

Pacific Journal of Mathematics

TENSOR PRODUCTS OVER H^* -ALGEBRAS

LARRY CHARLES GROVE

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Throughout, A , B , and C denote (semi-simple) H^* -algebras whose respective decompositions into minimal closed ideals are $A = \Sigma \oplus A_\alpha$, $B = \Sigma \oplus B_\beta$, and $C = \Sigma \oplus C_\gamma$. It is assumed that A is a right C -module and B is a left C -module. We define a tensor product $A \otimes_\sigma B$ that is again an H^* -algebra, and show that it is isometric and isomorphic with an ideal in $A \otimes B \otimes C$. As a corollary, $A \otimes_\sigma B$ is strongly semi-simple if A , B , and C are each strongly semi-simple. The converse to the corollary is shown to be false. When A , B , and C are closed ideals in some H^* -algebra, with ordinary multiplication as the module action, then $A \otimes_\sigma B$ is shown to be isomorphic with the direct sum of all the one-dimensional ideals in $A \cap B \cap C$. When $A = L^2(G)$, $B = L^2(H)$, and $C = L^2(K)$, for suitable related compact groups G , H , and K , then the module actions are defined, and $A \otimes_\sigma B$ can be constructed. When $G = H = K$, it is shown that $A \otimes_\sigma B \cong L^2(G/N)$, where N is the closure of the commutator subgroup of G . A conjecture is stated that would generalize this result to the case where K is a closed subgroup of $G \cap H$.

Since $A \otimes_\sigma B$ will be represented in terms of ordinary tensor products $A \otimes B$ of H^* -algebras, the requisite facts concerning $A \otimes B$ are stated here (details may be found in [2]).

$A \otimes B$ is the Hilbert space completion of the space $A \otimes' B$ of all conjugate bilinear functionals T on $A \times B$ of the form $T = \sum_{i=1}^n a_i \otimes b_i$, where $T(a, b) = \Sigma (a_i, a)(b_i, b)$ (see [3]). We define $(a \otimes b)(c \otimes d) = ac \otimes bd$, and extend by linearity and continuity to multiplication on $A \otimes B$. Then

I. $A \otimes B$ is an H^* -algebra and each $A_\alpha \otimes B_\beta$ may be identified with a closed ideal in $A \otimes B$.

II. $A \otimes B = \Sigma \otimes (A_\alpha \otimes B_\beta)$ is the decomposition of $A \otimes B$ into minimal closed ideals.

III. $A \otimes B$ is strongly semi-simple (see [5], p. 59) if and only if both A and B are strongly semi-simple.

1. Tensor products.

DEFINITION. $F'_\sigma(A, B)$ will denote the collection of all finite formal

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sums of the form

$\sum_{i=1}^n c_i(a_i, b_i)$, with $a_i \in A, b_i \in B$, and $c_i \in C$; i.e. $F_o(A, B)$ is the free C -module generated by $A \times B$.

$F_o(A, B)$ becomes an algebra and a pseudo-inner product space if the operations are defined by the formulas:

$$\begin{aligned} (c(a, b)) \cdot (c'(a', b')) &= cc'(aa', bb') , \\ \lambda \Sigma c_i(a_i, b_i) &= \Sigma (\lambda c_i)(a_i, b_i), \lambda \text{ complex, and} \\ (c(a, b), c'(a', b')) &= (c, c')(a, a')(b, b') \end{aligned}$$

(the first and third must be extended by linearity). The positive semi-definiteness of the pseudo-inner product follows from the fact that $(c(a, b), c'(a', b')) = (a \otimes b \otimes c, a' \otimes b' \otimes c')$; the other properties required of an inner product obviously hold.

