MAXIMAL CONVEX FILTERS IN A LOCALLY CONVEX SPACE

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Let $E$ be a locally convex space, $\mathcal{B}$ a saturated covering of $E$ by bounded sets, and $E'$ the topological dual of $E[\mathcal{T}]$. Let $\mathcal{F}_B$ be the topology on $E'$ of uniform convergence on sets of $\mathcal{B}$ and $E''$ the topological dual of $E'[\mathcal{F}_B]$. We assume $E''$ has the natural topology $\mathcal{T}_n$—that of uniform convergence on the equicontinuous sets of $E'$.

This article includes the following: (1) an intrinsic characterization for a bounded convex set $B$ of $E$ of the closure $\overline{B}$ of $B$ in $E''$; (2) an intrinsic characterization of the closure $\overline{E}$ of $E$ in $E''$; and (3) necessary and sufficient conditions that $\overline{E}$ be $E''$.

The spaces $\beta$. Let $\mathfrak{M}$ be the class of all closed convex neighborhoods of 0 in $E[\mathcal{T}]$, and $B \in \mathcal{B}$. A filter $\mathcal{F}$ on $B$ is called a convex filter if, for every $F \in \mathcal{F}$, there exist $M, N \in \mathfrak{M}$ and $\chi \in E$ such that $\hat{M} \supseteq N$, $F \supseteq (M + \chi) \cap B$, and $(N + \chi) \cap B \in \mathcal{F}$. Clearly if $\mathcal{F}$ and $\mathcal{G}$ are two convex filters on $B$, such that every set of $\mathcal{F}$ meets every set of $\mathcal{G}$, then the least upper bound filter of $\mathcal{F}$ and $\mathcal{G}$ on $B$ is also convex. Furthermore:

**Lemma 1.** For $M, N \in \mathfrak{M}$, if $\hat{M} \supseteq N$, then there exists $K \in \mathfrak{M}$ such that $\hat{M} \supseteq K \supseteq K \supseteq N$.

*Proof.* If $p$ and $q$ are the distance functions of $M$ and $N$, then $1/2(p + q)$ is the distance function of such a $K$.

**Theorem 1.** A convex filter $\mathcal{F}$ on $B$ is a maximal convex filter on $B$ if and only if, for every two closed convex bodies $K$ and $L$ of $E$ such that $\hat{K} \supseteq L$, either $K \cap B \in \mathcal{F}$ or $B \setminus L \in \mathcal{F}$.

*Proof.* Assume $\mathcal{F}$ is maximal and let $K$ and $L$ be as above, and let $B \setminus L \in \mathcal{F}$. Let $x \in \hat{L}$ and define a sequence $\{M_n\}$ in $\mathfrak{M}$ so that

$$K - x \supseteq M_1 \supseteq M_1 \supseteq L - x \quad \text{and} \quad M_n \supseteq M_{n+1} \supseteq L - x \quad (n \geq 1).$$

Then the filter $\mathcal{G}$ on $B$ with base $\{(M_n + x) \cap B \mid n = 1, 2, 3, \ldots\}$ is

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1 The notation and definitions are principally those of Gottfried Köthe, Topologische lineare Räume I, Springer-Verlag, Berlin, 1960.

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convex and $K \cap B \in \mathcal{G} \subset \mathcal{F}$.

Conversely let $\mathcal{F}$ and $\mathcal{G}$ be two convex filters on $B$ such that $\mathcal{F}$ is strictly weaker than $\mathcal{G}$. Let $G \in \mathcal{G}$, $M, N \in \mathcal{M}$, and $x \in E$ such that $G \in \mathcal{F}$, $M \supseteq N$, $G \supseteq (M + x) \cap B$, and $(N + x) \cap B \in \mathcal{G}$. Then neither $(M + x) \cap B$ nor $B \setminus (L + x) \in \mathcal{F}$.

**Remarks 1.** For every $x \in B$, $\mathcal{V}_x(x) = \{V \cap B \subset B \mid V$ a neighborhood of $x$ in $E\}$ is a maximal convex filter on $B$.

2. For a maximal convex filter $\mathcal{F}$ on $B$, there is $x \in B$ such that $\mathcal{F} = \mathcal{V}_x(x)$ if and only if $\mathcal{F}$ has nonempty intersection.

