

Pacific Journal of Mathematics

**THE UNIFORMIZING FUNCTION FOR CERTAIN SIMPLY
CONNECTED RIEMANN SURFACES**

HOWARD BENTON CURTIS, JR.

THE UNIFORMIZING FUNCTION FOR CERTAIN SIMPLY CONNECTED RIEMANN SURFACES

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This paper contains a definition of a class of simply connected Riemann surfaces, the determination of the type of a surface from this class, and a representation of the uniformizing function and its derivative as infinite products of quotients as well as quotients of infinite products.

Definition of the class of surfaces. Let $\{a_{2n-1}\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that for $n \geq 1$,

$$0 < a_{2n-1} < b_{2n-1} < b_{2n}$$

and $b_{2n+1} < b_{2n}$. A surface F of the class to be discussed consists of sheets S_n , $n = 1, 2, 3, \dots$, over the w -sphere, where for S_n a copy of the w -sphere,

- (a) S_1 is slit along the real axis from a_1 to b_1 .
- (b) For $n \geq 1$, S_{2n} is slit along the real axis from a_{2n-1} to b_{2n-1} and from b_{2n} to $+\infty$.
- (c) For $n \geq 1$, S_{2n+1} is slit along the real axis from a_{2n+1} to b_{2n+1} and from b_{2n} to $+\infty$.
- (d) For $n \geq 1$, S_n is joined to S_{n+1} along the slits to make the b_n coincide and to form first order branch points at the end-points of the slits.

The uniformizing function. Because F is simply connected and noncompact, there exists a unique function g which maps F schlichtly and conformally onto $\{|z| < R \leq \infty\}$, where for $f(z) = g^{-1}(z)$, $f(0) = 0 \in S_1$ and $f'(0) = 1$. Two surfaces of hyperbolic type are obtained by slitting each sheet of F along the uncut parts of the real axis, and an application of the reflection principle to the uniformizing function of one of these surfaces shows that $f(z)$ is real for real z . Let $f(\alpha_{2k-1}) = a_{2k-1}$, $f(-\beta_k) = b_k$, $f(\gamma_{2k}) = \infty \in S_{2k}$ and S_{2k+1} , $f(-\gamma_1) = \infty \in S_1$, and $f(\delta_k) = 0 \in S_k$. The image of F in the z -plane satisfies the following properties. The image of S_n is a region which is symmetric about the real axis. S_1 is mapped onto a domain containing the origin and bounded by a simple closed curve C_1 which intersects the real axis at $-\beta_1$ and α_1 . For $n \geq 2$, S_n is mapped onto an annular region about the origin and bounded by two simple closed curves C_{n-1} and C_n , which

Received July 24, 1964. Research for this paper was supported by a grant from the University of Texas Research Institute and revision of the paper was completed while the author was a Research Fellow in Mathematics at Rice University.

are images of cuts. For n odd, C_n intersects the real axis at $-\beta_n$ and α_n , while for n even, C_n intersects the real axis at $-\beta_n$ and γ_n . Furthermore, for $k \geq 1$,

$$-\beta_{k+1} < -\beta_k < -\gamma_1 < 0 < \alpha_{2k-1} < \delta_{2k} < \gamma_{2k} < \delta_{2k+1} < \alpha_{2k+1}.$$

The approximating closed surfaces. Let F_n be the surface formed from the first $2n + 2$ sheets of F with the slit in S_{2n+2} from b_{2n+2} to ∞ deleted, so that F_n is a compact, simply connected surface.

NOTATION. $\alpha_\varphi^* = 1 - z/\alpha_\varphi$, $\beta_\varphi^* = 1 + z/\beta_\varphi$,
 $\gamma_\varphi^* = 1 - z/\gamma_\varphi$, $\delta_\varphi^* = 1 - z/\delta_\varphi$.