Let I'_1 be the ideal in $F_o(A, B)$ spanned by the set of all elements of the following forms:

- (1) $c(a_1 + a_2, b) - c(a_1, b) - c(a_2, b)$,
- (2) $c(a, b_1 + b_2) - c(a, b_1) - c(a, b_2)$,
- (3) $(c_1 + c_2)(a, b) - c_1(a, b) - c_2(a, b)$,
- (4) $\lambda c(a, b) - c(\lambda a, b)$, and
- (5) $\lambda c(a, b) - c(a, \lambda b)$

for arbitrary $a, a_i \in A; b, b_i \in B; c, c_i \in C$; and complex numbers λ . Let I'_2 be the ideal in $F_o(A, B)$ generated by the set of all elements of the forms:

- (6) $c_1 c_2(a, b) - c_1(a c_2, b)$, and
- (7) $c_1 c_2(a, b) - c_1(a, c_2 b)$

for arbitrary $a \in A, b \in B$, and $c_i \in C$. Then let $I' = I'_1 \vee I'_2 = I'_1 + I'_2$, the ideal generated by the set of all elements of the forms (1)-(7).

PROPOSITION 1. $I'_1 = \{X \in F_o(A, B): (X, X) = 0\}$.

Proof. Straightforward computations show that $(X, Y) = 0$ if X is of one of the forms (1)-(5) and $Y = c'(a', b')$. Extending by linearity we have immediately that $(X, Y) = 0$ for all $X \in I'_1, Y \in F_o(A, B)$. Suppose then that $X = \sum_{i=1}^n c_i(a_i, b_i)$ and that $(X, X) = 0$. It must be shown that $X \in I'_1$.

If $\{c_i\}_{i=1}^n$ is not linearly independent, then we may assume that $c_n = \sum_{i=1}^{n-1} \lambda_i c_i$, and so

$$\begin{aligned} X &= \sum_{i=1}^{n-1} c_i(a_i, b_i) + \left(\sum_{i=1}^{n-1} \lambda_i c_i \right) (a_n, b_n) \\ &= \sum_{i=1}^{n-1} c_i(a_i, b_i) + \sum_{i=1}^{n-1} c_i(\lambda_i a_n, b_n) \\ &\quad + \left[\left(\sum_{i=1}^{n-1} \lambda_i c_i \right) (a_n, b_n) - \sum_{i=1}^{n-1} c_i(\lambda_i a_n, b_n) \right]. \end{aligned}$$

The expression in brackets is clearly an element of I'_1 , call it γ_1 . Thus we have

$$X = \sum_{j=1}^2 \sum_{i=1}^{n-1} c_i(a_{ij}, b_{ij}) + \gamma_1,$$

where $a_{i1} = a_i$, $a_{i2} = \lambda_i a_n$, $b_{i1} = b_i$, $b_{i2} = b_n$. Repeating the process as many times as is necessary we obtain

$$X = \sum_{j=1}^{2^p} \left(\sum_{i=1}^{n-2^p} c_i(a_{ij}, b_{ij}) \right) + \gamma_p,$$

where $\gamma_p \in I'_1$ and $\{c_i\}_{i=1}^{n-2^p}$ is linearly independent. Then, for each fixed index i , by using an argument similar to the one above, we can write

$$\sum_{j=1}^{2^p} c_i(a_{ij}, b_{ij}) = \sum_{k=1}^{2q(i)} \left(\sum_{j=1}^{2^{p-q(i)}} c_i(a_{ij}, b_{ijk}) \right) + \gamma_{iq(i)},$$

where $\gamma_{iq(i)} \in I'_1$ and $\{a_{ij}; j = 1, \dots, 2^p - q(i)\}$ is linearly independent. As a result, we have

$$X = \sum_{i=1}^{n-2^p} \sum_{j=1}^{2^{p-q(i)}} \sum_{k=1}^{2q(i)} c_i(a_{ij}, b_{ijk}) + \gamma,$$

where $\{c_i\}$ is linearly independent, $\{a_{ij}\}$ is linearly independent for each fixed i , and $\gamma \in I'_1$.

