ON THE STRICT AND UNIFORM CONVEXITY OF CERTAIN
BANACH SPACES

KONDAGUNTA SUNDARESAN
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Let \((X, S, \mu)\) be a \(\sigma\)-finite non-atomic measure space let \(N\) be a real valued continuous convex even function defined on the real line such that

1. \(N(u)\) is nondecreasing for \(u \geq 0\),
2. \(\lim_{u \to \infty} N(u)/u = \infty\),
3. \(\lim_{u \to 0} N(u)/u = 0\).

Let \(L_N\) be the set of all real valued \(\mu\)-measurable functions \(f\) such that \(\int_X N(f) d\mu < \infty\). It is known that if there exists a constant \(k\) such that \(N(2u) \leq kN(u)\) for all \(u \geq 0\) then \(L_N\) is a linear space; in fact, \(L_N\) is a B-Space if a norm \(\| \cdot \|\) is defined by setting

\[ \|f\| = \inf \left\{ 1/\zeta \mid \zeta > 0, \int_X N(\eta, f) d\mu \leq 1 \right\}. \]

Denoting the B-space \((L_N, \| \cdot \|)\) by \(L_N^*\) it is proposed to obtain the necessary and sufficient conditions in order that \(L_N^*\) may be (1) Strictly Convex (2) Uniformly Convex.

The linear space \(L_N\) admits another norm \(\| \cdot \|_{(N)}\) known as the Orlicz norm defined by setting

\[ \|f\|_{(N)} = \sup \int_X |fg| d\mu \]

for such that \(\int_X M(|g|) d\mu \leq 1\), \(M\) being the function complementary to \(N\) in the sense of Young. For a discussion of this class of Banach spaces we refer to Mazur and Orlicz [2]. Convexity properties of the Orlicz norm have been studied in Milnes [3].

The space \(L_N^*\) may be considered as a modulared linear space defined in Nakano [4]. A nonnegative extended real valued function \(m\) defined on a linear space is called a modular if

1. \(m(0) = 0\);
2. for any \(x \in L\) there exists \(\xi > 0\) such that \(m(\xi x) < \infty\);
3. \(m(\xi x) = 0\) for all \(\xi > 0\) implies \(x = 0\);
4. \(m(x) = \sup_{0 \leq \xi \leq 1} m(\xi x)\);
5. \(m\) is convex (i.e., \(\alpha \geq 0\), \(\beta \geq 0\), \(\alpha + \beta = 1\), \(x, y \in L\) imply \(m(\alpha x + \beta y) \leq \alpha m(x) + \beta m(y)\)).

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The modulared linear space may be considered as a normed linear space if a norm $\| \cdot \|$ is defined by setting

\[(**) \quad \| x \| = \inf \{1/\xi | \xi > 0 \text{ and } m(\xi x) \leq 1\}.\]

We note that the linear space $L_N$ is a modulared space if

\[m(f) = \int_x N(f) d\mu,\]

and the norm $\| \cdot \|$ defined by (**) is the same as the norm defined in $\ast$. In fact, the modulared space $L_N$ is a finite modulared space, meaning that $m(f) < \infty$, for all $f \in L_N$.

A Banach space $B$ is said to be strictly convex if $x, y \in B, \| x \| = \| y \| = \| (x + y)/2 \| = 1$ imply $x = y$. It is uniformly convex if to each $\varepsilon, 0 < \varepsilon \leq 2$, there corresponds a $\delta(\varepsilon) > 0$ such that conditions $\| x \| = \| y \| = 1, \| x - y \| \geq \varepsilon$ imply that $\| x + y \| < 2 - \delta(\varepsilon)$.

We shall start by characterizing the strict convexity of $L_N^\ast$.

**LEMMA 1.** The modulared norm defined in (**) associated with a finite modulared space is strictly convex if and only if $m(x) = m(y) = m((x + y)/2) = 1$ imply $x = y$.

The proof is an easy consequence of the fact that in a finite modulared space, $m(x) = 1$ if and only if $\| x \| = 1$ where $\| \cdot \|$ is the related modulared norm.

**THEOREM.** The Banach space $L_N^\ast$ is strictly convex if and only if the $N$-function $N$ is strictly convex; i.e.,

\[N\left(\frac{u + v}{2}\right) < \frac{1}{2} [N(u) + N(v)]\]

for all real $u, v$ such that $u \neq v$.

