

Pacific Journal of Mathematics

**L^2 EXPANSIONS IN TERMS OF GENERALIZED HEAT
POLYNOMIALS AND OF THEIR APPELL TRANSFORMS**

DEBORAH TEPPER HAIMO

L^2 EXPANSIONS IN TERMS OF GENERALIZED HEAT POLYNOMIALS AND OF THEIR APPELL TRANSFORMS

DEBORAH TEPPER HAIMO

The object of this paper is to characterize functions which have L^2 expansions in terms of polynomial solutions $P_{n,\nu}(x, t)$ of the generalized heat equation

$$(*) \quad \left[\frac{\partial^2}{\partial x^2} + \frac{2\nu}{x} \frac{\partial}{\partial x} \right] u(x, t) = \frac{\partial}{\partial t} u(x, t).$$

and in terms of the Appell transforms $W_{n,\nu}(x, t)$ of the $P_{n,\nu}(x, t)$. H^* denotes the C^2 class of functions $u(x, t)$ which, for $a < t < b$, satisfy (*) and for which

$$u(x, t) = \int_0^\infty G(x, y; t - t') u(y, t') d\mu(y),$$

$$d\mu(x) = 2^{(1/2)-\nu} \left[\Gamma\left(\nu + \frac{1}{2}\right) \right]^{-1} x^{2\nu} dx,$$

for all $t, t', a < t' < t < b$, the integral converging absolutely, where $G(x, y; t)$ is the source solution of (*). The principal results are the following:

THEOREM. Let $u(x, t) \in H^*$, $-\sigma \leq t < 0$, and

$$u(x, t) [G(x; -t)]^{\frac{1}{2}} \in L^2$$

for each fixed t $-\sigma \leq t < 0$, $0 \leq x < \infty$. Then, for $-\sigma \leq t < 0$,

$$\lim_{N \rightarrow \infty} \int_0^\infty G(x; -t) \left| u(x, t) - \sum_{n=0}^N a_n P_{n,\nu}(x, -t) \right|^2 d\mu(x) = 0,$$

and

$$\int_0^\infty G(x; -t) |u(x, t)|^2 d\mu(x) = \sum_{n=0}^\infty |a_n|^2 b_n^{-1} t^{2n},$$

where

$$b_n = [2^{4n} n!]^{-1} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2} + n\right)},$$

and

$$a_n = b_n \int_0^\infty u(y, t) W_{n,\nu}(y, -t) d\mu(y).$$

Received January, 16, 1965. The research was supported by a National Science Foundation Fellowship at Harvard University.

THEOREM. If $u(x, t) \in H^*$, $0 < t \leq \sigma$, and if

$$u(ix, t)[G(x; t)]^{(1/2)} \in L^2$$

for each fixed t , $0 < t \leq \sigma$, $0 \leq x < \infty$, then, for $0 < t \leq \sigma$,

$$\lim_{N \rightarrow \infty} \int_0^\infty G(x; t) \left| u(ix, t) - \sum_{n=0}^N a_n P_{n,\nu}(x, -t) \right|^2 d\mu(x) = 0,$$

and

$$\int_0^\infty G(x; t) |u(ix, t)|^2 d\mu(x) = \sum_{n=0}^\infty |a_n|^2 b_n^{-1} t^{2n},$$

where b_n is given above and

$$a_n = b_n \int_0^\infty u(ix, t) W_{n,\nu}(x, t) d\mu(x).$$

THEOREM. If $u(x, t) \in H^*$, $0 < \sigma \leq t$, and if

$$u(x, t)[G(ix; t)]^{(1/2)} \in L^2$$

for each fixed t , $0 < \sigma \leq t$, $0 \leq x < \infty$, then, for $0 < \sigma \leq t$,

$$\lim_{N \rightarrow \infty} \int_0^\infty G(ix; t) \left| u(x, t) - \sum_{n=0}^N a_n W_{n,\nu}(x, t) \right|^2 d\mu(x) = 0,$$

and

$$\int_0^\infty G(ix; t) |u(x, t)|^2 d\mu(x) = \sum_{n=0}^\infty t^{-2n} b_n^{-1} (2t)^{-2\nu-1} |a_n|^2,$$

where b_n is given above, and

$$a_n = b_n \int_0^\infty u(x, t) P_{n,\nu}(x, -t) d\mu(x).$$

The theory is an extension, in part, of recent results of P. C. Rosenbloom and D. V. Widder.

