ON REAL EIGENVALUES OF COMPLEX MATRICES

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This paper contains many inter-related results dealing with the general question of determination of real eigenvalues of complex matrices. We first discuss the relationship between the number of elementary divisors associated with real eigenvalues of a matrix $A$ and the signature of a Hermitian matrix $H$ when $AH$ is also Hermitian. We then obtain sets of equivalent conditions for a matrix to be similar to a real matrix; for a matrix to be symmetrizable; and for a matrix to be similar to a real diagonal matrix. As corollaries we obtain results on the eigenvalues and elementary divisors of products of two Hermitian matrices. Some of the results are not new; these are included to give a more complete survey of what is known in these particular areas.

Recently a theorem on the stability of complex matrices, due to Lyapunov, has been generalized by Taussky [15, 16], and independently, by Ostrowski and Schneider [12]. Their result may be stated as follows: Given a complex matrix $A$, there exists a Hermitian $H$ for which $AH + HA^* > 0$ (positive definite) if and only if $A$ has no imaginary eigenvalues. Further, if $AH + HA^* > 0$, the numbers of eigenvalues of $A$ with positive and negative real parts equal respectively the numbers of eigenvalues of $H$ which are positive and negative.

Further generalizations of these results have been obtained by Schneider and this author [4, 6], under the condition that $AH + HA^* \geq 0$ (positive semi-definite). This paper will use these results and methods to prove the theorems mentioned in the synopsis above.

I wish to acknowledge with thanks the contribution of Professor Emilie Haynsworth, who pointed out to me the connection between [7] and my results, and thus sparked this investigation. I also wish to thank the referee for many helpful comments, and for references to several related papers, especially [13], [14], and [17], with which I had not been familiar.

2. Definitions. We define the inertia of a complex matrix $A$ to be $\text{In} A = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$, $\nu(A)$, $\delta(A)$ are respectively the number of eigenvalues of $A$ with positive, negative, and zero real parts. We shall always let $G$, $H$ and $K$ represent Hermitian matrices; we denote the signature of $H$ by $\sigma(H) = \pi(H) - \nu(H)$. We shall define, as in [12], $R(AH) = \frac{1}{2}(AH + HA^*)$. 

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We shall use several simple propositions throughout the paper. The first two assume a matrix of the form (we let $\cdot$ denote a zero matrix)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \cdot & A_{22} \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} A_{11} & \cdot \\ A_{21} & A_{22} \end{bmatrix}.$$ 

We let $\lambda$ be an eigenvalue of $A$, and let

$$a_1 \geq a_2 \geq \cdots \geq a_r, \ b_1 \geq b_2 \geq \cdots \geq b_s, \ c_1 \geq c_2 \cdots \geq c_t$$

be the degrees of the elementary divisors associated with $\lambda$ in respectively $A$, $A_{11}$, and $A_{22}$ (let $a_i = 0$ for $i > r$, etc.).

**Proposition 1.** $a_i \leq b_i + c_i$ for all $i$.

**Proposition 2.** $b_i \leq a_i$ for all $i$, and $s \leq r$; if $c_1 = c_2 = \cdots = 0$, then $b_i = a_i$ for all $i$.

**Proofs.** Proposition 2 follows immediately from the more precise inequality

$$a_{i+1} \leq b_i \leq a_i \quad \text{for all } i$$

proved in Appendix A of [5].

To prove Proposition 1, we first assume $\lambda = 0$ (if $\lambda \neq 0$, we consider $A - \lambda I$ in place of $A$). Let $\{x_j, j = 1, \cdots, a_i\}$ be a lower Jordan chain (see [8], p. 201) associated with $\lambda = 0$ and define $x_0 = 0$; we have $Ax_j = x_{j-1}$, $j = 1, \cdots, a_i$. Let

$$x_j = \begin{bmatrix} y_j \\ z_j \end{bmatrix} ; \quad \text{then} \quad Ax_j = \begin{bmatrix} A_{11}y_j + A_{12}z_j \\ A_{22}z_j \end{bmatrix} = \begin{bmatrix} y_{j-1} \\ z_{j-1} \end{bmatrix}.$$ 