Fix any pair $\langle i, j \rangle$ of indices. By the Hahn-Banach Theorem and the Riesz Theorem there exist $a' \in A$ and $c' \in C$ such that

$$\|c'\| = \|a'\| = 1, (c_i, c') = d_i > 0, (a_{ij}, a') = d_{ij} > 0,$$

$(c_{i'}, c') = 0$ if $i' \neq i$, and $(a_{ij'}, a') = 0$ if $j' \neq j$. Since $F'_\sigma(A, B)$ is a pseudo-inner product space, the Schwarz inequality holds. Thus if we let $b' = \sum \{b_{ijk}; k = 1, \dots, 2^{q(i)}\}$, we have

$$|(X, c'(a', b'))| \leq (X, X)(c'(a', b'), c'(a', b')) = 0.$$

On the other hand,

$$\begin{aligned} (X, c'(a', b')) &= \sum_{m,n,k} (c_m, c')(a_{mn}, a')(b_{mnk}, b') \\ &= d_i d_{ij} \|b'\|^2 = 0, \end{aligned}$$

so that $b' = 0$. If we now write

$$\begin{aligned}\sum_k c_i(a_{ij}, b_{ijk}) &= c_i(a_{ij}, \sum_k b_{ijk}) + [\sum_k c_i(a_{ij}, b_{ijk}) - c_i(a_{ij}, \sum_k b_{ijk})] \\ &= c_i(a_{ij}, 0) + \gamma'_{ij},\end{aligned}$$

where γ'_{ij} is the expression in brackets, which is clearly an element of I'_1 , then we have

$$X = \sum_{i,j} c_i(a_{ij}, 0) + \gamma',$$

where $\gamma' = \sum_{i,j} \gamma'_{ij}$, and so $X \in I'_1$.

$F_c(A, B)$ is a pseudo-normed space, with $\|X\|^2 = (X, X)$. Let us denote by $\mathcal{F}_c(A, B)$ its pseudo-normed completion, i.e. the collection of all Cauchy sequences from $F_c(A, B)$. Define a mapping

$$\varphi: F_c(A, B) \rightarrow A \otimes B \otimes C$$

as follows:

$$\varphi(\sum c_i(a_i, b_i)) = \sum a_i \otimes b_i \otimes c_i.$$

It is immediate that φ is a linear, homogeneous, multiplicative isometry, and that its range is dense. Thus φ can be extended to an isometric homomorphism on $\mathcal{F}_c(A, B)$ onto $A \otimes B \otimes C$. Note that $\|XY\| \leq \|X\| \|Y\|$ for all $X, Y \in F_c(A, B)$, since $A \otimes B \otimes C$ is a Banach algebra. Thus the operations defined on $F_c(A, B)$ can be extended to $\mathcal{F}_c(A, B)$, as usual.

Let I_1, I_2 , and I denote the closures, in $\mathcal{F}_c(A, B)$, of I'_1, I'_2 , and I' , respectively. It is obvious from Proposition 1 that $I_1 = \{X \in \mathcal{F}_c(A, B): \|X\| = 0\}$, i.e. I_1 is the closure of (0) . Thus I_1 is a subset of every closed subspace of $\mathcal{F}_c(A, B)$, which means, in particular, that $I = I_2$. In other words, I can be described quite simply as the closed ideal of $\mathcal{F}_c(A, B)$ generated by the collection of all elements of the forms (6) and (7).

DEFINITION. $A \otimes_c B$, the tensor product of A and B , over C , is the quotient algebra $\mathcal{F}_c(A, B)/I$.

$A \otimes_c B$ is a normed space (as is always the case when a pseudo-normed space is factored by a closed subspace). We proceed to identify it with an ideal in $A \otimes B \otimes C$. Let $D = \varphi(I)$ and define a map $\gamma: A \otimes_c B \rightarrow (A \otimes B \otimes C)/D$ by the formula $\gamma(X + I) = \varphi(X) + D$. It is clear that γ is linear, and since $\gamma(I) = \varphi(0) + D = D$, γ is well defined; it is multiplicative since φ is multiplicative. Finally, γ is an isometry. For if $T = X + I \in A \otimes_c B$, then

$$\begin{aligned}\|\gamma T\| &= \|\varphi X + D\| = \inf \{\|\varphi X + Z\|: Z \in D\} \\ &= \inf \{\|\varphi X + \varphi Y\|: Y \in I\} \\ &= \inf \{\|X + Y\|: Y \in I\} = \|T\|,\end{aligned}$$

since φ is an isometric homomorphism.