LEMMA 1. Let R_n be the unique rational function which maps the z -sphere one-to-one onto the simply connected compact surface F_n with $R_n(0) = 0 \in S_1$, $R'_n(0) = 1$, and $R_n(\infty) = \infty \in S_{2n+2}$. Then

$$R_n(z) = [z/(1 + z/\gamma_{1,n})] \left[\prod_{k=2}^{2n+2} \delta_{k,n}^* \right] / \left[\prod_{k=1}^n (\gamma_{2k,n}^*)^2 \right]$$

and

$$R'_n(z) = [1/(1 + z/\gamma_{1,n})^2] \left[\prod_{k=0}^n \alpha_{2k+1,n}^* \right] \left[\prod_{k=1}^{2n+1} \beta_{k,n}^* \right] / \left[\prod_{k=1}^n (\gamma_{2k,n}^*)^3 \right].$$

Proof. The representations of R_n and R'_n must contain factors shown and can contain no more. The $\alpha_{2k+1,n}$, $-\beta_{k,n}$, $\gamma_{2k,n}$, and $\delta_{k,n}$, which are ordered in the same manner as the α_{2k+1} , $-\beta_k$, γ_{2k} , and δ_k , are images of a_{2k+1} , b_k , ∞ , and 0 , respectively, under R_n^{-1} .

LEMMA 2. F is parabolic.

Proof. Suppose that F is hyperbolic, and thus g maps F onto $\{|z| < R < \infty\}$. If D_n is the z -plane slit along the real axis from $-\beta_{2n+1,n}$ to $-\infty$, then $\zeta = \psi_n(z) = g[R_n(z)]$ defines a Schlicht mapping of D_n onto a simply connected region Δ_n of the ζ -plane bounded by C_{2n+2} and the segment $(-\beta_{2n+2}, -\beta_{2n+1})$. If $T_n(z) = z(1 - z/4\beta_{2n+1,n})^{-2}$, then $\zeta = \psi_n[T_n(z)]$ defines a properly normalized, Schlicht mapping of $\{|z| < 4\beta_{2n+1,n}\}$ onto Δ_n such that if the Koebe Distortion Theorem is applied to this map, then $\beta_{2n+1,n} \leq d(0, C_{2n+2}) \leq R < \infty$, where $d(0, C_{2n+2})$ is the distance from $\zeta = 0$ to the curve C_{2n+2} . Thus there exists a subsequence $\{\beta_{2n_j+1,n_j}\}$ such that $\beta_{2n_j+1,n_j} \rightarrow A \leq R$ as $j \rightarrow \infty$, and ψ_{n_j} is a Schlicht mapping of D_{n_j} onto Δ_{n_j} . If D is the z -plane slit along the negative real axis from $-A$ to $-\infty$, then $\{\psi_{n_j}\}$ forms a family of functions which is normal in D , and hence there exists a subsequence $\{\psi_i\}$ such that as $i \rightarrow \infty$, $\psi_i(z) \rightarrow \psi(z)$ uniformly on any compact sub-

set of D . Because $D_i \rightarrow D$ and $\psi_i(z) \rightarrow \psi(z)$ as $i \rightarrow \infty$, then $\Delta_i \rightarrow \{|z| < R\}$ and ψ maps D onto $\{|\zeta| < R\}$ in a one-to-one manner. ([1], p. 18). Then $R_i(z) = f[\psi_i(z)] \rightarrow f[\psi(z)] = H(z)$ uniformly on any compact subset of D as $i \rightarrow \infty$, where H is meromorphic in D , while $H(z) \not\equiv \infty$ because $R_i(0) = 0$. H maps D onto F .

Now let D^* be the z -plane slit along the real axis from $-A$ to $+\infty$. For i sufficiently large, $R_i(z)$ assumes no negative real values in any compact subset of D^* , and thus $\{R_i\}$ is a family of functions which is normal in D^* . Therefore, there exists a subsequence $\{R_m\}$ of $\{R_i\}$ such that as $m \rightarrow \infty$, $R_m(z) \rightarrow G(z)$ uniformly on any compact subset of D^* . H and G have a common domain of convergence, so that G is the analytic continuation of H . Then $w = G(z)$ defines a mapping of the z -plane punched at $z = A$ and ∞ one-to-one and conformally onto an open doubly connected Riemann surface F^* of which F is a subsurface obtained by inserting some slits in F^* over the real axis. This is impossible, as is clear from the definition of F . Hence $R = \infty$.