**Lemma 2.** Every maximal convex filter on $B$ is a weak Cauchy filter.

**Proof.** Let $\mathcal{F}$ be a maximal convex filter on $B$,

$$u \in E', M = \{x \in E \mid |ux| \leq 1/2\} \quad \text{and} \quad N = \{x \in E \mid |ux| \leq 1/4\}.$$  

Then $M, N \in \mathcal{M}$ and $M \supseteq N$. Since $B$ is weakly precompact, there exist $x_1, x_2, \ldots, x_n \in E$ such that $\bigcup_{i=1}^n (N + x_i) \supseteq B$, and so $(M + x_i) \cap B \in \mathcal{F}$ for some $1 \leq i \leq n$. For $x, y \in (M + x_i) \cap B$, we have $|ux - uy| \leq 1$.

For a maximal convex filter $\mathcal{F}$ on $B$ and $u \in E'$, let $\mathcal{F}(u)$ denote the limit of the restriction of $u$ to $B$ according to the filter $\mathcal{F}$.

**Lemma 3.** For every maximal convex filter $\mathcal{F}$ on $B$, the mapping $u \mapsto \mathcal{F}(u)$ on $E'$ is linear and $\mathcal{T}_\mathcal{F}$ continuous.

**Proof.** Linearity is easily proved. Also let $V$ be the polar set of the absolutely convex hull of $2B$, $u \in V$, and $F \in \mathcal{F}$ such that $|ux - \mathcal{F}(u)| \leq 1/2$ for every $x \in F$. Then, for such an $x$, we have $|\mathcal{F}(u)| \leq |\mathcal{F}(u) - ux| + |ux| \leq 1$.

We shall denote by $\beta = \beta_B$ the set of all maximal convex filters on $B$. By Lemma 3 there is a mapping $\pi_B$ from $\beta_B$ into $E''$ such that $\pi_B(\mathcal{F})(u) = \mathcal{F}(u)$ for every $\mathcal{F} \in \beta_B$ and $u \in E'$.

**Theorem 2.** If either $\mathcal{T}$ is the weak topology or $B$ is convex, then $\pi_B$ is a one-to-one mapping of $\beta_B$ onto the $\mathcal{T}_\mathcal{F}$-closure $\beta$ of $B$ in $E''$.

**Proof.** For $\mathcal{F} \in \beta_B$, $\pi_B(\mathcal{F})$ is in the weak closure of $B$ in $E''$. For
given \( u_1, \ldots, u_n \in E' \) and \( \varepsilon > 0 \), let \( F_1, \ldots, F_n \in \mathcal{F} \) such that 
\[ |u_ix - \mathcal{F}(u_i)| \leq \varepsilon (1 \leq i \leq n) \] and \( x \in \bigcap_{i=1}^n F_i \). Then \( \mathcal{F}(u_i) - u_ix \leq \varepsilon \), \( (1 \leq i \leq n) \).

Also, if \( B \) is convex, \( \pi_B(\mathcal{F}) \) is in the \( \mathcal{T}_n \)-closure \( \tilde{B} \) of \( B \) in \( E'' \). Suppose the contrary. Then there is a continuous real linear functional \( w \) on \( E'' \) and a real number \( r \) such that \( w(\pi_B(\mathcal{F})) < r \) and \( wz > r \) for every \( z \in \tilde{B} \).

Assume first that \( E \) is a real vector space. Let \( u \) be the restriction of \( w \) to \( E' \), so \( u \in E \). Let \( F \in \mathcal{F} \) such that \( |ux - \mathcal{F}(u)| < r - w(\pi_B(\mathcal{F})) \) for every \( x \in F \). Then, for such an \( x \), we have \( ux = ux - \mathcal{F}(u) + \mathcal{F}(u) < r \). But \( x \in B \).

Now let \( E \) be a complex vector space. Then there is a complex linear functional \( v \) on \( E'' \) such that \( w = \overline{\mathbb{R}}v \). Let \( u \) be the restriction of \( v \) to \( E' \) and \( F \in \mathcal{F} \) such that \( |ux - \mathcal{F}(u)| \leq r - w(\pi_B(\mathcal{F})) \) for every \( x \in F \). Then for such an \( x \) we have \( wx = \mathbb{R}(vx) = \mathbb{R}(ux - \mathcal{F}(u)) + \mathbb{R}(\mathcal{F}(u)) < r \). Again, we have a contradiction.

Thus \( \pi_B(\beta_B) \subset \tilde{B} \) if \( \mathcal{T} \) is the weak topology or \( B \) is convex.

