \( \| XY \| \leq \| X \| \| Y \| \) for all \( X, Y \in F_c(A, B) \), since \( A \otimes B \otimes C \) is a Banach algebra. Thus the operations defined on \( F_c(A, B) \) can be extended to \( \mathcal{F}_c(A, B) \), as usual.

Let \( I_1, I_2, \) and \( I \) denote the closures, in \( \mathcal{F}_c(A, B) \), of \( I'_1, I'_2, \) and \( I' \), respectively. It is obvious from Proposition 1 that \( I = \{ X \in \mathcal{F}_c(A, B): \| X \| = 0 \} \), i.e. \( I \) is the closure of \( (0) \). Thus \( I \) is a subset of every closed subspace of \( \mathcal{F}_c(A, B) \), which means, in particular, that \( I = I_2 \). In other words, \( I \) can be described quite simply as the closed ideal of \( \mathcal{F}_c(A, B) \) generated by the collection of all elements of the forms (6) and (7).

**DEFINITION.** \( A \otimes_c B \), the tensor product of \( A \) and \( B \), over \( C \), is the quotient algebra \( \mathcal{F}_c(A, B)/I \).

\( A \otimes_c B \) is a normed space (as is always the case when a pseudo-normed space is factored by a closed subspace). We proceed to identify it with an ideal in \( A \otimes B \otimes C \). Let \( D = \varphi(I) \) and define a map \( \gamma: A \otimes_c B \to (A \otimes B \otimes C)/D \) by the formula \( \gamma(X + I) = \varphi(X) + D \). It is clear that \( \gamma \) is linear, and since \( \gamma(I) = \varphi(0) + D = D \), \( \gamma \) is well defined; it is multiplicative since \( \varphi \) is multiplicative. Finally, \( \gamma \) is an isometry. For if \( T = X + I \in A \otimes_c B \), then

\[ \| \gamma T \| = \| \varphi X + D \| = \inf \{ \| \varphi X + Z \|: Z \in D \} \]

\[ = \inf \{ \| \varphi X + \varphi Y \|: Y \in I \} \]

\[ = \inf \{ \| X + Y \|: Y \in I \} = \| T \|, \]
since $\varphi$ is an isometric homomorphism.

Since $D$ is a closed ideal in the $H^*$-algebra $A \otimes B \otimes C$, $(A \otimes B \otimes C)/D$ is isomorphic and isometric with the closed ideal $D^\perp$, which we shall denote by $E$. We summarize the foregoing information in the next theorem.

**Theorem.** There is an isometric isomorphism from $A \otimes_c B$ into $A \otimes B \otimes C$; its range is the closed ideal $E$ which is the orthogonal complement of the closed ideal $D$ generated by all elements of the forms

(i) $a \otimes b \otimes c, c_2 - ac_1 \otimes b \otimes c_1$, 
(ii) $a \otimes b \otimes c, c_2 - a \otimes c_1 \otimes b \otimes c_1$.

Consequently, $A \otimes_c B$ is an $H^*$-algebra; its minimal closed ideals can be identified with those minimal closed ideals $A_x \otimes B_\beta \otimes C_\gamma$ of $A \otimes B \otimes C$ that are orthogonal to $D$.

**Corollary.** If $A$, $B$, and $C$ are strongly semi-simple, then $A \otimes_c B$ is strongly semi-simple.

The following proposition provides means by which it is easy to construct examples for which the converse to the above corollary is false.

**Proposition 2.** If $A_\alpha \otimes B_\beta \otimes C_\gamma$ is a minimal closed ideal in $E$, then $C_\gamma$ is of dimension one.

**Proof.** Choose a canonical basis $\{a_{ij} \otimes b_{kl} \otimes c_{mn}\}$ for $A_\alpha \otimes B_\beta \otimes C_\gamma$ (see [2]). Since $a_{ij} \otimes b_{kl} \otimes c_{mn} \in E$, it must be orthogonal to

$$a_{ij} \otimes b_{kl} \otimes c_{mp} - a_{ij} \otimes b_{kl} \otimes c_{mp}.$$ 

If the dimension of $C_\gamma$ were greater than one, then it would be possible to choose $n \neq p$, and we would have

$$0 = (a_{ij} \otimes b_{kl} \otimes c_{mn}, a_{ij} \otimes b_{kl} \otimes c_{mn} - a_{ij} \otimes b_{kl} \otimes c_{mp}) = \|a_{ij}\|^2 \|b_{kl}\|^2 \|c_{mn}\|^2,$$

since $(c_{mn}, c_{mp}) = 0$. This, of course, is a contradiction.

**Corollary.** If $C$ has no one-dimensional minimal ideals, then $A \otimes_c B = (0)$.

2. Examples. Perhaps the easiest method of obtaining examples of $H^*$-algebras $A$, $B$, and $C$ related as above is to let $A$, $B$, and $C$ be
closed ideals in some $H^*$-algebra $\mathcal{A}$. The structure of $A \otimes_c B$, under such circumstances, is described in the next proposition.