LEMMA 3. $R_n(z) \rightarrow f(z)$ uniformly on any compact subset of the z -plane as $n \rightarrow \infty$.

Proof. Because $\Delta_n \rightarrow \{|\zeta| < \infty\}$ as $n \rightarrow \infty$, it follows ([1], p. 18) that $z = R_n^{-1}[f(\zeta)] \rightarrow \zeta = g[R_n(z)]$ uniformly on any compact subset of the ζ -plane as $n \rightarrow \infty$. Also, $D_n \rightarrow \{|z| < \infty\}$ and $R_n(z) \rightarrow f(z)$ uniformly on any compact subset of the z -plane as $n \rightarrow \infty$.

LEMMA 4. $\alpha_{2k-1,n} \rightarrow \alpha_{2k-1}$, $\beta_{k,n} \rightarrow \beta_k$, $\gamma_{2k,n} \rightarrow \gamma_{2k}$, and $\delta_{k,n} \rightarrow \delta_k$ as $n \rightarrow \infty$.

Proof. This is a consequence of Hurwitz's Theorem.

LEMMA 5. The infinite product

$$\pi(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^* \delta_{2k+1}^* / (\gamma_{2k}^*)^2]$$

converges uniformly on any compact subset of the z -plane.

Proof. Since $\gamma_{2k} \rightarrow \infty$ and $\delta_k \rightarrow \infty$ as $k \rightarrow \infty$, then for any $R > 0$, there exists $n_0 = n_0(R)$ such that for $k \geq n_0$, $\delta_k > R$ and $\gamma_{2k} > R$. Then consider

$$M_p(z) = \prod_{k=n_0}^{n_0+p} [\delta_{2k}^* \delta_{2k+1}^* / (\gamma_{2k}^*)^2].$$

M_p is holomorphic for $|z| \leq R$ and $M_p(z) \neq 0$ for $|z| \leq R$. A sufficient

condition for the uniform convergence of $M_p(z)$ in $E = \{ |z| \leq R \}$ as $p \rightarrow \infty$ is the uniform convergence in E of

$$\sum_{k=n_0}^{n_0+p} \log [\delta_{2k}^* \delta_{2k+1}^* / (\gamma_{2k}^*)^2] \text{ as } p \rightarrow \infty ,$$

where each logarithm is the principal value. By the Cauchy criterion, this last sequence converges uniformly in E provided for $z \in E$ and for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for $n > N(\varepsilon)$ and $p > 0$,

$$\left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \delta_{2k+1}^* / (\gamma_{2k}^*)^2] \right| < \varepsilon .$$

Now since $\delta_{2k} < \delta_{2k+1}$ and since $\gamma_{2k} < \delta_{2k+2} < \delta_{2k+3}$, then for $m \geq 1$ and $p > 0$,

$$0 < \sum_{k=n_0+n}^{n_0+n+p} [1/(\delta_{2k})^m + 1/(\delta_{2k+1})^m - 2/(\gamma_{2k})^m] < 2/(\delta_{2n_0+2n})^m .$$

Then for all $p > 0$ and $z \in E$,

$$\begin{aligned} \left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \delta_{2k+1}^* / (\gamma_{2k}^*)^2] \right| &= \left| -\sum_{m=1}^{\infty} [z^m/m] \sum_{k=n_0+n}^{n_0+n+p} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right| \\ &\leq \sum_{m=1}^{\infty} [R^m/m] [2/(\delta_{2n_0+2n})^m] \leq 2 \sum_{m=1}^{\infty} [R/(\delta_{2n_0+2n})]^m = 2R/(\delta_{2n_0+2n} - R) . \end{aligned}$$

Since $\delta_{2n_0+2n} \rightarrow \infty$ as $n \rightarrow \infty$, the Cauchy criterion is satisfied and M_p converges uniformly in E . Thus $\Pi(z)$ converges uniformly in any compact subset of the z -plane.