1. Preliminary results. The generalized heat polynomial $P_{n,\nu}(x, t)$ is a polynomial defined by

$$(1.1) \quad P_{n,\nu}(x, t) = \sum_{k=0}^n 2^{2k} \binom{n}{k} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2} + n - k\right)} x^{2n-2k} t^k,$$

ν a fixed positive number. Note that when $\nu = 0$, $P_{n,0}(x, t) = v_{2n}(x, t)$, the ordinary heat polynomials defined in [8; p. 222]. For $t > 0$, $P_{n,\nu}(x, t)$ has the following integral representation.

$$(1.2) \quad P_{n,\nu}(x, t) = \int_0^\infty y^{2n} G(x, y; t) d\mu(y),$$

$$d\mu(y) = 2^{(1/2)-\nu} \left[\Gamma\left(\nu + \frac{1}{2}\right) \right]^{-1} x^{2\nu} dx.$$

As may readily be verified, for $-\infty < x, t < \infty$, $P_{n,\nu}(x, t)$ satisfies the generalized heat equation

$$(1.3) \quad \Delta_x u(x, t) = \frac{\partial}{\partial t} u(x, t) ,$$

where $\Delta_x f(x) = f''(x) + (2\nu/x) f'(x)$. We denote by H the class of all C^2 functions which satisfy (1.3). The source solution of (1.3) is given by $G(x; t)$, where

$$(1.4) \quad G(x, y; t) = \left(\frac{1}{2t}\right)^{\nu+\frac{1}{2}} \exp\left(-\frac{x^2 + y^2}{4t}\right) \mathcal{I}\left(\frac{xy}{2t}\right) ,$$

with $\mathcal{I}(z) = C_\nu z^{(1/2)-\nu} I_{\nu-(1/2)}(z)$, $C_\nu = 2^{(1/2)-\nu} \Gamma(\nu + (1/2))$, $I_r(z)$ being the Bessel function of imaginary argument of order r , and where $G(x; t) = G(x, 0; t)$. For a detailed study of the properties of $G(x, y; t)$ see [1].

Corresponding to the generalized heat polynomial $P_{n,\nu}(x, t)$ is its Appell transform $W_{n,\nu}(x, t)$ defined by

$$(1.5) \quad W_{n,\nu}(x, t) = G(x, t) P_{n,\nu}\left(\frac{x}{t}, -\frac{1}{t}\right), \quad t > 0, \quad n = 0, 1, 2, \dots ,$$

which is also a solution of (1.3). It follows readily from the definition of $P_{n,\nu}(x, t)$ that

$$(1.6) \quad W_{n,\nu}(x, t) = t^{-2n} G(x, t) P_{n,\nu}(x, -t), \quad t > 0, \quad n = 0, 1, 2, \dots .$$

The importance of $P_{n,\nu}(x, t)$ and $W_{n,\nu}(x, t)$ in our theory is that they form a biorthogonal system on $0 \leq x < \infty$. We have, for $t > 0$,

$$(1.7) \quad \int_0^\infty W_{n,\nu}(x, t) P_{m,\nu}(x, -t) d\mu(x) = \frac{1}{b_n} \delta_{mn} ,$$

where

$$(1.8) \quad b_n = \Gamma\left(\nu + \frac{1}{2}\right) / \left[2^{4n} n! \Gamma\left(\nu + \frac{1}{2} + n\right)\right] .$$

