

Pacific Journal of Mathematics

**MAXIMAL CONVEX FILTERS IN A LOCALLY CONVEX
SPACE**

FRANK J. WAGNER

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Let $E[\mathcal{S}]$ be a locally convex space, \mathfrak{B} a saturated covering of E by bounded sets, and E' the topological dual of $E[\mathcal{S}]$. Let $\mathcal{S}_{\mathfrak{B}}$ be the topology on E' of uniform convergence on sets of \mathfrak{B} and E'' the topological dual of $E'[\mathcal{S}_{\mathfrak{B}}]$. We assume E'' has the natural topology \mathcal{S}_n —that of uniform convergence on the equicontinuous sets of E' .

This article includes the following: (1) an intrinsic characterization for a bounded convex set B of E of the closure \bar{B} of B in E'' ; (2) an intrinsic characterization of the closure \bar{E} of E in E'' ; and (3) necessary and sufficient conditions that \bar{E} be E'' .

The spaces β . Let \mathfrak{M} be the class of all closed convex neighborhoods¹ of 0 in $E[\mathcal{S}]$, and $B \in \mathfrak{B}$. A filter \mathfrak{F} on B is called a *convex filter* if, for every $F \in \mathfrak{F}$, there exist $M, N \in \mathfrak{M}$ and $\chi \in E$ such that $\overset{\circ}{M} \supset N$, $F \supset (M + \chi) \cap B$, and $(N + \chi) \cap B \in \mathfrak{F}$. Clearly if \mathfrak{F} and \mathfrak{G} are two convex filters on B , such that every set of \mathfrak{F} meets every set of \mathfrak{G} , then the least upper bound filter of \mathfrak{F} and \mathfrak{G} on B is also convex. Furthermore:

LEMMA 1. For $M, N \in \mathfrak{M}$, if $\overset{\circ}{M} \supset N$, then there exists $K \in \mathfrak{M}$ such that $\overset{\circ}{M} \supset K \supset \overset{\circ}{K} \supset N$.

Proof. If p and q are the distance functions of M and N , then $1/2(p + q)$ is the distance function of such a K .

THEOREM 1. A convex filter \mathfrak{F} on B is a maximal convex filter on B if and only if, for every two closed convex bodies K and L of E such that $\overset{\circ}{K} \supset L$, either $K \cap B \in \mathfrak{F}$ or $B \setminus L \in \mathfrak{F}$.

Proof. Assume \mathfrak{F} is maximal and let K and L be as above, and let $B \setminus L \notin \mathfrak{F}$. Let $x \in \overset{\circ}{L}$ and define a sequence $\{M_n\}$ in \mathfrak{M} so that

$$\overset{\circ}{K} - x \supset M_1 \supset \overset{\circ}{M}_1 \supset L - x \text{ and } \overset{\circ}{M}_n \supset M_{n+1} \supset \overset{\circ}{M}_{n+1} \supset L - x \quad (n \geq 1).$$

Then the filter \mathfrak{G} on B with base $\{(M_n + x) \cap B \mid n = 1, 2, 3, \dots\}$ is

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¹ The notation and definitions are principally those of Gottfried Köthe, *Topologische Lineare Räume I*, Springer-Verlag, Berlin, 1960.

convex and $K \cap B \in \mathfrak{G} \subset \mathfrak{F}$.

Conversely let \mathfrak{F} and \mathfrak{G} be two convex filters on B such that \mathfrak{F} is strictly weaker than \mathfrak{G} . Let $G \in \mathfrak{G}$, $M, N \in \mathfrak{M}$, and $x \in E$ such that $G \notin \mathfrak{F}$, $\overset{\circ}{M} \supset N$, $G \supset (M + x) \cap B$, and $(N + x) \cap B \in \mathfrak{G}$. Then neither $(M + x) \cap B$ nor $B \setminus (L + x) \in \mathfrak{F}$.

REMARKS 1. For every $x \in B$, $\mathfrak{B}_B(x) = \{V \cap B \subset B \mid V \text{ a neighborhood of } x \text{ in } E\}$ is a maximal convex filter on B .

2. For a maximal convex filter \mathfrak{F} on B , there is $x \in B$ such that $\mathfrak{F} = \mathfrak{B}_B(x)$ if and only if \mathfrak{F} has nonempty intersection.

LEMMA 2. Every maximal convex filter on B is a weak Cauchy filter.