LEMMA 6. $\pi(z) = f(z)$.

Proof. As a consequence of Lemma 4, there exists $r > 0$ such that $R_n(z)/z \neq 0$ and $\pi(z)/z \neq 0$ for $|z| < r$, while each of these quotients defines a function which is holomorphic for $|z| < r$ and takes the value 1 at $z = 0$. Thus using the principal value of the logarithm, for $|z| < r$,

$$\begin{aligned} \log [R_n(z)/z] - \log [\pi(z)/z] &= \log [R_n(z)/\pi(z)] = \log [(1 + z/\gamma_1)/(1 + z/\gamma_{1,n})] \\ &- \sum_{m=1}^{\infty} \{z^m/m\} \left\{ \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^n (1/\delta_{2k+1,n}^m) - \sum_{k=1}^n (2/\gamma_{2k,n}^m) \right. \\ &\left. - \sum_{k=1}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right\} . \end{aligned}$$

Therefore, for $n_0 > 2$, as $n \rightarrow \infty$,

$$0 \leq \lim \sup \left| \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^n (1/\delta_{2k+1,n}^m) \right|$$

$$\begin{aligned}
 & - \sum_{k=1}^n (2/\gamma_{2k,n}^m) - \sum_{k=1}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \Big| \\
 & \leq \lim \sup \left| \sum_{k=n_0}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=n_0}^n (1/\delta_{2k+1,n}^m) - \sum_{k=n_0}^n (2/\gamma_{2k,n}^m) \right. \\
 & \quad \left. - \sum_{k=n_0}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right| \\
 & \leq \lim \sup [(1/\delta_{2n_0,n}^m) + (1/\delta_{2n_0+1,n}^m) + (1/\delta_{2n_0}^m) + (1/\delta_{2n_0+1}^m)] \\
 & = (2/\delta_{2n_0}^m) + (2/\delta_{2n_0+1}^m) .
 \end{aligned}$$

Since $\delta_{2n_0} \rightarrow \infty$ and $\delta_{2n_0+1} \rightarrow \infty$ as $n_0 \rightarrow \infty$, it follows that the limit as $n \rightarrow \infty$ of each coefficient of the preceding expansion of $\log [R_n(z)/\pi(z)]$ is zero. Furthermore, because as $n \rightarrow \infty$, $\{\log [R_n(z)/\pi(z)]\}_{n=1}^{\infty}$ converges uniformly on $\{|z| < r\}$, then $\log [R_n(z)/\pi(z)] \rightarrow 0$ as $n \rightarrow \infty$. Thus $\pi(z) = \lim_{n \rightarrow \infty} R_n(z) = f(z)$.

LEMMA 7. $\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$, $\sum_{k=1}^{\infty} 1/\beta_k < \infty$, $\sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty$, and $\sum_{k=2}^{\infty} 1/\delta_k < \infty$.

Proof. Again by Lemma 4, there exists $r > 0$ such that $f'(z) \neq 0$ and $R'_n(z) \neq 0$ for $|z| < r$. Since $R_n(z) \rightarrow f(z)$, it follows that $R'_n(z) \rightarrow f'(z)$ and thus $\log R'_n(z) \rightarrow \log f'(z)$ uniformly in $\{|z| < r\}$ as $n \rightarrow \infty$. Thus for $|z| < r$, $\log R'_n(z)$

$$\begin{aligned}
 & = \sum_{m=1}^{\infty} [z^m/m] \left[- \sum_{k=0}^n 1/\alpha_{2k+1,n}^m \right. \\
 & \quad \left. + \sum_{k=1}^{2n+1} (-1)^{m+1}/\beta_{k,n}^m + 2(-1)^m/\gamma_{1,n}^m + \sum_{k=1}^n 3/\gamma_{2k,n}^m \right] .
 \end{aligned}$$