Suppose $k$ is the minimal $j$ for which $z_j \neq 0$; then it follows from the above calculation that $A_{11}y_j = y_{j-1}, j = 1, \cdots, k - 1$, and $A_{22}z_j = z_{j-1}, j = k + 1, \cdots, a_i$ and $A_{22}z_k = 0$. Thus $\{y_j, j = 1, \cdots, k - 1\}$ and $\{z_j, j = k, \cdots, a_i\}$ form parts of Jordan chains for $A_{11}$ and $A_{22}$ respectively. As $b_1$ and $c_i$ are respectively the maximal lengths of such chains, $k - 1 \leq b_i$ and $a_i - k + 1 \leq c_i$, so that $a_i \leq b_i + c_i$.

Our third proposition is quite different. If $AH = K$, and $S$ is nonsingular, let $B = SAS^{-1}, H_0 = SHS^*$, and $K_0 = SKS^*$. Then $BH_0 = K_0$ and $B$ is similar to $A$; $lnH_0 = lnH$ and $lnK_0 = lnK$ (by Sylvester’s Law of Inertia). Thus we have

**Proposition 3.** If $AH = K$, we may replace $A$ by a matrix similar to it and either $H$ or $K$ by a matrix complex-congruent to it, leaving invariant the eigenvalues and elementary divisors of $A$
and the inertias of $H$ and $K$. In particular, we could assume $A$ to be in some variant of Jordan canonical form or $H = H_n \oplus 0$, where $H_n$ is nonsingular. We shall also always assume that $A$, $H$, and $K$ are partitioned conformably into submatrices.

It will be convenient to make the following definitions. Let $\gamma(A)$ be the number of $a_i$ which are associated with real eigenvalues $\lambda$. Let $\gamma_0(A)$ be the number of $a_i$ which are odd and associated with real $\lambda$. Let $\gamma_1(A)$ be the number of $a_i$ which are either odd and associated with real nonzero $\lambda$, or even and associated with $\lambda = 0$. Let $\sigma(A)$ be the sum of all $a_i$ associated with real $\lambda$.

One last comment: throughout the paper we shall discuss matrices of the form $AH$ or $H^2$, where (say) $H_2$ has some special property. All our results will remain true if we replace $AH$ by $HA$, or assume $H_1$ has the desired special property instead of $H_2$.

3. Elementary divisors associated with real eigenvalues.

Drazin and Haynsworth in [7] proved that a necessary and sufficient condition that $A$ have (at least) $m$ elementary divisors associated with real eigenvalues is that there exists an $H \geq 0$, of rank $m$, for which $AH$ is Hermitian. Our first theorem generalizes the conditions on $H$.

**Theorem 1.** Let $A$ be a complex matrix. A necessary and sufficient condition that $\gamma(A) \geq m$ is that there exists a Hermitian $H$ for which

$$|\sigma(H)| = m$$

and $AH$ is Hermitian.

If $AH$ is Hermitian and $H$ is nonsingular, then

$$|\sigma(H)| \leq \gamma_0(A),$$

$$|\sigma(AH)| \leq \gamma_1(A).$$

**Proof.** The necessity of the first assertion is contained in the Drazin-Haynsworth theorem (If $H \geq 0$, of rank $m$, then $|\sigma(H)| = m$). To prove sufficiency, we assume that $AH$ is Hermitian (i.e., $AH - HA^* = 0$) for some $H$ satisfying (1). Then if $A = iB$, $\gamma(A)$ is also the number of elementary divisors associated with imaginary eigenvalues of $B$. We have $R(BH) = \frac{1}{2}(BH + HB^*) = 0$. We may apply Theorem 3 of [4], which gives a set of bounds on the inertia of $H$ when $R(BH) \geq 0$, of rank $r$ (obviously here $r = 0$). For $r = 0$, bound (14) of [4] becomes $|\sigma(H)| \leq \gamma(A)$. As $|\sigma(H)| = m$ was assumed, the sufficiency is proved.