A consequence of (1.7) is a fundamental generating function for the biorthogonal set $P_{n,\nu}(x, -t)$, $W_{n,\nu}(x, t)$. We have, for $0 \leq x, y < \infty$, $-s < t < s$, $s > 0$,

$$(1.9) \quad G(x, y; s + t) = \sum_{n=0}^\infty b_n W_{n,\nu}(y, s) P_{n,\nu}(x, t) .$$

2. Inversion. For $t > s$, let us set

$$(2.1) \quad \mathcal{H}(x, y; s, t) = \sum_{n=0}^\infty b_n \left(\frac{t}{s}\right)^{(\nu/2)+(1/4)} e^{(x^2/8t)-(y^2/8s)} W_{n,\nu}(x, t) P_{n,\nu}(y, -s) ,$$

where b_n is defined by (1.8). Then, as a consequence of the definitions and of (1.9), we have

$$(2.2) \quad \mathcal{K}(x, y; s, t) = \left(\frac{t}{s}\right)^{(\nu/2)+(1/4)} e^{-(x^2(t-s))/(8t(t+s))} G(x\sqrt{2s/(t+s)}, y\sqrt{(t+s)/2s}; t-s).$$

From the well known properties of $G(x, y; t)$ – see [1; § 4] – the following results are immediate.

LEMMA 2.1.

(2.3) (a) $\mathcal{K}(x, y; s, t) \geq 0, 0 \leq x, y < \infty, s < t,$

(2.4) (b) $\lim_{y \rightarrow \infty} \mathcal{K}(x, y; s, t) = 0, 0 \leq x < \infty, s < t,$

(2.5) (c) $\lim_{s \rightarrow t^-} \mathcal{K}(x, y; s, t) = 0$ uniformly $0 \leq x, y < \infty,$
 $|y - x| \geq \delta > 0, \delta$ any fixed positive number.

(d) For x fixed, $0 \leq x < \infty,$

$$(2.6) \quad \begin{aligned} \lim_{s \rightarrow t^-} \int_a^b \mathcal{K}(x, y; s, t) d\mu(y) &= 1, & 0 \leq a < x < b \leq \infty, \\ &= 0, & 0 \leq a \leq b < x < \infty, \\ &= 0, & 0 \leq x < a < b \leq \infty. \end{aligned}$$

It is now easy to establish the following fundamental inversion theorem.

THEOREM 2.2. *If φ belongs to $L^1(0, \infty)$ and is continuous at x , then*

$$(2.7) \quad \lim_{s \rightarrow t^-} \int_0^\infty \mathcal{K}(x, y; s, t) \varphi(y) d\mu(y) = \varphi(x).$$

3. The Huygens property. A function $u(x, t)$ is said to have the Huygens property for $a < t < b$ if and only if $u(x, t) \in H$ there and for every $t, t', a < t' < t < b,$

$$(3.1) \quad u(x, t) = \int_0^\infty G(x, y; t - t') u(y, t') d\mu(y),$$

the integral converging absolutely. We denote the class of all functions with the Huygens property by H^* . Functions of class H^* have a complex integral representation as given in the following result.

LEMMA 3.1. *If $u(x, t) \in H^*, a < t < b,$ then for $a < t < t' < b,$*

$$(3.2) \quad u(x, t) = \int_0^\infty G(ix, y; t - t')u(iy, t')d\mu(y) .$$

The fact that $P_{n,\nu}(x, t) \in H^*$ for $-\infty < t < \infty$, and $W_{n,\nu}(x, t) \in H^*$ for $0 < t < \infty$ enables us to conclude that certain integrals involving functions of H^* are constant. A general result was proved in [5], but we state here the specific forms required in this theory.

THEOREM 3.2. *If $u(x, -t) \in H^*$ for $0 < t < \infty$, then*

$$(3.3) \quad \int_0^\infty u(x, -t) W_{n,\nu}(x, t)d\mu(x)$$

is a constant.

THEOREM 3.3. *If $u(x, t) \in H^*$ for $0 < t < \infty$, then*

$$(3.4) \quad \int_0^\infty u(ix, t) W_{n,\nu}(x, t)d\mu(x)$$

is a constant.