Proof. Let \mathfrak{F} be a maximal convex filter on B ,

$$u \in E', M = \{x \in E \mid |ux| \leq 1/2\} \quad \text{and} \quad N = \{x \in E \mid |ux| \leq 1/4\}.$$

Then $M, N \in \mathfrak{M}$ and $\overset{\circ}{M} \supset N$. Since B is weakly precompact, there exist $x_1, x_2, \dots, x_n \in E$ such that $\bigcup_{i=1}^n (N + x_i) \supset B$, and so $(M + x_i) \cap B \in \mathfrak{F}$ for some $1 \leq i \leq n$. For $x, y \in (M + x_i) \cap B$, we have $|ux - uy| \leq 1$.

For a maximal convex filter \mathfrak{F} on B and $u \in E'$, let $\mathfrak{F}(u)$ denote the limit of the restriction of u to B according to the filter \mathfrak{F} .

LEMMA 3. For every maximal convex filter \mathfrak{F} on B , the mapping $u \rightarrow \mathfrak{F}(u)$ on E' is linear and $\mathcal{T}_{\mathfrak{B}}$ continuous.

Proof. Linearity is easily proved. Also let V be the polar set of the absolutely convex hull of $2B$, $u \in V$, and $F \in \mathfrak{F}$ such that $|ux - \mathfrak{F}(u)| \leq 1/2$ for every $x \in F$. Then, for such an x , we have $|\mathfrak{F}(u)| \leq |\mathfrak{F}(u) - ux| + |ux| \leq 1$.

We shall denote by $\beta = \beta_B$ the set of all maximal convex filters on B . By Lemma 3 there is a mapping π_B from β_B into E'' such that $\pi_B(\mathfrak{F})(u) = \mathfrak{F}(u)$ for every $\mathfrak{F} \in \beta_B$ and $u \in E'$.

THEOREM 2. If either \mathcal{T} is the weak topology or B is convex, then π_B is a one-to-one mapping of β_B onto the \mathcal{T}_n -closure \bar{B} of B in E'' .

Proof. For $\mathfrak{F} \in \beta_B$, $\pi_B(\mathfrak{F})$ is in the weak closure of B in E'' . For

given $u_1, \dots, u_n \in E'$ and $\varepsilon > 0$, let $F_1, \dots, F_n \in \mathfrak{F}$ such that $|u_i x - \mathfrak{F}(u_i)| \leq \varepsilon$ ($1 \leq i \leq n$) and $x \in \bigcap_{i=1}^n F_i$. Then $|\mathfrak{F}(u_i) - u_i x| \leq \varepsilon$, ($1 \leq i \leq n$).

Also, if B is convex, $\pi_B(\mathfrak{F})$ is in the \mathcal{S}_n -closure \bar{B} of B in E'' . Suppose the contrary. Then there is a continuous real linear functional w on E'' and a real number r such that $w(\pi_B(\mathfrak{F})) < r$ and $wz > r$ for every $z \in \bar{B}$.

Assume first that E is a real vector space. Let u be the restriction of w to E' , so $u \in E$. Let $F \in \mathfrak{F}$ such that $|ux - \mathfrak{F}(u)| < r - w(\pi_B(\mathfrak{F}))$ for every $x \in F$. Then, for such an x , we have $wx = ux - \mathfrak{F}(u) + \mathfrak{F}(u) < r$. But $x \in B$.

Now let E be a complex vector space. Then there is a complex linear functional v on E'' such that $w = \Re v$. Let u be the restriction of v to E and $F \in \mathfrak{F}$ such that $|ux - \mathfrak{F}(u)| \leq r - w(\pi_B(\mathfrak{F}))$ for every $x \in F$. Then for such an x we have $wx = \Re(vx) = \Re(ux - \mathfrak{F}(u)) + \Re(\mathfrak{F}(u)) < r$. Again, we have a contradiction.

Thus $\pi_B(\beta_B) \subset \bar{B}$ if \mathcal{S} is the weak topology or B is convex.