If $H$ is nonsingular, then (2) is merely a restatement of the second display of § 9 of [4]. Also, (3) is a restatement of a theorem by
4. Complex matrices similar to real matrices. The next result contains a conjugate-transpose analogue for complex matrices of a theorem proved by Taussky and Zassenhaus [18]: That every matrix with elements in a field $F$ may be taken into its transpose by similarity transformation by a matrix symmetric in $F$.

**THEOREM 2.** Let $A$ be a complex matrix. The following four conditions are equivalent:

(i) $A$ is similar to a real matrix.

(ii) There exists a Hermitian $H$ for which $H^{-1}AH = A^*$. 

(iii) There exists a nonsingular Hermitian $H$ for which $AH$ is Hermitian.

(iv) There exist Hermitian matrices $G$ and $H$, with $H$ nonsingular, for which $A = GH$.

Proof. Suppose there exists a matrix $T$ so that $B = T^{-1}AT$, where $B$ is real. By the Taussky-Zassenhaus theorem, there exists a real symmetric $S$ so that $S^{-1}BS = B' = B^*$. Calculation shows that $(TST^*)^{-1} A(TST^*) = A^*$; clearly $TST^*$ is Hermitian. Conversely, if $H^{-1}AH = A^*$, $A$ is similar to $A^*$, and conjugate eigenvalues of $A$ must have elementary divisors with identical degrees. Thus $A$ must be similar to a real matrix.

The equivalence of (ii), (iii), and (iv) is obvious.

5. Products of Hermitian matrices. As corollaries of the Drazin-Haynsworth theorem and our previous theorems, we obtain results on the eigenvalues and elementary divisors of products of two Hermitian matrices. Some are not new; some are easily proved independently. They are all presented, however, as, taken together they give a fairly complete description of the eigenvalues of a product of two Hermitian matrices.

**Corollary 1** extends a result credited by MacDuffee [11, p. 65] to Klein [9].

**Corollary 1.** If $H_1$ and $H_2$ are Hermitian, then

$$
\gamma(H_1H_2) \geq |\sigma(H_2)|,
$$

$$
\alpha(H_1H_2) \geq |\sigma(H_2)| + \delta(H_2).
$$

If $H_2$ is nonsingular, then

$$
\gamma_0(H_1H_2) \geq |\sigma(H_2)|, \quad \gamma_1(H_1H_2) \geq |\sigma(H_1)|.
$$
Proof. We first suppose $H_2$ is nonsingular. Then for $A = H_1 H_2$, and $H = H_2^{-1}$, $AH = H_1$ is Hermitian and (4) and (6) follow from Theorem 1 (as $In H_2 = In H_2^{-1}$). This completes the proof of (6).

If $H_2$ is singular, we may assume $H_2 = K_{11} \oplus 0$, where $K_{11}$ is nonsingular (there exists a unitary $U$ so that $U^* H_2 U = K_{11} \oplus 0$; we write $H_i$ for $UH_i U^*$). Then

$$H_1 H_2 = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} K_{11} & \\ & \end{bmatrix} = \begin{bmatrix} H_{11} K_{11} & \\ & \end{bmatrix}.$$  

As $H_{11}$ and $K_{11}$ are Hermitian and $K_{11}$ is nonsingular, we may apply (4) to $H_{11}$ and $K_{11}$ to obtain

$$\gamma(H_{11} K_{11}) \geq \sigma(K_{11}) .$$  

As $\pi(K_{11}) = \pi(H_2)$ and $\nu(K_{11}) = \nu(H_2)$, we have

$$|\sigma(K_{11})| = |\sigma(H_2)| .$$  

By Proposition 2 applied to all real eigenvalues,

$$\gamma(H_1 H_2) \geq \gamma(H_{11} K_{11}) .$$  

Combining (8), (9), and (10) we have proved (4) for all $H_1$ and $H_2$. It is clear from our definitions that