Hence, for $m = 1$,

$$\lim_{n \rightarrow \infty} \left| - \sum_{k=0}^n 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n} + \sum_{k=1}^n 3/\gamma_{2k,n} \right| < \infty .$$

Because $0 < \gamma_{1,n} < \beta_{1,n}$ and $0 < \gamma_{2k,n} < \alpha_{2k+1,n}$, then

$$\begin{aligned}
 0 & < \sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^n 2/\gamma_{2k,n} \\
 & < - \sum_{k=0}^n 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n} \\
 & \quad + \sum_{k=1}^n 3/\gamma_{2k,n} + 1/\alpha_{1,n} + 2/\gamma_{1,n} .
 \end{aligned}$$

Therefore, as $n \rightarrow \infty$,

$$0 \leq \lim \sup \left[\sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^n 2/\gamma_{2k,n} \right] < \infty ,$$

$$\limsup \sum_{k=1}^{2n+1} 1/\beta_{k,n} < \infty, \text{ and } \limsup \sum_{k=1}^n 1/\gamma_{2k,n} < \infty .$$

Furthermore, because for

$$k \geq 1, \gamma_{2k,n} < \delta_{2k+1,n} < \alpha_{2k+1,n} < \delta_{2k+2,n} ,$$

$$\limsup_{n \rightarrow \infty} \sum_{k=3}^{2n+2} 1/\delta_{k,n} \leq \limsup_{n \rightarrow \infty} \sum_{k=1}^n 2/\gamma_{2k,n} < \infty$$

and

$$\limsup_{n \rightarrow \infty} \sum_{k=1}^n 1/\alpha_{2k+1,n} \leq \limsup_{n \rightarrow \infty} \sum_{k=1}^n 1/\gamma_{2k,n} < \infty .$$

Hence

$$\limsup_{n \rightarrow \infty} \sum_{k=0}^n 1/\alpha_{2k+1,n} < \infty \text{ and } \limsup_{n \rightarrow \infty} \sum_{k=2}^{2n+2} 1/\delta_{k,n} < \infty .$$

For all $N > 0$, as $n \rightarrow \infty$,

$$\sum_{k=0}^N 1/\alpha_{2k+1} = \sum_{k=0}^N \lim 1/\alpha_{2k+1,n} \leq \limsup \sum_{k=0}^n 1/\alpha_{2k+1,n} < \infty ,$$

and thus $\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$. The convergence of the other series is established in a similar manner.

LEMMA 8. *Each of the three infinite products in*

$$P(z) = [1/(1 + z/\gamma_1)^2] \left[\prod_{k=0}^{\infty} \alpha_{2k+1}^* \prod_{k=1}^{\infty} \beta_k^* / \prod_{k=1}^{\infty} (\gamma_{2k}^*)^3 \right]$$

converges uniformly on any compact subset of the z-plane.

Proof. This is a consequence of Lemma 7.

LEMMA 9. $f'(z) = [\exp(\delta z)][P(z)]$ where δ is real.

Proof. By Lemma 4, there exists $r > 0$ such that for $|z| < r$, $R'_n(z) \neq 0$ and $f'(z) \neq 0$. For $m \geq 1$, consider the coefficient of z^m/m in the Taylor expansion of $\log [R'_n(z)/P(z)]$ about $z = 0$ for $|z| < r$. Because of Lemma 7, there exists $M > 0$ such that for all $n \geq 1$,

$$\sum_{k=1}^n 1/\gamma_{2k,n} < M \text{ and } \sum_{k=1}^{\infty} 1/\gamma_{2k} < M .$$

Then because of the ordering of the $\gamma_{k,n}$ and γ_k , for each $k < n$, $k/\gamma_{2k,n} < M$ and $k/\gamma_{2k} < M$. Thus for each $N > 1$, as $n \rightarrow \infty$,

$$\begin{aligned} & \lim \sup \left| \sum_{k=1}^n 1/\gamma_{2k,n}^m - \sum_{k=1}^{\infty} 1/\gamma_{2k}^m \right| \\ & \leq \lim \sup \left| \sum_{k=N}^n 1/\gamma_{2k,n}^m - \sum_{k=N}^{\infty} 1/\gamma_{2k}^m \right| \leq 2M^m \sum_{k=N}^{\infty} 1/k^m, \end{aligned}$$

which implies for $m \geq 2$, as $n \rightarrow \infty$

$$\lim \left[\sum_{k=1}^n 1/\gamma_{2k,n}^m - \sum_{k=1}^{\infty} 1/\gamma_{2k}^m \right] = 0 .$$