**Proof.** Let $N$ be a strictly convex $N$-function. Let $f, g \in L_N^\ast$ such that

\[m(f) = m(g) = m\left(\frac{f + g}{2}\right) = 1.\]

By definition of $m$ it follows that

\[\int_x \left[\frac{N(f) + N(g)}{2} - N\left(\frac{f + g}{2}\right)\right] d\mu = 0.\]

whence the convexity of $N$ together with the restrictions on $f$, and $g$ imply that $f = g$ a.e. Thus by Lemma 1, $L_N^\ast$ is strictly convex.
To prove the "only if" part, let \( L^*_X \) be strictly convex. If possible let \( N \) be not strictly convex so that there exist \( a, b \geq 0 \) \( a \neq b \) such that \( N((a + b)/2) = 1/2 [N(a) + N(b)] \). The continuity of \( N \) together with the condition \( \lim_{u \to 0} N(u)/u = 0 \) imply that \( N \) is linear on the interval \([a, b]\) and \( a \neq 0, b \neq 0 \). For \( u \in [a, b] \) let \( N(u) = pu + q \), where \( p \) and \( q \) are reals.

Since \( \mu \) is a nonatomic positive measure there exist pairwise disjoint measurable sets \( A, B, C \) of arbitrarily small measure such that \( \mu(A) = \mu(B) = \mu(C) \).

Let us define functions \( f, g \) as follows. Let \( f(x) = a \) for \( x \in A \), \( f(x) = b \) for \( x \in B \), and \( f(x) = 0 \) for all \( x \notin A \cup B \). Let \( g(x) = b \) for \( x \in A \), \( g(x) = a \) for \( x \in B \), and \( g(x) = 0 \) for \( x \notin A \cup B \), and \( g(x) = 0 \) for \( x \in A \cup B \). Then

\[
\begin{align*}
m(f) &= \int_X N(f) d\mu = [p(a + b) + 2q] \mu(A) , \\
m(g) &= \int_X N(g) d\mu = [p(a + b) + 2q] \mu(B) , \\
m\left(\frac{f + g}{2}\right) &= \frac{1}{2} [m(f) + m(g)] ,
\end{align*}
\]

and \( m(f) = m(g) = m((f + g)/2) \). By a suitable choice of \( A, B, C \) we can assume that

\[
m(f) = m(g) = m\left(\frac{f + g}{2}\right) = K < \frac{1}{2} .
\]

Now let \( h \) be a function on \( X \) defined by setting

\[
h(x) = 0 \text{ if } x \in C , \quad h(x) = t \text{ if } x \in C
\]

where \( t \) is such that \( N(t)\mu(C) = 1 - K \). Let \( f_1 = h + f \), and \( g_1 = h + g \); since \( h \wedge f = 0 \) \( h \wedge g \), we obtain

\[
m(f_1) = m(h) + m(f) = (1 - K) + K = 1 .
\]

Similarly \( m(g_1) = 1 \), and further

\[
m\left(\frac{f_1 + g_1}{2}\right) = m\left(\frac{f + g}{2} + h\right) = m\left(\frac{f + g}{2}\right) + m(h) = 1 .
\]

Thus we have \( f_1, g_1 \in L^*_X \) and \( m(f_1) = m(g_1) = m((f_1 + g_1)/2) = 1 \); however \( f_1 \neq g_1 \). Thus \( L^*_X \) is not strictly convex, a contradiction.

We next proceed to characterize the uniform convexity of \( L^*_X \).

It is known [5] that in a module\(\mathcal{S}ed\) semiordered linear space, the modular norm is uniformly convex if and only if the associated norm
is uniformly convex. The modulared linear spaces $L_N$ are modulared semiordered linear spaces under the natural pointwise ordering, and the above two norms are respectively the norms $\| \cdot \|_{(N)}$ and $\| \|_{(N)}\cdot \|$.

With this remark we conclude that the Theorem 8 in Milnes [3] which characterizes the uniform convexity of the norm $\| \|_{(N)}\cdot \|$ also characterizes the uniform convexity of the norm $\| \cdot \|_{(N)}$.

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