On the other hand, if \( z \in \tilde{B} \), then:

\[ \mathcal{B}(z) = \{ V \cap B \subset B \mid V \text{ a neighborhood of } z \text{ in } E''[\mathcal{T}_n] \in \beta_B \] and \( \pi_B(\mathcal{B}(z)) = z \). Let \( V \) be a neighborhood of \( z \) in \( E''[\mathcal{T}_n] \), and let \( U \) and \( W \) be closed convex neighborhoods of \( 0 \) in \( E''[\mathcal{T}_n] \) such that \( U \supset W \) and \( U + U \subset V - z \). Let \( \chi \in (U + z) \cap (-W + z) \cap B \), \( M = U \cap E \), and \( N = V \cap E \). Then \( M \), \( N \in \mathcal{W} \) and \( M \supset N \), \( V \supset (M + \chi) \cap B \), and \( (N + \chi) \cap B = (W + \chi) \cap B \in \mathcal{B}(z) \). Thus \( \mathcal{B}(z) \) is convex.

Let \( K \) and \( L \) be closed convex bodies of \( E \) such that \( \hat{K} \supset L \). Let \( x \in \hat{L} \), \( M = K - x \), and \( N = L - x \). Either \( z \) in interior \( M^\circ + x \)—in which case \( K \cap B = (M + x) \cap B = (M^\circ + x) \cap B \in \mathcal{B}(z) \)—or \( z \in N^\circ + x \)—in which case \( E'' \setminus (N^\circ + x) \) is a neighborhood of \( z \) in \( E'' \) and so \( B \cap [E'' \setminus (N^\circ + x)] \cap B \in \mathcal{B}(z) \). Thus \( \mathcal{B}(z) \in \beta_B \).

Finally, let \( u \in E' \), \( \varepsilon > 0 \), and \( F \in \mathcal{B}(z) \) such that \( |ux - \mathcal{B}(z)(u)| \leq \varepsilon/2 \) for every \( x \in F \). Let \( V = \{ w \in E'' \mid |wu - zu| \leq \varepsilon/2 \} \). Then, for \( x \in F \cap V \), we have \( |\mathcal{B}(z)(u) - zu| \leq |\mathcal{B}(z)(u) - ux| + |ux - zu| \leq \varepsilon \). Therefore, \( \pi_B(\mathcal{B}(z))(u) = zu \) for \( u \in E'' \), and so \( \pi_B(\mathcal{B}(z)) = z \).

REMARK. Thus \( \pi_B(\mathcal{B}(z)) = z \) for \( z \in \tilde{B} \) and \( \mathcal{F} = \mathcal{B}(\pi_B(\mathcal{F})) \) for \( \mathcal{F} \in \beta_B \).

COROLLARY 1. If either \( \mathcal{T} \) is the weak topology or \( B \) is convex, then every maximal convex filter on \( B \) is a \( \mathcal{T} \)-Cauchy filter.

COROLLARY 2. If either \( \mathcal{T} \) is the weak topology or \( B \) is convex,
then for every $\mathcal{F} \in \beta_B$ and $M \in \mathcal{W}$, there exist $x \in B$ such that $(M + x) \cap B \in \mathcal{F}$.

**Proof.** Let $F \in \mathcal{F}$ such that $F - F \subset M$ and $x \in F$.

For $M \in \mathcal{W}$ and $x \in B$ we define:

$$
\nu_B(M, x) = \{ \mathcal{F} \in \beta_B | (\mathcal{F} + x) \cap B \in \mathcal{F} \}
$$

$$
\mu_B(M, x) = \{ \mathcal{F} \in \beta_B | \pi_B(\mathcal{F}) \in \text{interior } M^\circ + x \}.
$$

For $M, N \in \mathcal{W}$ and $x, y \in B$, if $z \in (\mathcal{F} + x) \cap (N + y) \cap B$ and $K = (M + x - z) \cap (N + y - z)$, then $\nu_B(M, x) \cap \nu_B(N, y) = \nu_B(K, z)$ and $\mu_B(M, x) \cap \mu_B(N, y) = \mu_B(K, z)$. Hence the class of all sets of the form $\nu_B(M, x)$ and the class of all sets of the form $\mu_B(M, x)$ (for $M \in \mathcal{W}$ and $x \in B$) form bases of topologies, called the $\nu$- and $\mu$-topologies respectively, on $\beta_B$.

**Theorem 3.** If $\pi_B(\beta_B) \subset \bar{B}$ (in particular if either $\mathcal{F}$ is the weak topology or $B$ is convex), then $\nu$- and $\mu$-topologies coincide and $\pi_B$ is a homeomorphism of $\beta_B$ onto $\bar{B}$.

**Proof.** If $\pi_B(\beta_B) \subset \bar{B}$, then, for $M \in \mathcal{W}$ and $x \in B$, we have $\mu_B(M, x) \subset \nu_B(M, x)$, and so the identity mapping of $\beta_B$ with the $\mu$-topology onto $\beta_B$ with the $\nu$-topology is continuous.

