

Pacific Journal of Mathematics

FIXED POINTS IN A TRANSFORMATION GROUP

HSIN CHU

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In this paper, the following result is proved: "Let (X, T, π) be a transformation group, where X is a Peano continuum with an end point fixed under T . If the group T is one of the following two types: (1) It contains a subgroup R^n such that G/R^n is compact or (2) It contains a subgroup $Z \cdot R^n$ such that $G/(Z \cdot R^n)$ is compact, where Z is isomorphic to the discrete additive group of all integers, then T has another fixed point."

Professor A. D. Wallace, in [4], proved the following: "Let (X, Z, π) be a transformation group, where $Z =$ the discrete additive group of all integers. If X is a Peano continuum with a fixed end point under Z , then Z has another fixed point." An interesting question, (See [5]) has been raised by Wallace: "Can one reach the same conclusion about either compact groups or abelian groups"? In the case of compact groups, Professor H. C. Wang answered the question in the affirmative (See [6]). We also give an affirmative answer to the question in the case of abelian groups when the abelian group is of the type either $R^n \cdot K$ or $Z \cdot R^n \cdot K$ where R^n is a vector group of dimension n and K is a compact abelian group. Actually, we also cover the case of non-abelian groups. The same conclusion can be reached if the group, G , is one of the following two types:

- (1) It contains a subgroup R^n such that G/R^n is compact or
- (2) It contains a subgroup $Z \cdot R^n$ such that $G/(Z \cdot R^n)$ is compact.

2. We divide that proof of our main result into several steps.

LEMMA 1. *Let (X, T, π) be a transformation group, where X is an arcwise connected Hausdorff space with an end point e fixed under T . If X has a closed invariant set A under T which does not contain e then T has another fixed point. Let $1(t), 0 \leq t \leq 1$, be an arc connecting e and some point x in A such that $1(0) = e$ and $1(1) = x$. All the points which separate e and A lie on $1(t)$. Let S be the set of all those points. S is not empty. Introduce a linear ordering in $1(t), 0 \leq t \leq 1$, by the natural linear ordering of t . Then the upper limit point of S is a fixed point, other than e , under T .*

Proof. The first part of the lemma is an equivalent statement of a theorem, in [6], of Professor H. C. Wang. Under the same assumption

Received June 8, 1964. This work was supported by Contract NAS8-1646, with the George C. Marshall Space Flight Center, NASA, Huntsville, Alabama.

as our lemma, Wang's conclusion is that T has no other fixed point if and only if, given any neighborhood U of e , the orbit UT under T coincides with the whole space X . We notice that if S is a closed invariant set under T which does not contain e , then $U = X - S$ is a neighborhood of e and $UT = U$ which does not coincide with the whole space X and vice versa.

The proof of the second part of this lemma can be obtained from the proof of Wang's theorem. (See [6]).

LEMMA 2. *Let (X, Z, π) be a transformation group. If X is a compact, connected, Hausdorff space which is more than a point and has a fixed end point e , then there is a closed set $H \subset X - e$, which is invariant under Z .*

Proof. This is a theorem by Wallace, See [4].

By Lemma 1 and Lemma 2, we obtain Wallace's result.

LEMMA 3. *Let (X, Z, π) be a transformation group. If X is a Peano continuum with a fixed end point e under Z , then Z has another fixed point.*

LEMMA 4. *Let (X, T, π) be a transformation group. If X is a Peano continuum with a fixed end point e under T and T contains a syndetic subgroup Z (i.e. T contains a integer group Z such that T/Z is a compact set), then T has another fixed point. If, furthermore, T is connected, then the assumption on the given end point being fixed under T is not necessary.*

Proof. Consider the transformation group (X, Z, π) induced by (X, T, π) . From Lemma 3, we know that there is another fixed point p under Z . Since Z is syndetic, there is a compact subset K in T such that $T = Z \cdot K$. Consequently, $pT = (pZ)K = pK$ which is compact and therefore, is closed. It is clear that $e \notin pK$. We know pK is closed and invariant under T . By Lemma 1, X has another fixed point q under T .

If T is connected, it is easy to see that every end point is fixed under T (See [5]). Suppose e is an end point and $e \neq et$ for some $t \in T$. Then, because e is an end point and eT is connected, there is $s \in eT$ such that s separates e and et . Consequently, there exists some $t' \in T$ such that $s = et'$. It follows that as t' is a homeomorphism of X , et' is also an end point as well as a cut point. A contradiction!