$$\alpha(A) \geq \gamma(A)$$  

for any $A$; hence (5) follows from (4) when $H_2$ is nonsingular (i.e. $\delta(H_2) = 0$). When $H_2$ is singular we again assume, as in (7), $H_1 H_2 = H_{11} (K_{11} \oplus 0)$, where the zero matrix has order $\delta(H_2)$; clearly

$$\alpha(H_1 H_2) = \alpha(H_{11} K_{11}) + \delta(H_2) .$$  

From (8), (9), and (11) (with $A = H_{11} K_{11}$), we have

$$\alpha(H_{11} K_{11}) \geq \gamma(H_{11} K_{11}) \geq |\sigma(K_{11})| = |\sigma(H_2)| ,$$  

and substituting this in (12) we obtain (5). We have proved Corollary 1.

**Corollary 2.** If $H_1$ and $H_2$ are Hermitian, then $H_1 H_2$ is similar to a real matrix.

Proof. If $H_2$ is nonsingular, this is part of Theorem 2. If $H_2$ is singular, as in Corollary 1 we may assume $H_2 = K_{11} \oplus 0$, with $K_{11}$ nonsingular. Now $H_1 H_2$ is given by (7).

We shall use Proposition 2 for $A = H_1 H_2$ and $A_{11} = H_{11} K_{11}$. To avoid confusion, we attach a superscript ($\lambda$) to each $a_i$ and $b_i$ associated with the eigenvalue $\lambda$. As $H_{11} K_{11}$ is similar to a real matrix by
Theorem 2, and $A_{22} = 0$, we have for all nonreal $\lambda$,
\[
a_i^{(\lambda)} = b_i^{(\lambda)} = b_i^{(\overline{\lambda})} = a_i^{(\overline{\lambda})}.
\]
This implies that $H_1H_2$ is similar to a real matrix.

**Remark 1.** Corollary 2 implies that $H_1H_2$ is similar to $H_1H_3$ (we have that $H_1H_2$ is similar to $H_1H_3$ and hence to $(H_1H_3)^* = H_2H_1$) for all Hermitian $H_1$ and $H_3$, a property not enjoyed by all pairs of matrices: for example take
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

**Remark 2.** We note that when $H_2$ is nonsingular the above result is trivial ($(H_1H_2)^* = H_2H_1 = H_2(H_1H_2H_1^{-1})$. However, the singular case cannot be handled in the obvious way by continuity arguments on $H_1 + \varepsilon I$, as the elementary divisors structure is not a continuous function of the elements of the matrix. The same comment applies to the two corollaries below.

**Corollary 3.** If $H_1$ and $H_2$ are Hermitian and $H_2 > 0$, then $H_1H_2$ is diagonalizable, with all real eigenvalues, and $\ln H_1H_2 = \ln H_1$.

**Proof.** Let $A = H_1H_2$ and $H = H_2^{-1} > 0$; then $AH = H_1$. By the Drazin-Haynsworth theorem, for $m = \text{order } A$, we have that $H_1H_2$ is diagonalizable, with all real eigenvalues. By Corollary 3 of [12], $\ln H_1H_2 = \ln H_2$.

**Remark 3.** Corollary 3 is well known; cf. [19, p. 108, problem 6].

Corollary 4 (below) has connections with previous work on symmetrizable operators; we shall discuss this further in § 6.

**Corollary 4.** If $H_1$ is Hermitian and $H_2 \succeq 0$, then $H_1H_2$ has all real eigenvalues; nonzero eigenvalues have linear elementary divisors, zero eigenvalues have elementary divisors of degree less than or equal two. We have
\[
\pi(H_1H_2) \leq \pi(H_1), \quad \nu(H_1H_2) \leq \nu(H_1).
\]

**Proof.** Let $H_2 = K_{11} \oplus 0$, where $K_{11} > 0$. Again $H_1H_2$ is given by (7). By Corollary 3, $H_1K_{11}$ is diagonalizable, with all real eigenvalues. As $(H_1H_2)_{22} = 0$, we have by Proposition 2
\[
a_i^{(\lambda)} = b_i^{(\lambda)} = 1
\]
for all $\lambda \neq 0$ and all nonzero $a_i^{(k)}$. We also have by Proposition 1
\[ a_i^{(0)} \leq b_i^{(0)} + c_i^{(0)} \leq 2 \]
for all nonzero $a_i^{(0)}$.