Since D is a closed ideal in the H^* -algebra $A \otimes B \otimes C$, $(A \otimes B \otimes C)/D$ is isomorphic and isometric with the closed ideal D^\perp , which we shall denote by E . We summarize the foregoing information in the next theorem.

THEOREM. *There is an isometric isomorphism from $A \otimes_\sigma B$ into $A \otimes B \otimes C$; its range is the closed ideal E which is the orthogonal complement of the closed ideal D generated by all elements of the forms*

- (i) $a \otimes b \otimes c_1 c_2 - a c_2 \otimes b \otimes c_1$,
- (ii) $a \otimes b \otimes c_1 c_2 - a \otimes c_2 b \otimes c_1$.

Consequently, $A \otimes_\sigma B$ is an H^* -algebra; its minimal closed ideals can be identified with those minimal closed ideals $A_\alpha \otimes B_\beta \otimes C_\gamma$ of $A \otimes B \otimes C$ that are orthogonal to D .

COROLLARY. *If A, B , and C are strongly semi-simple, then $A \otimes_\sigma B$ is strongly semi-simple.*

The following proposition provides means by which it is easy to construct examples for which the converse to the above corollary is false.

PROPOSITION 2. *If $A_\alpha \otimes B_\beta \otimes C_\gamma$ is a minimal closed ideal in E , then C_γ is of dimension one.*

Proof. Choose a canonical basis $\{a_{ij} \otimes b_{kl} \otimes c_{mn}\}$ for $A_\alpha \otimes B_\beta \otimes C_\gamma$ (see [2]). Since $a_{ij} \otimes b_{kl} \otimes c_{mn} \in E$, it must be orthogonal to

$$a_{ij} \otimes b_{kl} \otimes c_{mp} c_{pn} - a_{ij} c_{pn} \otimes b_{kl} \otimes c_{mp}.$$

If the dimension of C_γ were greater than one, then it would be possible to choose $n \neq p$, and we would have

$$\begin{aligned} 0 &= (a_{ij} \otimes b_{kl} \otimes c_{mn}, a_{ij} \otimes b_{kl} \otimes c_{mn} - a_{ij} c_{pn} \otimes b_{kl} \otimes c_{mp}) \\ &= \|a_{ij}\|^2 \|b_{kl}\|^2 \|c_{mn}\|^2, \end{aligned}$$

since $(c_{mn}, c_{mp}) = 0$. This, of course, is a contradiction.

COROLLARY. *If C has no one-dimensional minimal ideals, then $A \otimes_\sigma B = (0)$.*

2. Examples. Perhaps the easiest method of obtaining examples of H^* -algebras A, B , and C related as above is to let A, B , and C be

closed ideals in some H^* -algebra \mathcal{A} . The structure of $A \otimes_o B$, under such circumstances, is described in the next proposition.

PROPOSITION 3. Suppose that A, B and C are closed ideals in an H^* -algebra \mathcal{A} . If A and B are viewed as C -modules with ordinary multiplication in \mathcal{A} as the module action, then $A \otimes_o B$ is isomorphic with the direct sum of all the one-dimensional minimal ideals in $A \cap B \cap C$. The isomorphism is an isometry if and only if the identity of each one-dimensional minimal ideal in $A \cap B \cap C$ has norm one.

