A SUBDETERMINANT INEQUALITY

Marvin David Marcus and H. Minc
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Let $A$ be an $n$-square positive semi-definite hermitian matrix and let $D_m(A)$ denote the maximum of all order $m$ principal subdeterminants of $A$. In this note we prove the inequality

$$(D_m(A))^{1/m} \geq (D_{m+1}(A))^{1/(m+1)}, \quad m = 1, \ldots, n - 1,$$

and discuss in detail the case of equality. This result is closely related to Newton’s and Szász’s inequalities.

Let $A = (a_{ij})$ be an $n$-square positive semi-definite hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. We introduce some notation. For $1 \leq m \leq n$ let $Q_{m,n}$ denote the set of all $\binom{n}{m}$ sequences $\omega = (\omega_1, \cdots, \omega_m)$, $1 \leq \omega_1 < \omega_2 < \cdots < \omega_m \leq n$. Let $A[\omega | \omega]$ denote the $m$-square submatrix of $A$ whose $(i,j)$ entry is $a_{\omega_i \omega_j}$, $i,j = 1, \cdots, m$.

**THEOREM.** If $A$ is a positive semi-definite hermitian matrix then

$$\max_{\alpha \in Q_{m,n}} (\det (A[\alpha | \alpha]))^{1/m} \geq \max_{\omega \in Q_{m+1,n}} (\det (A[\omega | \omega]))^{1/(m+1)}, \quad m = 1, \ldots, n - 1.$$  

Equality holds for a given $m$ if and only if either $A$ has rank less than $m$ or $A[\omega^0 | \omega^0]$ is a multiple of the identity, where the sequence $\omega^0 \in Q_{m+1,n}$ is one that satisfies

$$\det (A[\omega^0 | \omega^0]) = \max_{\omega \in Q_{m+1,n}} \det A[\omega | \omega].$$

There are two classical results that are closely related to the inequalities (1). These are Szász’s inequalities and the Newton inequalities. Szász proved that [1, p. 119]

$$(\prod_{\alpha \in Q_{m,n}} (\det (A[\alpha | \alpha]))^{1/(m)})^{1/m} \geq \left( \prod_{\omega \in Q_{m+1,n}} (\det (A[\omega | \omega]))^{1/(m+1)} \right)^{1/(m+1)}.$$  

Newton’s inequalities [1, p. 106] state that if $E_m(A)$ is the $m$th elementary symmetric function of the nonnegative numbers $\lambda_1, \cdots, \lambda_n$ then

Received August 27, 1964.

1 The research of this author was supported by N. S. F. Grant G. P. 1085.

2 The research of this author was supported by U. S. Air Force Grant No. AF-AFOSR-432-63.
However, 

\[ E_m(A) = \sum_{\alpha \in \mathcal{P}_{m,n}} \det(A[\alpha | \alpha]) \]

and hence (4) can be written

\[
\left( \sum_{\alpha \in \mathcal{P}_{m,n}} \det(A[\alpha | \alpha]) \right)^{1/m} \geq \left( E_{m+1}(A) \right)^{1/(m+1)}.
\]

Notice that (3) compares the geometric mean of the principal subdeterminants of order \( m \) with the geometric mean of the principal subdeterminants of order \( m + 1 \). Also (6) makes the same kind of comparison for the arithmetic means of these quantities. The result (1) compares the maxima of the two sets of subdeterminants.

To prove the theorem we state and prove a preliminary lemma.

**Lemmas.** If \( A \) is a positive semi-definite \( n \)-square hermitian matrix then

\[
\max_{\alpha \in \mathcal{P}_{m,n}} \det(A[\alpha | \alpha]) \geq (\det(A))^{m/n}, \quad 1 \leq m \leq n.
\]

Equality holds if and only if either the rank of \( A \) is less than \( m \) or \( A \) is a multiple of the identity matrix.