**Proposition 3.** Suppose that $A$, $B$ and $C$ are closed ideals in an $H^*$-algebra $\mathcal{A}$. If $A$ and $B$ are viewed as $C$-modules with ordinary multiplication in $\mathcal{A}$ as the module action, then $A \otimes_c B$ is isomorphic with the direct sum of all the one-dimensional minimal ideals in $A \cap B \cap C$. The isomorphism is an isometry if and only if the identity of each one-dimensional minimal ideal in $A \cap B \cap C$ has norm one.

**Proof.** Choose a canonical basis $\{u_{pq}^b\}$ for $\mathcal{A}$. Then $\{a_{\alpha}\} = A = \cap \{u_{pq}^b\}$, $\{b_{kl}^c\} = B \cap \{u_{pq}^b\}$, and $\{c_{mn}^\gamma\} = C \cap \{u_{pq}^b\}$ are canonical bases for $A$, $B$, and $C$, respectively and $\{a_{\alpha}^a \otimes b_{kl}^c \otimes c_{mn}^\gamma\}$ is a canonical basis for $A \otimes B \otimes C$. If $a_{\alpha}^a \otimes b_{kl}^c \otimes c_{mn}^\gamma \in E$, then, by Proposition 2, $c_{mn}^\gamma = c'$ is the identity of a one-dimensional minimal ideal. If $\alpha \neq \gamma$, then

$$a_{\alpha}^a \otimes b_{kl}^c \otimes c' = a_{\alpha}^a b_{kl}^c \otimes c' = a_{\alpha}^a b_{kl}^c \otimes c' \in D.$$ 

Similarly, if $\beta \neq \gamma$, then $a_{\beta}^a \otimes b_{kl}^c \otimes c' \in D$. Thus if an element of a canonical basis is to be in $E$ it must be of the form $c' \otimes c' \otimes c'$. Relatively straightforward computations show that each such basis element is orthogonal to $D$, and the proof is completed.

Suppose now that $G$, $H$, and $K$ are compact groups, and that $\Theta: K \to G$ and $\phi: K \to H$ are continuous homomorphisms. Then $\Theta(K)$ and $\phi(K)$ are closed subgroups of $G$ and $H$, respectively, $L^1(G)$ and $L^1(H)$ become modules over $L^1(K)$, with the module action defined by:

$$g \ast h(x) = \int_K g(x(\theta z)^{-1})h(z)dz,$$

$$k \ast h(y) = \int_K k(z)h((\phi z)^{-1}y)dz,$$

for all $g \in L^1(G)$, $h \in L^1(H)$, $k \in L^1(K)$, $x \in G$, and $y \in H$ (all integrations are with respect to normalized Haar measures). If we let $A = L^1(G)$, $B = L^1(H)$, $C = L^1(K)$, then $A \otimes_c B$ is a well-defined $H^*$-algebra. As was remarked in [2], $A \otimes B \otimes C$ can be identified with $L^1(G \times H \times K)$, and so, by the Theorem of §1, $A \otimes_c B$ can be identified with a closed ideal $J$ in $L^1(G \times H \times K)$. At one extreme, suppose $\Theta$ and $\phi$ map $K$ onto the identities of $G$ and $H$, respectively. It is not difficult to see that in this case $A \otimes_c B$ can be identified with $L^1(G \times H)$.

At what might be considered another extreme, suppose that $G$ and $H$ are closed subgroups of some compact group, that $K$ is a closed subgroup of $G \cap H$, and that $\Theta$ and $\phi$ are the inclusion maps. Define an equivalence relation on $G \times H \times K$ as follows: $(x, y, z) \sim (u, v, w)$...
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if and only if \( F(x, y, z) = F(u, v, w) \) for all \( F \in J \). Then \( M = \{(x, y, z) : (x, y, z) \sim (e, e, e)\} \) is a closed normal subgroup of \( G \times H \times K \), and its cosets are the equivalence classes of \( \sim \). All functions \( F \in J \) are thus constant on the cosets of \( M \), providing a mapping \( \psi \) from \( J \) to \( \mathcal{L}^2((G \times H \times K)/M) \). The map \( \psi \) is an isometric isomorphism and its image is an ideal. On the basis of the Tannaka Duality Theorem (see [4], p. 439) it seems reasonable to conjecture that \( \psi \) is surjective, so that \( A \otimes_0 B \) can be identified with \( \mathcal{L}^2((G \times H \times K)/M) \). The conjecture has not been settled in general, but let us consider the very special case where \( G = H = K \). Then, by Proposition 3, \( A \otimes_0 B \) can be identified with the direct sum of all one-dimensional minimal ideals in \( \mathcal{L}^1(G) \), which in turn is isomorphic and isometric with \( \mathcal{L}^1(G/N) \), where \( N \) is the closure of the commutator subgroup of \( G \). Since \( G/N \) and \( (G \times G \times G)/M \) are isomorphic via the mapping \( xN \rightarrow (x, e, e)M \), the conjecture is verified in this special case.

**References**


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