All that remains to be proved is (13); but this follows from Corollary 4 of [12].

6. Symmetrizable matrices. In a recent paper [14] Silberstein discusses symmetrizable operators in unitary spaces. (Other results on symmetrizable operators may be found in Reid [13] and Zaanen [20]). We specialize his definition to our finite dimensional setting:

**Definition.** The complex matrix $A$ is symmetrizable if there exists a Hermitian $H$, $H \geq 0$, for which

(i) $Hx = 0$ implies $Ax = 0$.

(ii) $HA$ is Hermitian.

**Theorem 3.** Let $A$ be a complex matrix. The following conditions are equivalent:

(i) $A$ has all real eigenvalues; nonzero eigenvalues have linear elementary divisors, zero eigenvalues have elementary divisors of degree less than or equal two.

(ii) $A$ is symmetrizable.

(iii) There exists a nonsingular $H$ for which $AH \geq 0$.

(iv) There exist Hermitian matrices $H$ and $K$, with $K \geq 0$, for which $A = HK$.

(v) There exist Hermitian matrices $H$ and $K$, with $H$ nonsingular and $K \geq 0$, for which $A = HK$.

**Proof.** That (i) $\Rightarrow$ (ii) is due to Silberstein (Theorems 3.1 and 3.2 of [14]). That (iii) $\Rightarrow$ (i) follows from our Corollary 4. We however, shall prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

Suppose $A$ satisfies (i). Then for some $P$, $P^{-1}AP = J = D \oplus 0$ $\oplus$ ($\sum_i J_i$), where $D$ is a real non-singular diagonal and each
\[ J_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \]

Define $K = D \oplus I \oplus (\sum_i K_i)$, where each
\[ K_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad J_i K_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]

Then $K$ is nonsingular, Hermitian, and $JK \geq 0$. We define $H = PKP^*$, and then $AH = (PJP^{-1}) (PKP^*) \geq 0$.

Suppose (iii) holds; then $A = HK$, where $K = H^{-1}A = H^{-1}(AH)H^{-1}$. 

...
0. That $(v) \implies (iv)$ is obvious. Suppose $(iv)$ holds, and $A = HK$, where $K \succeq 0$. Then $KA = KHK$ is Hermitian, and $Kx = 0 \implies Ax = HKx = 0$. The proof is complete.

As a corollary we give a slight generalization of a result due (in a more general setting) to Reid [13].

**COROLLARY 5.** If $AH \succeq 0$ for some $H$, rank $H = r \neq 0$, then $\alpha(A) \geq r$. In particular, if $H \neq 0$, $A$ has a real eigenvalue.

**Proof.** We may assume by Proposition 3 that $H = H_{11} \oplus 0$, $H_{11}$ nonsingular. Then

$$AH = \begin{bmatrix} A_{11}A_{12} & H_{11} \\ A_{21}A_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \end{bmatrix} = \begin{bmatrix} A_{11}H_{11} \\ A_{21}H_{11} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \end{bmatrix}.$$  

As $AH$ is Hermitian, $A_{21}H_{11} = 0$; as $H_{11}$ nonsingular, $A_{21} = 0$. Now by Theorem 3, as $A_{11}H_{11} \succeq 0$, $A_{11}$ has all real eigenvalues. As $A_{21} = 0$, obviously $\alpha(A) \geq \alpha(A_{11}) = r$.

7. Elementary divisors associated with positive and negative eigenvalues. We give a corollary to Theorem 3 similar in nature to Theorem 1.