Also $\pi_B$ from $\beta_B$ with the $\nu$-topology onto $\bar{B}$ is continuous. Let $\mathcal{F} \in \beta_B$ and $V$ a neighborhood of $\pi_B(\mathcal{F})$ in $E''[\mathcal{T}_n]$. Let $U$ be a closed convex neighborhood of 0 in $E''$ such that $U + U \subset V - \pi_B(\mathcal{F})$, $M = U \cap E$, and $x \in (\mathcal{U} + \pi_B(\mathcal{F})) \cap B$. Then $(\mathcal{F} + x) \cap B = \beta_B(\pi_B(\mathcal{F})) = \mathcal{F}$, and so $\mathcal{F} \in \nu_B(M, x)$. Also if $\mathcal{G} \in \nu_B(M, x)$, there is a neighborhood $W$ of $\pi_B(\mathcal{G})$ such that $W \cap B = (M + x) \cap B = (U + x) \cap B$ so

$$
\pi_B(\mathcal{G}) \in W \cap B \subset U + x \subset U + U + \pi_B(\mathcal{F}) \subset V.
$$

Finally $\pi_B^1$ from $\bar{B}$ onto $\beta_B$ with the $\mu$-topology is continuous by the definition of the sets $\mu$.

**Corollary 1.** If either $\mathcal{F}$ is the weak topology or $B$ is convex, then $B$ is closed in $E''[\mathcal{T}_n]$ if and only if every maximal convex filter on $B$ has nonempty intersection.

**Corollary 2.** $B$ is weakly compact if and only if every maximal weakly convex filter on $B$ has nonempty intersection.

2. The space $\eta$. Let $\mathcal{A}$ denote the class of all convex sets of $\mathcal{B}$ and $\alpha = \bigcup_{B \in \mathcal{A}} \beta_B$ the topological union of the spaces $\beta_B$. Let $\pi$ be
the continuous function from $\alpha$ into $E''[\mathcal{T}_n]$ defined by $\pi(\mathcal{F}) = \pi_B(\mathcal{F})$ if $\mathcal{F} \in \beta_B$. For $A, B \in \mathfrak{A}$ such that $A \subset B$, define a mapping $g_{BA}$ from $\beta_A$ into $\beta_B$ by $g_{BA}(\mathcal{F}) = \mathcal{B}_A(\pi_A(\mathcal{F}))$ (for $\mathcal{F} \in \beta_A$). Then $g_{BA} = \pi_B^1 \pi_A$ and consequently is a homeomorphism of $\beta_A$ into $\beta_B$. Also, if $A \subset B \subset C$, then $g_{CB} = g_{CB} g_{BA}$.

**Theorem 4.** Let $A, B \in \mathfrak{A}$ such that $A \subset B$, and let $\mathcal{F} \in \beta_A$ and $\mathcal{G} \in \beta_B$. The following three conditions are equivalent:

(a) $\mathcal{G} = g_{BA}(\mathcal{F})$;
(b) $\pi(\mathcal{F}) = \pi(\mathcal{G})$;
(c) Every set of $\mathcal{G}$ contains a set of $\mathcal{F}$.

**Proof.** $\mathcal{F} = \mathcal{B}_A(\pi_A(\mathcal{F}))$, $\mathcal{G} = \mathcal{B}_B(\pi_B(\mathcal{G}))$, and $g_{BA}(\mathcal{F}) = \mathcal{B}_B(\pi_B(\mathcal{F}))$. Hence (a) and (b) are equivalent. Also (b) implies (c): Given $G \in \mathcal{G}$ there is a neighborhood $V$ of $\pi(\mathcal{G}) = \pi(\mathcal{F})$ such that $G = V \cap B \supset V \cap A \in \mathcal{F}$. Also (c) implies (b): If $\pi(\mathcal{F}) \neq \pi(\mathcal{G})$, then $\pi(\mathcal{F})$ and $\pi(\mathcal{G})$ have disjoint neighborhoods $V$ and $W$ in $E''$, and so $W \cap A$ is a set of $\mathcal{G}$ containing no set of $\mathcal{F}$.

**Corollary.** Let $A$ and $B \in \mathfrak{A}$, $\mathcal{F} \in \beta_A$, and $\mathcal{G} \in \beta_B$. The following three conditions are equivalent:

(a) $\pi(\mathcal{F}) = \pi(\mathcal{G})$.
(b) There exists $C \in \mathfrak{A}$ such that $C \supset A \cup B$ and $g_{CA}(\mathcal{F}) = g_{CB}(\mathcal{G})$.
(c) There exists $C \in \mathfrak{A}$ and $\mathcal{H} \in \beta_C$ such that $C \supset A \cup B$ and every set of $\mathcal{H}$ contains a set of $\mathcal{F}$ and a set of $\mathcal{G}$.