As a direct consequence of Lemma 4, we have:

LEMMA 5. *Let (X, R, π) be a transformation group. If X is a Peano continuum with an end point, then R has another fixed point.*

LEMMA 6. *Let (X, R^n, π) be a transformation group where n is a positive integer. If X is a Peano continuum with an end point e , then R^n has another fixed point.*

Proof. By Lemma 4, we know that the end point e is fixed under R^n for all n . The proof of this lemma is by induction. Suppose the statement is true for $n = k$. Consider $n = k + 1$. Let $(x_1, \dots, x_k, x_{k+1})$ be a coordinate system of R^{k+1} . Let A and B be the closed subgroups determined by $x_1 = 0$ and $x_2 = 0$ respectively. Then $A \cong B \cong R^k$. Let the transformation groups (X, A, π) and (X, B, π) both be induced by (X, R^{k+1}, π) . By the inductive assumption, we know there are two points p and q such that p is invariant under A and q is invariant under B . Both p and q are distinct from e . Let C_1 be the subgroup of R^{k+1} determined by $x_2 = 0, \dots, x_{k+1} = 0$. Let C_2 be the subgroup of R^{k+1} determined by $x_1 = 0, x_3 = 0, \dots, x_{k+1} = 0$. Then $C_1 \cong C_2 \cong R$ and, as direct products $R^{k+1} = C_1 \cdot A = C_2 \cdot B$. Consider the orbit, $(p)R^{k+1}$, of p under R^{k+1} and the orbit, $(q)R^{k+1}$, of q under R^{k+1} . It is clear that $(p)R^{k+1} = (p)C_1$ and $(q)R^{k+1} = (q)C_2$, where $(p)C_1$ and $(q)C_2$ both are connected.

We know both $cl((p)C_1)$ and $cl((q)C_2)$ are invariant under R^{k+1} . If e is not in either $cl((p)C_1)$ or $cl((q)C_2)$, then, by Lemma 1, R^{k+1} has another fixed point. Suppose e is in both $cl((p)C_1)$ and $cl((q)C_2)$. This implies that every neighborhood of e contains points from both $(p)C_1$ and $(q)C_2$.

Let U_e be a neighborhood of e such that $\{p, q\} \cap U_e = \phi$. Since e is a fixed end point, there exists $x \in U_e$ such that $X - x = X_1 \cup X_2$ for some sets X_1 and X_2 with the properties:

$$X_1 \cap cl(X_2) = cl(X_1) \cap X_2 = \phi \quad \text{and} \quad e \in X_1 \subset U_e.$$

Consequently, $\{p, q\} \subset X_2$. Notice that X_1 is open in X . It follows that X_1 contains points from both $(p)C_1$ and $(q)C_2$. Since both $(p)C_1$ and $(q)C_2$ are connected, it follows that $x \in (p)C_1 \cap (q)C_2$. Since R^{k+1} is abelian, we have $p = q$ and p is a fixed point under R^{k+1} other than e . Complete the proof by Lemma 5.

LEMMA 7. *Let $(X, Z \cdot R^n, \pi)$ be a transformation group. If X is a Peano continuum with a fixed end point e under $Z \cdot R^n$, then $Z \cdot R^n$ has another fixed point.*

Proof. If $n = 0$, the statement of this lemma is the same as Lemma 3. Let $n > 0$. Let (X, A, π) be a transformation group induced

by $(X, Z \cdot R^n, \pi)$ where $A = Z \cdot R^{n-1}$ is a subgroup of $Z \cdot R^n$. Let $B \cong R$ be a subgroup of $Z \cdot R^n$ such that $Z \cdot R^n = A \cdot B$. Prove this lemma by induction on n . Suppose (X, A, π) has a fixed point, p , other than e , under A . Consider the orbit $(p)(Z \cdot R^n)$. It is clear that $(p)(Z \cdot R^n) = (p)B$, which is connected. The orbit-closure $cl((p)(Z \cdot R^n))$ is a connected compact Hausdorff space. Obviously, $cl((p)(Z \cdot R^n))$ is invariant under $Z \cdot R^n$. If e is not in $cl((p)(Z \cdot R^n))$, then, by Lemma 1, $Z \cdot R^n$ has another fixed point. Suppose $e \in cl((p)(Z \cdot R^n))$. Let Z' be an integer group of B . Then e is a fixed end point of the transformation group $(cl((p)(Z \cdot R^n)), Z', \pi)$. By Lemma 2, there is a Z' -invariant closed subset H of $cl((p)(Z \cdot R^n))$ such that $e \notin H$. Consider the transformation group (X, Z', π) , induced by $(X, Z \cdot R^n, \pi)$. Choose a point $q \in H$ and connect e and q by an arc $1(t)$, $0 \leq t \leq 1$ on which $1(0) = e$ and $1(1) = q$. Let S be the set of all points which separate e and H . By Lemma 1 the upper limit point, r , of S is a fixed point, other than e , under Z' . Since $cl((p)(Z \cdot R^n))$ is connected, we have $S \subset cl((p)(Z \cdot R^n))$. Consequently, $r \in cl((p)(Z \cdot R^n))$. Since the points in $(p)(Z \cdot R^n)$ are fixed under A , the points in $cl((p)(Z \cdot R^n))$ are also fixed under A . It follows that r is fixed under both A and Z' . Let $B = Z'K'$ for some compact set K' . Then $(r)(Z \cdot R^n) = (r)K'$ which is compact. It is obvious $e \notin (r)K'$. By Lemma 1, $(Z \cdot R^n)$ has another fixed point. Complete the proof by induction.