On the other hand, if $z \in \bar{B}$, then :

$$\mathfrak{B}_B(z) = \{V \cap B \subset B \mid V \text{ a neighborhood of } z \text{ in } E''[\mathcal{S}_n]\} \in \beta_B$$

and $\pi_B(\mathfrak{B}_B(z)) = z$. Let V be a neighborhood of z in $E''[\mathcal{S}_n]$, and let U and W be closed convex neighborhoods of 0 in $E''[\mathcal{S}_n]$ such that $\dot{U} \supset W$ and $U + U \subset V - z$. Let $\chi \in (U + z) \cap (-\dot{W} + z) \cap B$, $M = U \cap E$, and $N = V \cap E$. Then $M, N \in \mathfrak{M}$ and $\dot{M} \supset N$, $V \supset (M + \chi) \cap B$, and $(N + \chi) \cap B = (W + \chi) \cap B \in \mathfrak{B}_B(z)$. Thus $\mathfrak{B}_B(z)$ is convex.

Let K and L be closed convex bodies of E such that $\dot{K} \supset L$. Let $x \in \dot{L}$, $M = K - x$, and $N = L - x$. Either $z \in \text{interior } M^{\circ\circ} + x$ —in which case $K \cap B = (M + x) \cap B = (M^{\circ\circ} + x) \cap B \in \mathfrak{B}_B(z)$ —or $z \notin N^{\circ\circ} + x$ —in which case $E'' \setminus (N^{\circ\circ} + x)$ is a neighborhood of z in E'' and so $B \setminus L = [E'' \setminus (N^{\circ\circ} + x)] \cap B \in \mathfrak{B}_B(z)$. Thus $\mathfrak{B}_B(z) \in \beta_B$.

Finally, let $u \in E'$, $\varepsilon > 0$, and $F \in \mathfrak{B}_B(z)$ such that $|ux - \mathfrak{B}_B(z)(u)| \leq \varepsilon/2$ for every $x \in F$. Let $V = \{w \in E'' \mid |wu - zu| \leq \varepsilon/2\}$. Then, for $x \in F \cap V$, we have $|\mathfrak{B}_B(z)(u) - zu| \leq |\mathfrak{B}_B(z)(u) - ux| + |ux - zu| \leq \varepsilon$. Therefore, $\pi_B(\mathfrak{B}_B(z))(u) = zu$ for $u \in E''$, and so $\pi_B(\mathfrak{B}_B(z)) = z$.

REMARK. Thus $\pi_B(\mathfrak{B}_B(z)) = z$ for $z \in \bar{B}$ and $\mathfrak{F} = \mathfrak{B}_B(\pi_B(\mathfrak{F}))$ for $\mathfrak{F} \in \beta_B$.

COROLLARY 1. If either \mathcal{S} is the weak topology or B is convex, then every maximal convex filter on B is a \mathcal{S} -Cauchy filter.

COROLLARY 2. If either \mathcal{S} is the weak topology or B is convex,

then for every $\mathfrak{F} \in \beta_B$ and $M \in \mathfrak{M}$, there exist $x \in B$ such that $(M + x) \cap B \in \mathfrak{F}$.

Proof. Let $F \in \mathfrak{F}$ such that $F - F \subset M$ and $x \in F$.
 For $M \in \mathfrak{M}$ and $x \in B$ we define:

$$\begin{aligned} \nu_B(M, x) &= \{\mathfrak{F} \in \beta_B \mid (\overset{\circ}{M} + x) \cap B \in \mathfrak{F}\} \\ \mu_B(M, x) &= \{\mathfrak{F} \in \beta_B \mid \pi_B(\mathfrak{F}) \in \text{interior } M^{\circ\circ} + x\}. \end{aligned}$$

For $M, N \in \mathfrak{M}$ and $x, y \in B$, if $z \in (\overset{\circ}{M} + x) \cap (\overset{\circ}{N} + y) \cap B$ and $K = (M + x - z) \cap (N + y - z)$, then $\nu_B(M, x) \cap \nu_B(N, y) = \nu_B(K, z)$ and $\mu_B(M, x) \cap \mu_B(N, y) = \mu_B(K, z)$. Hence the class of all sets of the form $\nu_B(M, x)$ and the class of all sets of the form $\mu_B(M, x)$ (for $M \in \mathfrak{M}$ and $x \in B$) form bases of topologies, called the ν - and μ -topologies respectively, on β_B .

THEOREM 3. *If $\pi_B(\beta_B) \subset \bar{B}$ (in particular if either \mathcal{T} is the weak topology or B is convex), then ν - and μ -topologies coincide and π_B is a homeomorphism of β_B onto \bar{B} .*

Proof. If $\pi_B(\beta_B) \subset \bar{B}$, then, for $M \in \mathfrak{M}$ and $x \in B$, we have $\mu_B(M, x) \subset \nu_B(M, x)$, and so the identity mapping of β_B with the μ -topology onto β_B with the ν -topology is continuous.