**COROLLARY 6.** A necessary and sufficient condition for $A$ to have at least $p$ and $q$ elementary divisors associated with, respectively, positive and negative eigenvalues is that there exist a Hermitian $H$ for which $\pi(H) = p$, $\nu(H) = q$, and $AH \succeq 0$, of rank $p + q$.

**Proof.** The proof of the necessity is modeled after the corresponding proof of the Drazin-Haynsworth theorem. Let $\beta_1, \cdots, \beta_{p+q}$ be real eigenvalues of $A$, of which $p$ are positive and $q$ are negative; let $V_1, \cdots, V_{p+q}$ be a set of linearly independent associated eigenvectors for $\beta_1, \cdots, \beta_{p+q}$. Let $D = \text{diag} (\beta_1, \cdots, \beta_{p+q})$ and let $V = (V_1, \cdots, V_{p+q})$. We have $AV =VD$ and

$$AVDV^* = VD^2V^* = VDV^*A^*.$$  

We take $H = VDV^*$; clearly $\pi(H) = p$, $\nu(H) = q$. As $D^2 > 0$, of order $p + q$, $AH = HA^* = VD^2V^* \succeq 0$, of rank $p + q$.

To prove sufficiency, we assume by Proposition 3 that $H = H_{11} \oplus 0$, where $H_{11}$ is nonsingular. As in Corollary 5, $A_{21} = 0$ and $AH = A_{11}H_{11} \oplus 0$. We have

$$\text{rank } H_{11} = \text{rank } H = p + q = \text{rank } AH = \text{rank } A_{11}H_{11}$$  

and $A_{11}H_{11} \succeq 0$; therefore $A_{11}H_{11} > 0$.

By Theorem 1 of [12], $\text{Im}A_{11} = \text{Im}H_{11} = (p, q, 0)$. By our Theorem
3, as \( A_n \) and \( H_n \) are nonsingular, \( A_n \) has all real nonzero eigenvalues with linear elementary divisors. Thus \( A_n \) has \( p \) and \( q \) elementary divisors associated with respectively positive and negative eigenvalues. By Proposition 2, as \( A_{21} = 0 \), \( A \) has at least as many of each.

8. Matrices similar to real diagonal matrices. We next give conditions for a complex matrix to be diagonalizable, with all real eigenvalues. For the case when \( A \) is real, some of these have been given by Taussky [17].

**Corollary 7.** The matrix \( A \) is diagonalizable, with all real nonzero eigenvalues, if and only if there exists an \( H \) for which \( AH > 0 \). If \( AH > 0 \), then \( \text{In}A = \text{In}H \).

*Proof.* This is Corollary 6 for \( p + q = \text{order} \ A \).

**Corollary 8.** Let \( A \) be a complex matrix. The following conditions are equivalent:

(i) \( A \) is diagonalizable, with all real eigenvalues,

(ii) There exists an \( H \) for which \( AH \geq 0 \), with \( \text{rank} \ A = \text{rank} \ H = \text{rank} \ AH \),

(iii) There exists an \( H > 0 \) for which \( AH \) is Hermitian,

(iv) There exists an \( H \geq 0 \) for which \( AH \) is Hermitian and \( \text{rank} \ A = \text{rank} \ AH \),

(v) There exist Hermitian matrices \( G \) and \( H \), with \( H > 0 \), for which \( A = GH \).

Further, if (ii) holds, then \( \text{In}A = \text{In}H \). If either (iii) or (iv) holds, then \( \text{In}A = \text{In}AH \). If (v) holds, then \( \text{In}A = \text{In}G \).

**Remark 4.** If \( A \) is a real matrix, all of the Hermitian matrices \( G \) and \( H \) of Corollaries 7 and 8 may be chosen to be real symmetric. In fact, all constructions of \( G \) and \( H \), given in proof or referred to, may be used to obtain such real \( G \) and \( H \). The real case of Corollary 8, (i) \( \iff \) (v), is known; cf. [17, p. 133].