Now let $R$ be the equivalence relation $\pi(\mathcal{F}) = \pi(\mathcal{G})$ on $\alpha, \eta$ the quotient space $\alpha/R$, $\rho$ the canonical mapping of $\alpha$ onto $\eta$, and $\sigma$ the mapping from $\eta$ into $E''$ such that $\pi = \sigma \rho$.

**Theorem 5.** $\sigma$ is a homeomorphism of $\eta$ onto the $\mathcal{T}_n$-closure $\bar{E}$ of $E$ in $E''$.

**Proof.** We need only prove $\sigma(\eta) = \pi(\alpha) \supset \bar{E}$. Consider the dual system $\langle E', \bar{E} \rangle$. Since every $u \in E'$ is uniformly continuous on $E$, the topology induced on $\bar{E}$ by $\mathcal{T}_n$ is admissible for this dual system. For $z \in \bar{E}$, there is a closed absolutely convex set $B \in \mathcal{B}$ such that $|zu| \leq 1$ for every $u \in B^\circ$. Hence, $z \in B^\circ = \text{the closure of } B$ in any admissible topology = the $\mathcal{T}_n$-closure $\bar{B}$ of $B$.

For $B \in \mathfrak{A}$, the weakest topology on $\beta_B$ for which every function of the form $\mathcal{F} \rightarrow \mathcal{F}(u)$ (for $u \in E'$) is continuous will be called the weak topology of $\beta_B$. Clearly $\beta_B$ in the weak topology is homeomorphic
with $\bar{B}$ in the topology induced on $\bar{B}$ by the weak-star topology of $E''$.

**Theorem 6.** The following three conditions are equivalent:

(a) $\bar{E} = E''$;
(b) $\bar{B}$ is weak-star compact for every $B \in \mathfrak{A}$;
(c) $\beta_B$ is weakly compact for every $B \in \mathfrak{A}$.

**Proof.** Clearly (b) and (c) are equivalent. Also (a) implies (b); by the Alaoglu—Bourbaki theorem, for $B \in \mathfrak{A}$, the weak-star closure of $B$ in $E'' = \bar{E}$ is weak-star compact; but since $\mathfrak{T}_n$ is an admissible topology for the dual system $\langle E', \bar{E} \rangle$, this weak-star closure is $\bar{B}$. Finally (b) implies (a); regarding $\mathfrak{B}$ as a total class of bounded subsets of $\bar{E}$, by the Mackey-Arens theorem $\mathfrak{T}_{\mathfrak{B}}$ is an admissible topology for the dual system $\langle E', \bar{E} \rangle$, and so $E'' = \bar{E}$.

**Theorem 7.** For $B \in \mathfrak{A}$, $\beta_B$ is weakly compact if and only if for every maximal weakly-convex filter $\mathfrak{F}$ on $B$, there is a maximal $\mathfrak{T}$-convex filter on $B$ which is stronger than $\mathfrak{F}$.

**Proof.** Let $\beta^n_B$ be the space of all maximal weakly convex filters on $B$ and $\pi^n_B$ the homeomorphism of $\beta^n_B$ into $E''$ with the weak-star topology. In general $B \subset \pi^n_B(\beta_B) = \bar{B} \subset$ weak-star closure of $B = \pi^n_B(\beta^n_B)$. If $\beta_B$ is weakly compact, then $\pi^n_B(\beta^n_B) = \pi_B(\beta_B) = \bar{B}$. So, for $\mathfrak{F} \in \beta^n_B$, $\pi^n_B(\mathfrak{F}) \in \bar{B}$ and hence $\mathfrak{B}_B(\pi^n_B(\mathfrak{F})) \in \beta_B$ is stronger than $\mathfrak{F}$.

Conversely, let $\mathfrak{F} \in \beta^n_B$ and $\mathcal{G} \in \beta_B$ stronger than $\mathfrak{F}$. Then $\pi^n_B(\mathfrak{F}) = \pi_B(\mathcal{G})$, and so $\pi^n_B(\beta^n_B) \subset \pi_B(\beta_B)$.

**Corollary.** $\bar{E} = E''$ if and only if, for every $B \in \mathfrak{A}$ and every maximal weakly-convex filter $\mathfrak{F}$ on $B$, there is a $\mathfrak{T}$-convex filter on $B$ stronger than $\mathfrak{F}$.

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