Proof. Choose a canonical basis $\{u_{pq}^\delta\}$ for \mathcal{A} . Then $\{a_{ij}\} = A = \cap \{u_{pq}^\delta\}$, $\{b_{kl}^\beta\} = B \cap \{u_{pq}^\delta\}$, and $\{c_{mn}^\gamma\} = C \cap \{u_{pq}^\delta\}$ are canonical bases for A, B , and C , respectively and $\{a_{ij}^\alpha \otimes b_{kl}^\beta \otimes c_{mn}^\gamma\}$ is a canonical basis for $A \otimes B \otimes C$. If $a_{ij}^\alpha \otimes b_{kl}^\beta \otimes c_{mn}^\gamma \in E$, then, by Proposition 2, $c_{mn}^\gamma = c^\gamma$ is the identity of a one-dimensional minimal ideal. If $\alpha \neq \gamma$, then

$$a_{ij}^\alpha \otimes b_{kl}^\beta \otimes c^\gamma c^\gamma - a_{ij}^\alpha c^\gamma \otimes b_{kl}^\beta \otimes c^\gamma = a_{ij}^\alpha \otimes b_{kl}^\beta \otimes c^\gamma \in D.$$

Similarly, if $\beta \neq \gamma$, then $a_{ij}^\alpha \otimes b_{kl}^\beta \otimes c^\gamma \in D$. Thus if an element of a canonical basis is to be in E it must be of the form $c^\gamma \otimes c^\gamma \otimes c^\gamma$. Relatively straightforward computations show that each such basis element is orthogonal to D , and the proof is completed.

Suppose now that G, H , and K are compact groups, and that $\theta: K \rightarrow G$ and $\varphi: K \rightarrow H$ are continuous homomorphisms. Then $\theta(K)$ and $\varphi(K)$ are closed subgroups of G and H , respectively, $L^2(G)$ and $L^2(H)$ become modules over $L^2(K)$, with the module action defined by:

$$g * k(x) = \int_K g(x(\theta z)^{-1})k(z)dz,$$

$$k * h(y) = \int_K k(z)h((\varphi z)^{-1}y)dz,$$

for all $g \in L^2(G), h \in L^2(H), k \in L^2(K), x \in G$, and $y \in H$ (all integrations are with respect to normalized Haar measures). If we let $A = L^2(G), B = L^2(H), C = L^2(K)$, then $A \otimes_o B$ is a well-defined H^* -algebra. As was remarked in [2], $A \otimes B \otimes C$ can be identified with $L^2(G \times H \times K)$, and so, by the Theorem of §1, $A \otimes_o B$ can be identified with a closed ideal J in $L^2(G \times H \times K)$. At one extreme, suppose θ and φ map K onto the identities of G and H , respectively. It is not difficult to see that in this case $A \otimes_o B$ can be identified with $L^2(G \times H)$.

At what might be considered another extreme, suppose that G and H are closed subgroups of some compact group, that K is a closed subgroup of $G \cap H$, and that θ and φ are the inclusion maps. Define an equivalence relation on $G \times H \times K$ as follows: $(x, y, z) \sim (u, v, w)$

if and only if $F(x, y, z) = F(u, v, w)$ for all $F \in J$. Then $M = \{(x, y, z): (x, y, z) \sim (e, e, e)\}$ is a closed normal subgroup of $G \times H \times K$, and its cosets are the equivalence classes of \sim . All functions $F \in J$ are thus constant on the cosets of M , providing a mapping ψ from J to $L^2((G \times H \times K)/M)$. The map ψ is an isometric isomorphism and its image is an ideal. On the basis of the Tannaka Duality Theorem (see [4], p. 439) it seems reasonable to conjecture that ψ is surjective, so that $A \otimes_o B$ can be identified with $L^2((G \times H \times K)/M)$. The conjecture has not been settled in general, but let us consider the very special case where $G = H = K$. Then, by Proposition 3, $A \otimes_o B$ can be identified with the direct sum of all one-dimensional minimal ideals in $L^2(G)$, which in turn is isomorphic and isometric with $L^2(G/N)$, where N is the closure of the commutator subgroup of G . Since G/N and $(G \times G \times G)/M$ are isomorphic via the mapping $xN \rightarrow (x, e, e)M$, the conjecture is verified in this special case.

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