**Proof.** We use some properties of the compound matrix of \( A \), denoted by \( C_m(A) \). The essential facts concerning \( C_m(A) \) are [1, pp. 17, 24, 70]:

1. \( \det(C_m(A)) = (\det(A))^{(n-1)/m} \) (Sylvester-Franke theorem);
2. if \( A \) is positive semi-definite hermitian, so is \( C_m(A) \);
3. the characteristic roots of \( C_m(A) \) are the \( \binom{n}{m} \) products
   \[
   \prod_{i=1}^{m} \lambda_{\omega_i}, \quad \omega \in \mathcal{Q}_{m,n}.
   \]

We want to prove that

\[
\max_{\alpha \in \mathcal{P}_{m,n}} \det(A[\alpha | \alpha]) \geq (\det(A))^{m/n}.
\]

If we apply the Hadamard determinant theorem [1, p. 114] to \( C_m(A) \) then we conclude from (i)

\[
\prod_{\alpha \in \mathcal{P}_{m,n}} \det(A[\alpha | \alpha]) \geq \det(C_m(A)) = (\det(A))^{(n-1)/m}.
\]
If for every \( \alpha \in Q_{m,n} \), \( \det (A[\alpha | \alpha]) \) were strictly less than \( (\det (A))^{m/n} \) then from (9) we would conclude that

\[
\text{(10)} \quad (\det (A))^{(n-1)/(m-1)} < ((\det (A))^{m/n})^{n} = (\det (A))^{(n-1)/(m-1)},
\]
a contradiction. Thus (8) holds. If (8) were equality suppose first that not all \( \det (A[\alpha | \alpha]), \alpha \in Q_{m,n} \) are equal. Then from (9) we would obtain the same contradiction (10). Thus for equality to hold in (8)

\[
\det (A[\alpha | \alpha]) = (\det (A))^{m/n}
\]
for all \( \alpha \in Q_{m,n} \). This means that all the main diagonal elements of \( C_m(A) \) are equal. If this common value is 0 then \( A \) has rank at most \( m - 1 \). If the common value is nonzero then (9) is equality throughout and as we know from the case of equality in the Hadamard determinant theorem \( C_m(A) \) is a multiple of the identity. Thus from (iii) we know that the characteristic roots

\[
\prod_{i=1}^{m} \lambda_{\alpha,i}, \quad \alpha \in Q_{m,n}, \quad m < n,
\]
are equal. But then it follows that \( \lambda_1 = \cdots = \lambda_n \) and hence \( A \) is a multiple of the identity, completing the proof of the lemma.

To prove the inequality (1) we apply the lemma to submatrices. Let \( \omega^0 \) be a sequence in \( Q_{m+1,n} \) for which

\[
\text{(11)} \quad \det (A[\omega^0 | \omega^0]) = \max_{\omega \in Q_{m+1,n}} \det (A[\omega | \omega]) .
\]

For \( \alpha \in Q_{m,n} \) and \( \alpha \) a subsequence of \( \omega^0 \), i.e., \( \alpha \subset \omega^0 \), we know that \( A[\alpha | \alpha] \) is an \( m \)-square submatrix of \( A[\omega^0 | \omega^0] \). Hence, by the lemma,

\[
\text{(12)} \quad \max_{\alpha \in Q_{m,n} \cdot \alpha \subset \omega^0} \det (A[\alpha | \alpha]) \geq (\det (A[\omega^0 | \omega^0]))^{m/(m+1)} .
\]

Thus a fortiori

\[
\text{(13)} \quad \max_{\alpha \in Q_{m,n}} \det (A[\alpha | \alpha]) \geq (\det (A[\omega^0 | \omega^0]))^{m/(m+1)} .
\]

Applying (11) we obtain the inequality (1) from (13).

If equality holds in (1) then (12) must be equality as well. Therefore either the rank of \( A[\omega^0 | \omega^0] \) is less than \( m \) or \( A[\omega^0 | \omega^0] \) is a multiple of the \((m + 1)\)-square identity matrix. If the former is the case then \( \det (A[\omega^0 | \omega^0]) = 0 \) and hence, since (13) is equality, every \( m \)th order principal subdeterminant of \( A \) is 0. Thus the rank of \( A \) is less than \( m \).
REFERENCE


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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Basil Gordon, Acting Managing Editor until February 1, 1966.
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