Also π_B from β_B with the ν -topology onto \bar{B} is continuous. Let $\mathfrak{F} \in \beta_B$ and V a neighborhood of $\pi_B(\mathfrak{F})$ in $E''[\mathcal{T}_n]$. Let U be a closed convex neighborhood of 0 in E'' such that $U + U \subset V - \pi_B(\mathfrak{F})$, $M = U \cap E$, and $x \in (\overset{\circ}{U} + \pi_B(\mathfrak{F})) \cap B$. Then $(\overset{\circ}{M} + x) \cap B \in \mathfrak{F}$, and so $\mathfrak{F} \in \nu_B(M, x)$. Also if $\mathfrak{G} \in \nu_B(M, x)$, there is a neighborhood W of $\pi_B(\mathfrak{G})$ such that $W \cap B = (M + x) \cap B = (U + x) \cap B$ so

$$\pi_B(\mathfrak{G}) \in \overline{W \cap B} \subset U + x \subset U + U + \pi_B(\mathfrak{F}) \subset V.$$

Finally π_B^{-1} from \bar{B} onto β_B with the μ -topology is continuous by the definition of the sets μ .

COROLLARY 1. *If either \mathcal{T} is the weak topology or B is convex, then B is closed in $E''[\mathcal{T}_n]$ if and only if every maximal convex filter on B has nonempty intersection.*

COROLLARY 2. *B is weakly compact if and only if every maximal weakly convex filter on B has nonempty intersection.*

2. The space η . Let \mathfrak{A} denote the class of all convex sets of \mathfrak{B} and $\alpha = \bigcup_{B \in \mathfrak{A}} \beta_B$ the topological union of the spaces β_B . Let π be

the continuous function from α into $E''[\mathcal{F}_n]$ defined by $\pi(\mathfrak{F}) = \pi_B(\mathfrak{F})$ if $\mathfrak{F} \in \beta_B$. For $A, B \in \mathfrak{A}$ such that $A \subset B$, define a mapping g_{BA} from β_A into β_B by $g_{BA}(\mathfrak{F}) = \mathfrak{B}_B(\pi_A(\mathfrak{F}))$ (for $\mathfrak{F} \in \beta_A$). Then $g_{BA} = \pi_B^{-1}\pi_A$ and consequently is a homeomorphism of β_A into β_B . Also, if $A \subset B \subset C$, then $g_{CA} = g_{CB}g_{BA}$.

THEOREM 4. *Let $A, B \in \mathfrak{A}$ such that $A \subset B$, and let $\mathfrak{F} \in \beta_A$ and $\mathfrak{G} \in \beta_B$. The following three conditions are equivalent;*

- (a) $\mathfrak{G} = g_{BA}(\mathfrak{F})$;
- (b) $\pi(\mathfrak{F}) = \pi(\mathfrak{G})$;
- (c) Every set of \mathfrak{G} contains a set of \mathfrak{F} .

Proof. $\mathfrak{F} = \mathfrak{B}_A(\pi_B(\mathfrak{F}))$, $\mathfrak{G} = \mathfrak{B}_B(\pi_B(\mathfrak{G}))$, and $g_{BA}(\mathfrak{F}) = \mathfrak{B}_B(\pi_B(\mathfrak{F}))$. Hence (a) and (b) are equivalent. Also (b) implies (c): Given $G \in \mathfrak{G}$ there is a neighborhood V of $\pi(\mathfrak{G}) = \pi(\mathfrak{F})$ such that $G = V \cap B \supset V \cap A \in \mathfrak{F}$. Also (c) implies (b): If $\pi(\mathfrak{F}) \neq \pi(\mathfrak{G})$, then $\pi(\mathfrak{F})$ and $\pi(\mathfrak{G})$ have disjoint neighborhoods V and W in E'' , and so $W \cap A$ is a set of \mathfrak{G} containing no set of \mathfrak{F} .