THEOREM. *Let (X, T, π) be a transformation group. If X is a Peano continuum with a fixed end point under T and T is one of the following two types:*

- (1) *It contains a subgroup R^n such that G/R^n is compact or*
- (2) *It contains a subgroup $Z \cdot R^n$ such that $G/Z \cdot R^n$ is compact.*

Proof. Complete the proof by Lemma 1, Lemma 6, Lemma 7 and a similar method used in the proof of Lemma 4.

COROLLARY 1. *Let (X, T, π) be a transformation group. If X is a Peano continuum with an end point and T is locally compact, connected, abelian group, then T has another fixed point.*

We have the following application in Topological Dynamics. (See [1]). The proof is similar to the one used for the theorem.

COROLLARY 2. *Let (X, T, π) be a transformation group. If X is arcwise connected, Hausdorff with a fixed end point e and a regularly almost periodic point p , other than e , then T has another fixed point.*

Proof. By the definition of regularly almost periodic point, for a closed neighborhood U of p such that $e \in U$, there exists a syndetic

subgroup A of T such $pA \subset U$. It follows that $cl(pA) \subset U$, and thereby, $e \notin cl(pA)$. It is clear that $cl(xA)$ is invariant under A . By Lemma 1, we have another fixed point q under A . Since A is syndetic, there exists a compact set K such that $T = A \cdot K$. From $qT = (qA)K = qK$, we know qT is compact and, therefore, is closed and $e \notin qT$. Since qT is invariant under T , by Lemma 1 we have another fixed point under T . The theorem is proved.

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50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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* Basil Gordon, Acting Managing Editor until February 1, 1966.

Pacific Journal of Mathematics

Vol. 15, No. 4

December, 1965

Robert James Blattner, <i>Group extension representations and the structure space</i>	1101
Glen Eugene Bredon, <i>On the continuous image of a singular chain complex</i>	1115
David Hilding Carlson, <i>On real eigenvalues of complex matrices</i>	1119
Hsin Chu, <i>Fixed points in a transformation group</i>	1131
Howard Benton Curtis, Jr., <i>The uniformizing function for certain simply connected Riemann surfaces</i>	1137
George Wesley Day, <i>Free complete extensions of Boolean algebras</i>	1145
Edward George Effros, <i>The Borel space of von Neumann algebras on a separable Hilbert space</i>	1153
Michel Mendès France, <i>A set of nonnormal numbers</i>	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i>	1171
Frederick Paul Greenleaf, <i>Norm decreasing homomorphisms of group algebras</i>	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i>	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i>	1239
James Patrick Jans, <i>Some aspects of torsion</i>	1249
Laura Ketchum Kodama, <i>Boundary measures of analytic differentials and uniform approximation on a Riemann surface</i>	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic functions</i>	1279
Ronald John Larsen, <i>Almost invariant measures</i>	1295
You-Feng Lin, <i>Generalized character semigroups: The Schwarz decomposition</i>	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i>	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i>	1319
Mitsuru Nakai, <i>Φ-bounded harmonic functions and classification of Riemann surfaces</i>	1329
L. G. Novoa, <i>On n-ordered sets and order completeness</i>	1337
Fredos Papangelou, <i>Some considerations on convergence in abelian lattice-groups</i>	1347
Frank Albert Raymond, <i>Some remarks on the coefficients used in the theory of homology manifolds</i>	1365
John R. Ringrose, <i>On sub-algebras of a C^*-algebra</i>	1377
Jack Max Robertson, <i>Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres</i>	1383
Zalman Rubinstein, <i>Some results in the location of zeros of polynomials</i>	1391
Arthur Argyle Sagle, <i>On simple algebras obtained from homogeneous general Lie triple systems</i>	1397
Hans Samelson, <i>On small maps of manifolds</i>	1401
Annette Sinclair, <i>$\varepsilon(z)$-closeness of approximation</i>	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional curvature</i>	1415
Earl J. Taft, <i>Invariant splitting in Jordan and alternative algebras</i>	1421
L. E. Ward, <i>On a conjecture of R. J. Koch</i>	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i>	1435
Horace C. Wisner, <i>Embedding a circle of trees in the plane</i>	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i>	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their application to learning theory</i>	1471
Charles Vernon Coffman, <i>Correction to: Non-linear differential equations on cones in Banach spaces</i>	1472
P. H. Doyle, III, <i>Correction to: A sufficient condition that an arc in S^n be cellular</i>	1474
P. P. Saworotnow, <i>Correction to: On continuity of multiplication in a complemented algebra</i>	1474
Basil Gordon, <i>Correction to: A generalization of the coset decomposition of a finite group</i>	1474