

Pacific Journal of Mathematics

TOPOLOGIES FOR LAPLACE TRANSFORM SPACES

DONALD EARL MYERS

TOPOLOGIES FOR LAPLACE TRANSFORM SPACES

DONALD E. MYERS

In this paper four topologies are compared:

- (i) an L_2 -type topology on the space of functions having bilateral transforms,
- (ii) an L_1 and (iii) an L_2 -type topology on the space of transforms, and
- (iv) finally that of one form of convergence of compact subsets for the space of analytic functions. It is shown that sequential convergence in (i) implies (iii) and (iv) and (ii) implies (i) and (iv) and hence (iii).

In an earlier paper [2]; the author used equivalence classes of analytic functions to construct an imbedding space for Schwartz Distributions. The mechanism for constructing the mapping was the bilateral Laplace Transform, in this way the traditional approach to operational calculus was preserved. In that paper a topology was imposed on the imbedding space from the space of analytic functions. We now obtain some additional results about the possible topologies defined on the space of analytic functions.

THEOREM 1. *Let $F_j(t)$, $j = 0, 1, 2, \dots$, $-\infty < t < \infty$ be real valued functions such that for each j*

$$d_1(F_j) = \left[\int_0^\infty |e^{-\sigma_1 t} F_j(t)|^2 dt \right]^{1/2} < \infty$$

and

$$d_2(F_j) = \left[- \int_0^{-\infty} |e^{-\sigma_2 t} F_j(t)|^2 dt \right]^{1/2} < \infty$$

where $-\infty < \sigma_1 < \sigma_2 < \infty$. If

$$d(F_j - F_0) = d_1(F_j - F_0) + d_2(F_j - F_0) \rightarrow 0$$

as $j \rightarrow \infty$ then

- (i) $f_j(z) \rightarrow f_0(z)$ uniformly on compact subsets of $\sigma_1 < R(z) < \sigma_2$ where $f_j(z) = \int_{-\infty}^\infty e^{-zt} F_j(t) dt$
- (ii) $\|f_j - f_0\|_x \rightarrow 0$

where

Received April 24, 1964. This research was partially supported by NSF Contract Grant NSF GP-1951.

$$\|f_j\|_x = \left[\int_{-\infty}^{\infty} |f_j(x + iy)|^2 dy \right]^{1/2}$$

for each

$$\sigma_1 < x < \sigma_2 .$$

Proof. From [3], pp 245, $d_1 < \infty, d_2 < \infty$ implies the existence of $f_j(z)$ with the integral defining $f_j(z)$ converging absolutely for $\sigma_1 < R(z) < \sigma_2$. Rewrite this integral as

$$f_j(z) = \int_{-\infty}^0 e^{-t(z-\sigma_2)} e^{-\sigma_2 t} F_j(t) dt + \int_0^{\infty} e^{-t(z-\sigma_1)} e^{-t\sigma_1} F_j(t) dt .$$

By the Cauchy-Schwarz Inequality

$$|f_j(z)| \leq \left[\frac{1}{2[\sigma_2 - R(z)]} \int_{-\infty}^0 |e^{-\sigma_2 t} F_j(t)|^2 dt \right]^{1/2} + \left[\frac{1}{2[R(z) - \sigma_1]} \int_0^{\infty} |e^{-\sigma_1 t} F_j(t)|^2 dt \right]^{1/2} .$$

Let

$$g_K(z) = \max_{z \in K} \left[\sqrt{\frac{1}{2[R(z) - \sigma_1]}} , \sqrt{\frac{1}{2[\sigma_2 - R(z)]}} \right]$$

for K an arbitrary subset of $-\sigma_1 < R(z) < \sigma_2$. Now

$$\max_{z \in K} |f_j(z) - f_0(z)| \leq g_K(z) d(F_j - F_0)$$

and (i) is established.