THEOREM 3.4. *If $u(x, t) \in H^*$ for $0 < t < \infty$, then*

$$(3.5) \quad \int_0^\infty u(x, t) P_{n,\nu}(x, -t)d\mu(x)$$

is a constant.

4. L^2 expansions. We establish criteria for a function $u(x, t)$ so that the series $\sum_{n=0}^\infty a_n P_{n,\nu}(x, -t)$ converges in mean, with weight functions $G(x, -t)$, to $u(x, t)$.

THEOREM 4.1. *Let $u(x, t) \in H^*$ for $-\sigma \leq t < 0$, and*

$$u(x, t)[G(x, -t)]^{1/2} \in L^2$$

for $-\sigma \leq t < 0, 0 \leq x < \infty$. Then, for $-\sigma \leq t < 0$,

$$(4.1) \quad \lim_{N \rightarrow \infty} \int_0^\infty G(x, -t) \left| u(x, t) - \sum_{n=0}^N a_n P_{n,\nu}(x, -t) \right|^2 d\mu(x) = 0$$

and

$$(4.2) \quad \int_0^\infty G(x, -t) |u(x, t)|^2 d\mu(x) = \sum_{n=0}^\infty \frac{|a_n|^2}{b_n} t^{2n} ,$$

where b_n is given by (1.8) and

$$(4.3) \quad a_n = b_n \int_0^\infty u(y, t) W_{n,\nu}(y, -t)d\mu(y) .$$

Proof. For t fixed, let $\phi(x, t)$ be a continuous function vanishing outside a finite interval and such that, for $\varepsilon > 0$,

$$(4.4) \quad \int_0^\infty |u(x, -t)[G(x, t)]^{1/2} - \phi(x, t)|^2 d\mu(x) < \varepsilon, \quad 0 < t \leq \sigma.$$

Now set

$$(4.5) \quad \psi_n(x, t) = P_{n,\nu}(x, -t)[G(x, t)]^{1/2}, \quad 0 < t \leq \sigma.$$

Then, by (2.1), we have

$$(4.6) \quad \mathcal{K}(x, y; s, t) = \sum_{n=0}^\infty b_n t^{-2n} \psi_n(x, t) \psi_n(y, s),$$

where b_n is defined by (1.8). Hence

$$\begin{aligned} \int_0^\infty \mathcal{K}(x, y; s, t) \phi(y, t) d\mu(y) &= \int_0^\infty \phi(y, t) d\mu(y) \sum_{n=0}^\infty b_n t^{-2n} \psi_n(x, t) \psi_n(y, s) \\ &= \sum_{n=0}^\infty b_n t^{-2n} \psi_n(x, t) \int_0^\infty \psi_n(y, s) \phi(y, t) d\mu(y). \end{aligned}$$

If we set

$$(4.7) \quad A_n(t) = b_n t^{-2n} \int_0^\infty \psi_n(y, t) \phi(y, t) d\mu(y),$$

and apply Theorem 2.2, we find that

$$(4.8) \quad \sum_{n=0}^\infty A_n(t) \psi_n(x, t) = \lim_{s \rightarrow t^-} \int_0^\infty \mathcal{K}(x, y; s, t) \phi(y, t) d\mu(y) = \phi(x, t).$$

If we multiply both sides of (4.8) by $\phi(x, t) d\mu(x)$ and integrate between 0 and ∞ , we obtain

$$\sum_{n=0}^\infty A_n(t) \int_0^\infty \psi_n(x, t) \phi(x, t) d\mu(x) = \int_0^\infty \phi^2(x, t) d\mu(x),$$

or, by (4.7),

$$(4.9) \quad \sum_{n=0}^\infty \frac{t^{2n}}{b_n} A_n^2(t) = \int_0^\infty \phi^2(x, t) d\mu(x).$$