**Remark 5.** That (i) \( \iff \) (iii) is contained in Theorem 3.3 of [14]. That (i) \( \iff \) (v) has been noted by Taussky (Amer. Math. Monthly 66 (1959), p. 427, problem 4846; published solution by Parker, Amer. Math. Monthly 67 (1960) p. 192). That (v) \( \iff \) (i) is our Corollary 3. Alternate proofs for Corollary 7 and Corollary 8, (i) \( \iff \) (iii), may be obtained using the equivalence of (i) and (v).

*Proof of Corollary 8.* (i) \( \implies \) (ii). Let \( S^{-1}AS = B = \text{diag} (\beta_1, \cdots, \beta_n) \), where the \( \beta_i \) are real. Obviously \( B^2 \geq 0 \), with \( \text{rank} \ B = \text{rank} \ B^2 \); and if we let \( H = SBS^* \), \( AH \geq 0 \), with \( \text{rank} \ A = \text{rank} \ H = \text{rank} \ AH \).

(ii) \( \implies \) (i). Assume \( H = H_{11} \oplus 0 \), \( H_{11} \) nonsingular. As in the proof
of Corollary 5, $A_{21} = 0$ and $AH = A_{11}H_{11} \oplus 0 \succeq 0$. As rank $H = \text{rank } AH$, $A_{11}H_{11} > 0$ and by Corollary 7, $A_{11}$ is diagonalizable, with all real nonzero eigenvalues. As rank $A = \text{rank } H = \text{rank } H_{11} = \text{rank } A_{11}$, and $A_{21} = 0$, we have $A_{22} = 0$. The proof of (i) now follows from Proposition 1. That $InA = InAH$ follows from (i) and Corollary V.2 of [6].

(i) $\iff$ (iii). This is the Drazin-Haynsworth theorem for $m = \text{order } A$. That $InA = InAH$ follows from Corollary 3 of [12].

(i) $\iff$ (iv). Assume $S^{-1}AS = B = \text{diag } (\beta_1, \ldots, \beta_n)$, where the $\beta_i$ are real. Let $H = SB^*S^* \succeq 0$.

Clearly $AH = SB^*S^*$ is Hermitian, rank $A = \text{rank } AH$.

(iv) $\implies$ (i). Suppose now $AH$ is Hermitian, rank $A = \text{rank } AH$, and $H = H_{11} \oplus 0$, where $H_{11} > 0$. By (iii), as $(AH)_{11} = A_{11}H_{11}$ is Hermitian, $A_{11}$ is diagonalizable, with all real eigenvalues, and $InA_{11} = In(A_{11}H_{11})$. As before, $A_{21} = 0$, and $AH = A_{11}H_{11} \oplus 0$. As rank $A = \text{rank } AH = \text{rank } A_{11}H_{11} = \text{rank } A_{11}$, we must have $A_{22} = 0$, so that all eigenvalues of $A$ are real. Further, all nonzero eigenvalues have linear elementary divisors by Proposition 1.

We now prove that zero eigenvalues also have linear elementary divisors. Let $S_{11}$ be a matrix for which $S_{11}^{-1}A_{11}S_{11} = D \oplus 0$, where $D$ is a nonsingular diagonal. Let $S = S_{11} \oplus I$. Then

$$S^{-1}AS = B = \begin{bmatrix} D & B_{13} \\ \vdots & \vdots & \vdots \\ B_{23} \end{bmatrix},$$

where $A_{22}$ corresponds to the zero matrix in the lower-right corner of $B$. As rank $D = \text{rank } A_{11} = \text{rank } A$, obviously $B_{23} = 0$ and all zero eigenvalues have linear elementary divisors, (again by Proposition 1).

As $A_{22} = 0$, $\pi(A) = \pi(A_{11}) = \pi(A_{11}H_{11}) = \pi(AH)$, and similarly $\nu(A) = \nu(AH)$. The proof is complete.

**References**

7. Michael Drazin, and Emilie Haynsworth, *Criteria for the reality of matrix*

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