To prove (ii) we use an additional consequence of the theorem from [3], pp 254, namely for each j

$$\int_{-\infty}^{\infty} |e^{-xt} F_j(t)|^2 dt = \int_{-\infty}^{\infty} |f_j(x + iy)|^2 dt$$

for $\sigma_1 < x < \sigma_2$. Since

$$\int_0^{\infty} |e^{-xt} F_j(t)|^2 dt \leq [d_1(F_j)]^2$$

$$\int_{-\infty}^0 |e^{-xt} F_j(t)|^2 dt \leq [d_2(F_j)]^2$$

it follows that

$$\|f_j - f_0\|_x^2 \leq [d_1(F_j - F_0)]^2 + [d_2(F_j - F_0)]^2 .$$

By definition $d_1(F_j) \geq 0, d_2(F_j) \geq 0$, therefore $d(F_j - F_0) \rightarrow 0$ implies $d_1(F_j - F_0) \rightarrow 0$ and $d_2(F_j - F_0) \rightarrow 0$ and (ii) is proved.

Although it is clear that $d(F)$ is a norm on the set of functions with $d < \infty$ and $\|f\|_x$ is a norm on a subset of the functions analytic for $\sigma_1 < R(z) < \sigma_2$, the topology of uniform convergence on compact sets gives only a countably normed space or a Fréchet metric. (See [1], pg. 139.)

THEOREM 2. *Let $f_j(z)$, $j = 0, 1, 2, \dots$ be functions analytic for $\sigma_1 < R(z) < \sigma_2$ such that for each j*

$$\int_{-\infty}^{\infty} |f_j(x + iy)| dy < \infty, \quad \sigma_1 < x < \sigma_2$$

and $\lim_{|y| \rightarrow \infty} f_j(x + iy) = 0$ uniformly on closed subintervals of $\sigma_1 < x < \sigma_2$. If $\|f_j - f\|_x \rightarrow 0$ as $j \rightarrow \infty$ for each $\sigma_1 < x < \sigma_2$. then

(i) $d'(F_j - F_0) \rightarrow 0$ as $j \rightarrow \infty$ where

$$d'(F_j - F_0) = d'_1(F_j - F_0) + d'_2(F_j - F_0)$$

and

$$(1) \quad F_j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{xt} e^{ity} f_j(x + iy) dy,$$

$\sigma_1 < x < \sigma_2$. (The prime denotes that the metric is the same as in Theorem 1, except that σ_1 is replaced by x_1 and σ_2 by x_2 where $\sigma_1 < x_1 < x_2 < \sigma_2$.)

(ii) $f_j(z) \rightarrow f_0(z)$ uniformly on compact subsets of $x_1 < R(z) < x_2$.

Proof. From [3], pg. 265 the hypotheses of the theorem are sufficient to insure that each $f_i(z)$ is a bilateral transform and also the validity of the inversion formula (1).

Since $|f_j(x + iy)| \rightarrow 0$ as $|y| \rightarrow \infty$ each $|f_j(x + iy)|$ is bounded for $\sigma_1 < x < \sigma_2$ if $|f_j(x + iy)| \leq M_j(x)$ then

$$\int_{-\infty}^{\infty} |f_j(x + iy)|^2 dy \leq M_j(x) \int_{-\infty}^{\infty} |f_j(x + iy)| dy < \infty.$$

Consider, for $\sigma_1 < x < x_1$

$$\begin{aligned} \int_0^{\infty} |e^{x_1 t} F_j(t)|^2 dt &= \frac{1}{(2\pi)^2} \int_0^{\infty} \left| e^{-t(x_1-x)} \int_{-\infty}^{\infty} e^{ity} f_j(x + iy) dy \right|^2 dt \\ &\leq \frac{1}{(2\pi)^2} \|f_j\|_x^2 \int_0^{\infty} e^{-2t(x_1-x)} dt \end{aligned}$$

but $0 < x_1 - x$ so that

$$C_1 = \frac{1}{2\pi} \int_0^{\infty} e^{-2t(x_1-x)} dt < \infty$$

and $d'_1(F_j) \leq C_1 \|f_j\|_x$. Likewise

$$\int_{-\infty}^0 |e^{-x_2 t} F_j(t)|^2 dt \leq \frac{1}{(2\pi)^2} \|f_j\|_{x'}^2 \int_{-\infty}^0 e^{t(x'-x_2)} dt$$

for $x_2 < x' < \sigma_2$, or

$$d'_2(F_j) \leq C_2 \|f_j\|_{x'}, \quad \text{where} \quad C_2 = \frac{1}{2\pi} \int_{-\infty}^0 e^{t(x'-x_2)} dt.$$

If $\|f_j - f_0\|_x \rightarrow 0$ as $j \rightarrow \infty$ for all $\sigma_1 < x < \sigma_2$ then

$$\begin{aligned} d'(F_j - F_0) &= d'_1(F_j - F_0) + d'_2(F_j - F_0) \\ &\leq C_1 \|f_j - f_0\|_x + C_2 \|f_j - f_0\|_{x'} \end{aligned}$$

or $d'(F_j - F_0) \rightarrow 0$.