Now, let

$$(4.10) \quad c_n(t) = b_n t^{-2n} \int_0^\infty u(y, -t)[G(y, t)]^{1/2} \psi_n(y, t) d\mu(y).$$

Consider

$$(4.11) \quad I = \int_0^\infty \left\{ u(x, -t)[G(x, t)]^{1/2} - \sum_{k=0}^n c_k(t) \psi_k(x, t) \right\}^2 d\mu(x).$$

Since, by 1.7, we have

$$(4.12) \quad \int_0^\infty \psi_n(x, t) \psi_m(x, t) d\mu(x) = \frac{t^{2n}}{b_n} \delta_{mn} ,$$

with b_n given in (1.8), it follows that

$$\begin{aligned} I &= \int_0^\infty [u(x, -t)]^2 G(x, t) d\mu(x) - \sum_{k=0}^n c_k^2(t) \frac{t^{2k}}{b_k} \\ &\leq \int_0^\infty [u(x, -t)]^2 G(x, t) d\mu(x) + \sum_{k=0}^n \frac{t^{2k}}{b_k} [A_k(t) - c_k(t)]^2 - \sum_{k=0}^n c_k^2(t) \frac{t^{2k}}{b_k} \\ &= \int_0^\infty [u(x, -t)]^2 G(x, t) d\mu(x) + \sum_{k=0}^n \frac{t^{2k}}{b_k} A_k^2(t) - 2 \sum_{k=0}^n \frac{t^{2k}}{b_k} A_k(t) c_k(t) \\ &= \int_0^\infty \left\{ u(x, -t) [G(x, t)]^{1/2} - \sum_{k=0}^n A_k(t) \psi_k(x, t) \right\}^2 d\mu(x) \\ &\leq 2 \int_0^\infty \{ u(x, -t) [G(x, t)]^{1/2} - \phi(x, t) \}^2 d\mu(x) \\ &\quad + 2 \int_0^\infty \left\{ \phi(x, t) - \sum_{k=0}^n A_k(t) \psi_k(x, t) \right\}^2 d\mu(x) . \end{aligned}$$

By (4.4), we have

$$\begin{aligned} I &< 2\varepsilon + 2 \int_0^\infty \phi^2(x, t) d\mu(x) + 2 \int_0^\infty \sum_{k=0}^n A_k^2(t) \psi_k^2(x, t) d\mu(x) \\ &\quad - 4 \int_0^\infty \phi(x, t) d\mu(x) \sum_{k=0}^n A_k(t) \psi_k(x, t) \\ &< 2\varepsilon + 2 \int_0^\infty \phi^2(x, t) d\mu(x) + 2 \sum_{k=0}^n A_k^2(t) \frac{t^{2n}}{b_n} \\ &\quad - 4 \sum_{k=0}^n A_k(t) \int_0^\infty \phi(x, t) \psi_k(x, t) d\mu(x) \\ &< 2\varepsilon + 2 \left\{ \int_0^\infty \phi^2(x, t) d\mu(x) - \sum_{k=0}^n A_k^2(t) \frac{t^{2n}}{b_n} \right\} . \end{aligned}$$

It follows, therefore, by (4.9), that if n is sufficiently large, $I < 4\varepsilon$.

Hence

$$(4.13) \quad \lim_{N \rightarrow \infty} \int_0^\infty \left| u(x, -t) [G(x, t)]^{1/2} - \sum_{k=0}^N c_k(t) \psi_k(x, t) \right|^2 d\mu(x) = 0 ,$$

or, by (4.5), we have (4.1) with $c_k(t) = a_k$. Theorem 3.4 establishes the fact that a_k is independent of t .

Parseval's equation (4.2) follows since

$$\begin{aligned} \int_0^\infty G(x, t) |u(x, -t)|^2 d\mu(x) &= \int_0^\infty \left| \sum_{n=0}^\infty c_n(t) \psi_n(x, t) \right|^2 d\mu(x) \\ &= \sum_{n=0}^\infty |a_n|^2 \frac{t^{2n}}{b_n} , \end{aligned}$$

with the last equality a result of (4.12).