Similarly, the other terms in the coefficient of z^m/m have a limit of zero for $m \geq 2$, and the coefficient of z is real. Then as $n \rightarrow \infty$, $\log [R'_n(z)/P(z)] \rightarrow \log [f'(z)/P(z)] = \delta z$, and thus $f'(z) = [\exp(\delta z)][P(z)]$.

LEMMA 10. $\delta = 0$.

Proof. Because the factors of $P(z)$ are canonical products of genus zero with real zeros, for $\varepsilon > 0$ and $0 < \rho \leq |\arg z| \leq \pi - \rho$, $P(z) = 0[\exp(\varepsilon|z|)]$ and $1/P(z) = 0[\exp(\varepsilon|z|)]$. Then if $\arg z$ satisfies the preceding conditions and $|z|$ is sufficiently large, then

$$\exp[\delta \mathcal{R}(z) - \varepsilon|z|] \leq |f'(z)| \leq \exp[\delta \mathcal{R}(z) + \varepsilon|z|] .$$

Let $A_1 = \{z \mid \pi/4 \leq \arg z \leq \pi/3\}$ and $A_2 = \{z \mid 2\pi/3 \leq \arg z \leq 3\pi/4\}$. If $\delta > 0$, then there exists $\varphi_1 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geq \exp(\varphi_1|z|)$ when $z \in A_1$ and $|f'(z)| \leq \exp(-\varphi_1|z|)$ when $z \in A_2$. Thus as $z \rightarrow \infty$ in A_2 , $f'(z) \rightarrow 0$, and because $f(z) \geq b_{2n} > 0$ for z on the curve C_{2n} , $f(z) \rightarrow k \geq 0$ as $z \rightarrow \infty$ in A_2 . Thus for n sufficiently large, $b_{2n} < k + 1$. Since $f'(z)dz > 0$ in the positive sense on the part of the curve C_{2n+1} in A_1 , $b_{2n+1} - a_{2n+1} \rightarrow \infty$ as $n \rightarrow \infty$, where $a_{2n+1} > 0$ and thus $b_{2n+1} \rightarrow \infty$ as $n \rightarrow \infty$. Because $b_{2n+1} < b_{2n}$, a contradiction has been reached and $\delta \not> 0$. If $\delta < 0$, then there exists $\varphi_2 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geq \exp(\varphi_2|z|)$ when $z \in A_2$ and $|f'(z)| \leq \exp(-\varphi_2|z|)$ when $z \in A_1$. Similarly, $\delta \not< 0$.

THEOREM. A Riemann surface of the class defined is parabolic and its mapping function f is given by

$$f(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^* \delta_{2k+1}^*/(\gamma_{2k}^*)^2]$$

where

$$f'(z) = [1/(1 + z/\gamma_1)^2] \left[\prod_{k=0}^{\infty} \alpha_{2k+1}^* \prod_{k=1}^{\infty} \beta_k^* / \prod_{k=1}^{\infty} (\gamma_{2k}^*)^3 \right] .$$

Furthermore,

$$\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty, \sum_{k=1}^{\infty} 1/\beta_k < \infty, \sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty, \text{ and } \sum_{k=2}^{\infty} 1/\delta_k < \infty .$$

REMARKS. Lemmas 5 and 6 establish the representation of $f(z)$ as the product of quotients, while Lemmas 8 and 9 show a representation of $f'(z)$ as a quotient of products. However, Lemma 7 can be used to show that the representation of $f(z)$ can also be considered as the quotient of products.

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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* Basil Gordon, Acting Managing Editor until February 1, 1966.

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