Part (ii) follows by applying Theorem 1 to part (i).

In a subsequent paper the author expects to extend these to multidimensional Laplace Transforms and to Laplace Transforms on Locally Compact Abelian Groups.

REFERENCES

1. W. Thron, *Introduction to the Theory of Functions of a Complex Variable*, John Wiley, New York, 1953.
2. D. E. Myers, *An imbedding space for Schwartz distributions*, Pacific J. Math. **11** (1961), 1467-1477.
3. D. V. Widder, *The Laplace Transform*, Princeton, 1946.

UNIVERSITY OF ARIZONA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California

R. M. BLUMENTHAL

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California
Los Angeles, California 90007

*RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Basil Gordon, Acting Managing Editor until February 1, 1966.

David R. Arterburn and Robert James Whitley, <i>Projections in the space of bounded linear operators</i>	739
Robert McCallum Blumenthal, Joram Lindenstrauss and Robert Ralph Phelps, <i>Extreme operators into $C(K)$</i>	747
L. Carlitz, <i>A note on multiple exponential sums</i>	757
Joseph A. Cima, <i>A nonnormal Blaschke-quotient</i>	767
Paul Civin and Bertram Yood, <i>Lie and Jordan structures in Banach algebras</i>	775
Luther Elic Claborn, <i>Dedekind domains: Overrings and semi-prime elements</i>	799
Luther Elic Claborn, <i>Note generalizing a result of Samuel's</i>	805
George Bernard Dantzig, E. Eisenberg and Richard Warren Cottle, <i>Symmetric dual nonlinear programs</i>	809
Philip J. Davis, <i>Simple quadratures in the complex plane</i>	813
Edward Richard Fadell, <i>On a coincidence theorem of F. B. Fuller</i>	825
Delbert Ray Fulkerson and Oliver Gross, <i>Incidence matrices and interval graphs</i>	835
Larry Charles Grove, <i>Tensor products over H^*-algebras</i>	857
Deborah Tepper Haimo, <i>L^2 expansions in terms of generalized heat polynomials and of their Appell transforms</i>	865
I. Martin (Irving) Isaacs and Donald Steven Passman, <i>A characterization of groups in terms of the degrees of their characters</i>	877
Donald Gordon James, <i>Integral invariants for vectors over local fields</i>	905
Fred Krakowski, <i>A remark on the lemma of Gauss</i>	917
Marvin David Marcus and H. Minc, <i>A subdeterminant inequality</i>	921
Kevin Mor McCrimmon, <i>Norms and noncommutative Jordan algebras</i>	925
Donald Earl Myers, <i>Topologies for Laplace transform spaces</i>	957
Olav Njstad, <i>On some classes of nearly open sets</i>	961
Milton Philip Olson, <i>A characterization of conditional probability</i>	971
Barbara Osofsky, <i>A counter-example to a lemma of Skornjakov</i>	985
Sidney Charles Port, <i>Ratio limit theorems for Markov chains</i>	989
George A. Reid, <i>A generalisation of W^*-algebras</i>	1019
Robert Wells Ritchie, <i>Classes of recursive functions based on Ackermann's function</i>	1027
Thomas Lawrence Sherman, <i>Properties of solutions of nth order linear differential equations</i>	1045
Ernst Snapper, <i>Inflation and deflation for all dimensions</i>	1061
Kondagunta Sundaresan, <i>On the strict and uniform convexity of certain Banach spaces</i>	1083
Frank J. Wagner, <i>Maximal convex filters in a locally convex space</i>	1087
Joseph Albert Wolf, <i>Translation-invariant function algebras on compact groups</i>	1093