An example illustrating the theorem is given by $u(x, t) = e^{a^2 t} \mathcal{F}(ax)$. This function satisfies the hypotheses for $-\infty < t < 0$ and we find that

$$(4.14) \quad \int_0^\infty G(x, t) \mathcal{F}^2(ax) e^{-2a^2 t} d\mu(x) = \mathcal{F}(2a^2 t), \quad 0 < t < \infty,$$

whereas

$$(4.15) \quad \sum_{n=0}^\infty |a_n|^2 \frac{t^{2n}}{b_n} = \sum_{n=0}^\infty b_n (a^2 t)^{2n} 2^{4n} \\ = \mathcal{F}(2a^2 t), \quad 0 < t < \infty,$$

since

$$(4.16) \quad a_n = b_n \int_0^\infty e^{-a^2 t} \mathcal{F}(ay) W_{n,\nu}(y, t) d\mu(y), \quad 0 < t < \infty \\ = (2a)^{2n} b_n.$$

Although, in this example, $u(x, t) \in H^*$ for $-\infty < t < \infty$, the expansion (4.1) does not hold in the extended strip. Note that, in this case, the requirement that $u(x, t)[G(x, -t)]^{1/2}$ be in L^2 fails for $0 < t < \infty$. A modification of Theorem 4.1 when $u(x, t) \in H^*$ for $0 < t \leq \sigma$ is given by the following result.

THEOREM 4.2. *If $u(x, t) \in H^*$ for $0 < t \leq \sigma$, and if*

$$u(ix, t)[G(x, t)]^{1/2} \in L^2$$

for each fixed t , $0 < t \leq \infty$, $0 \leq x < \infty$, then for $0 < t \leq \sigma$,

$$(4.17) \quad \lim_{N \rightarrow \infty} \int_0^\infty G(x, t) \left| u(ix, t) - \sum_{n=0}^N a_n P_{n,\nu}(x, -t) \right|^2 d\mu(x) = 0,$$

and

$$(4.18) \quad \int_0^\infty G(x, t) |u(ix, t)|^2 d\mu(x) = \sum_{n=0}^\infty \frac{t^{2n}}{b_n} |a_n|^2,$$

where b_n is given by (1.8) and

$$(4.19) \quad a_n = b_n \int_0^\infty u(ix, t) W_{n,\nu}(x, t) d\mu(x), \quad 0 < t \leq \sigma.$$

Proof. As in the preceding proof, we have

$$\lim_{N \rightarrow \infty} \int_0^\infty \left| u(ix, t)[G(x, t)]^{1/2} - \sum_{n=0}^N c_n(t) \psi_n(x, t) \right|^2 d\mu(x) = 0,$$

with

$$c_n(t) = b_n t^{-2n} \int_0^\infty u(iy, t) [G(y, t)]^{1/2} \psi_n(y, t) d\mu(y) .$$

Hence (4.17) holds with $c_n(t) = a_n$, which, by Theorem 3.5, is independent of t . Further,

$$\begin{aligned} \int_0^\infty G(x, t) |u(ix, t)|^2 d\mu(x) &= \int_0^\infty \left| \sum_{n=0}^\infty c_n(t) \psi_n(x, t) \right|^2 d\mu(x) \\ &= \sum_{n=0}^\infty \frac{t^{2n}}{b_n} |a_n|^2 \end{aligned}$$

which is the Parseval equation (4.18).

The example of the preceding theorem satisfies these hypotheses for $0 < t < \infty$, and we have, for $0 < t < \infty$,

$$\int_0^\infty G(x, t) e^{2a^2 t} \mathcal{F}^2(ix) d\mu(x) = \mathcal{F}(2a^2 t) ,$$

whereas

$$a_n = b_n \int_0^\infty e^{a^2 t} \mathcal{F}(ix) W_{n,\nu}(x, t) d\mu(x) ,$$

so that

$$\sum_{n=0}^\infty \frac{t^{2n}}{b_n} |a_n|^2 = \mathcal{F}(2a^2 t) .$$

Criteria for expansions in terms of $W_{n,\nu}(x, t)$ are given in the following result.