COROLLARY. *Let A and $B \in \mathfrak{A}$, $\mathfrak{F} \in \beta_A$, and $\mathfrak{G} \in \beta_B$. The following three conditions are equivalent:*

- (a) $\pi(\mathfrak{F}) = \pi(\mathfrak{G})$.
- (b) There exists $C \in \mathfrak{A}$ such that $C \supset A \cup B$ and $g_{CA}(\mathfrak{F}) = g_{CB}(\mathfrak{G})$.
- (c) There exists $C \in \mathfrak{A}$ and $\mathfrak{H} \in \beta_C$ such that $C \supset A \cup B$ and every set of \mathfrak{H} contains a set of \mathfrak{F} and a set of \mathfrak{G} .

Now let R be the equivalence relation $\pi(\mathfrak{F}) = \pi(\mathfrak{G})$ on α , η the quotient space α/R , ρ the canonical mapping of α onto η , and σ the mapping from η into E'' such that $\pi = \sigma\rho$.

THEOREM 5. *σ is a homeomorphism of η onto the \mathcal{F}_n -closure \bar{E} of E in E'' .*

Proof. We need only prove $\sigma(\eta) = \pi(\alpha) \supset \bar{E}$. Consider the dual system $\langle E', \bar{E} \rangle$. Since every $u \in E'$ is uniformly continuous on E , the topology induced on \bar{E} by \mathcal{F}_n is admissible for this dual system. For $z \in \bar{E}$, there is a closed absolutely convex set $B \in \mathfrak{B}$ such that $|zu| \leq 1$ for every $u \in B^\circ$. Hence, $z \in B^{\circ\circ} =$ the closure of B in any admissible topology = the \mathcal{F}_n -closure \bar{B} of B .

For $B \in \mathfrak{A}$, the weakest topology on β_B for which every function of the form $\mathfrak{F} \rightarrow \mathfrak{F}(u)$ (for $u \in E'$) is continuous will be called the *weak topology* of β_B . Clearly β_B in the weak topology is homeomorphic

with \bar{B} in the topology induced on \bar{B} by the weak-star topology of E'' .

THEOREM 6. *The following three conditions are equivalent:*

- (a) $\bar{E} = E''$;
- (b) \bar{B} is weak-star compact for every $B \in \mathfrak{A}$;
- (c) β_B is weakly compact for every $B \in \mathfrak{A}$.

Proof. Clearly (b) and (c) are equivalent. Also (a) implies (b); by the Alaoglu—Bourbaki theorem, for $B \in \mathfrak{A}$, the weak-star closure of B in $E'' = \bar{E}$ is weak-star compact; but since \mathcal{T}_n is an admissible topology for the dual system $\langle E', \bar{E} \rangle$, this weak-star closure is \bar{B} . Finally (b) implies (a): regarding \mathfrak{B} as a total class of bounded subsets of \bar{E} , by the Mackey-Arens theorem $\mathcal{T}_{\mathfrak{B}}$ is an admissible topology for the dual system $\langle E', \bar{E} \rangle$, and so $E'' = \bar{E}$.

THEOREM 7. *For $B \in \mathfrak{A}$, β_B is weakly compact if and only if for every maximal weakly-convex filter \mathfrak{F} on B , there is a maximal \mathcal{T} -convex filter on B which is stronger than \mathfrak{F} .*

Proof. Let β_B^w be the space of all maximal weakly convex filters on B and π_B^w the homeomorphism of β_B^w into E'' with the weak-star topology. In general $B \subset \pi_B(\beta_B) = \bar{B} \subset$ weak-star closure of $B = \pi_B^w(\beta_B^w)$.

If β_B is weakly compact, then $\pi_B^w(\beta_B^w) = \pi_B(\beta_B) = \bar{B}$. So, for $\mathfrak{F} \in \beta_B^w$, $\pi_B^w(\mathfrak{F}) \in \bar{B}$ and hence $\mathfrak{B}_B(\pi_B^w(\mathfrak{F})) \in \beta_B$ is stronger than \mathfrak{F} .

Conversely, let $\mathfrak{F} \in \beta_B^w$ and $\mathfrak{G} \in \beta_B$ stronger than \mathfrak{F} . Then $\pi_B^w(\mathfrak{F}) = \pi_B(\mathfrak{G})$, and so $\pi_B^w(\beta_B^w) \subset \pi_B(\beta_B)$.

COROLLARY. *$\bar{E} = E''$ if and only if, for every $B \in \mathfrak{A}$ and every maximal weakly-convex filter \mathfrak{F} on B , there is a \mathcal{T} -convex filter on B stronger than \mathfrak{F} .*

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