THEOREM 4.3. *If $u(x, t) \in H^*$ for $0 < \sigma \leq t$, and if*

$$u(x, t) [G(ix, t)]^{1/2} \in L^2$$

for each fixed t , $0 \leq \sigma < t$, $0 \leq x < \infty$, then for $0 < \sigma \leq t$,

$$(4.20) \quad \lim_{N \rightarrow \infty} \int_0^\infty G(ix, t) \left| u(x, t) - \sum_{n=0}^N a_n W_{n,\nu}(x, t) \right|^2 d\mu(x) = 0 ,$$

and

$$(4.21) \quad \int_0^\infty G(ix, t) |u(x, t)|^2 d\mu(x) = \sum_{n=0}^\infty \frac{t^{-2n}}{b_n} (2t)^{-2\nu-1} |a_n|^2 ,$$

where b_n is given by (1.8) and

$$(4.22) \quad a_n = b_n \int_0^\infty u(x, t) P_{n,\nu}(x, -t) d\mu(x) \quad \sigma \leq t < \infty ,$$

Proof. Again, as in Theorem 4.1, since $u(x, t) [G(ix, t)]^{1/2} \in L^2$, we have

$$(4.23) \quad \lim_{N \rightarrow \infty} \int_0^\infty \left| u(x, t) [G(ix, t)]^{1/2} - \sum_{n=0}^N c_n(t) \psi_n(x, t) \right|^2 d\mu(x) = 0,$$

with

$$(4.24) \quad c_n(t) = b_n t^{-2n} \int_0^\infty u(x, t) [G(ix, t)]^{1/2} \psi_n(x, t) d\mu(x).$$

Now, (4.23) can be written in the form

$$\lim_{N \rightarrow \infty} \int_0^\infty G(ix, t) \left| u(x, t) - \sum_{n=0}^N c_n(t) (2t)^{\nu+(1/2)} t^{2n} W_{n,\nu}(x, t) \right|^2 d\mu(x) = 0,$$

with (4.24) becoming

$$c_n(t) = b_n t^{-2n} (2t)^{-\nu-(1/2)} \int_0^\infty u(x, t) P_{n,\nu}(x, -t) d\mu(x).$$

Hence, if we set $a_n = c_n(t) t^{2n} (2t)^{\nu+(1/2)}$, a_n is independent of t , by Theorem 3.6, and (4.20) is established. Moreover, Parseval's formula is

$$\begin{aligned} \int_0^a u(ix, t) |u(x, t)|^2 d\mu(x) &= \sum_{n=0}^\infty |c_n^2(t)|^2 \frac{t^{2n}}{b_n} \\ &= \sum_{n=0}^\infty t^{-2n} (2t)^{-2\nu-1} \frac{|a_n|^2}{b_n}. \end{aligned}$$

Note that the function $u(x, t) = G(x, k; t)$ satisfies the conditions of the theorem for $0 < t < \infty$. In this case, we have

$$a_n = b_n k^{2n},$$

and hence

$$\sum_{n=0}^\infty t^{-2n} (2t)^{-2\nu-1} \frac{|a_n|^2}{b_n} = \left(\frac{1}{2t} \right)^{2\nu+1} \mathcal{J} \left(\frac{k^2}{2t} \right),$$

whereas

$$\int_0^\infty G(ix; t) |G(x, k; t)|^2 d\mu(x) = \left(\frac{1}{2t} \right)^{2\nu+1} \mathcal{J} \left(\frac{k^2}{2t} \right).$$

BIBLIOGRAPHY

1. F. M. Cholewinski and D. T. Haimo, *The Weierstrass-Hankel convolution transform*, J. d'Analyse Math. (to appear).
2. A. Erdelyi et al., *Higher transcendental functions*, vol. II, 1953.
3. D. T. Haimo, *Expansions in terms of generalized heat polynomials and of their Appell transforms*, J. Math. Mech. (to appear).
4. ———, *Integral equations associated with Hankel convolutions*, Trans. Amer. Math. Soc. (to appear).
5. ———, *Generalized temperature functions*, Duke Math. J. (to appear).
6. ———, *Functions with the Huygens property*, Bull. Amer. Math. Soc. **71** (1965), 528-532.

7. I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, J. d'Analyse Math. **8** (1960-61), 307-336.
8. P. C. Rosenbloom and D. V. Widder, *Expansions in terms of heat polynomials and associated functions*, Trans. Amer. Math. Soc. **92** (1959), 220-266.
9. E. C. Titchmarsh, *The theory of Fourier integrals*, 1937.
10. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge, 1958.

SOUTHERN ILLINOIS UNIVERSITY AND HARVARD UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California

R. M. BLUMENTHAL

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California
Los Angeles, California 90007

*RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Basil Gordon, Acting Managing Editor until February 1, 1966.

David R. Arterburn and Robert James Whitley, <i>Projections in the space of bounded linear operators</i>	739
Robert McCallum Blumenthal, Joram Lindenstrauss and Robert Ralph Phelps, <i>Extreme operators into $C(K)$</i>	747
L. Carlitz, <i>A note on multiple exponential sums</i>	757
Joseph A. Cima, <i>A nonnormal Blaschke-quotient</i>	767
Paul Civin and Bertram Yood, <i>Lie and Jordan structures in Banach algebras</i>	775
Luther Elic Claborn, <i>Dedekind domains: Overrings and semi-prime elements</i>	799
Luther Elic Claborn, <i>Note generalizing a result of Samuel's</i>	805
George Bernard Dantzig, E. Eisenberg and Richard Warren Cottle, <i>Symmetric dual nonlinear programs</i>	809
Philip J. Davis, <i>Simple quadratures in the complex plane</i>	813
Edward Richard Fadell, <i>On a coincidence theorem of F. B. Fuller</i>	825
Delbert Ray Fulkerson and Oliver Gross, <i>Incidence matrices and interval graphs</i>	835
Larry Charles Grove, <i>Tensor products over H^*-algebras</i>	857
Deborah Tepper Haimo, <i>L^2 expansions in terms of generalized heat polynomials and of their Appell transforms</i>	865
I. Martin (Irving) Isaacs and Donald Steven Passman, <i>A characterization of groups in terms of the degrees of their characters</i>	877
Donald Gordon James, <i>Integral invariants for vectors over local fields</i>	905
Fred Krakowski, <i>A remark on the lemma of Gauss</i>	917
Marvin David Marcus and H. Minc, <i>A subdeterminant inequality</i>	921
Kevin Mor McCrimmon, <i>Norms and noncommutative Jordan algebras</i>	925
Donald Earl Myers, <i>Topologies for Laplace transform spaces</i>	957
Olav Njstad, <i>On some classes of nearly open sets</i>	961
Milton Philip Olson, <i>A characterization of conditional probability</i>	971
Barbara Osofsky, <i>A counter-example to a lemma of Skornjakov</i>	985
Sidney Charles Port, <i>Ratio limit theorems for Markov chains</i>	989
George A. Reid, <i>A generalisation of W^*-algebras</i>	1019
Robert Wells Ritchie, <i>Classes of recursive functions based on Ackermann's function</i>	1027
Thomas Lawrence Sherman, <i>Properties of solutions of nth order linear differential equations</i>	1045
Ernst Snapper, <i>Inflation and deflation for all dimensions</i>	1061
Kondagunta Sundaresan, <i>On the strict and uniform convexity of certain Banach spaces</i>	1083
Frank J. Wagner, <i>Maximal convex filters in a locally convex space</i>	1087
Joseph Albert Wolf, <i>Translation-invariant function algebras on compact groups</i>	1093