THE UNIFORMIZING FUNCTION FOR CERTAIN SIMPLY CONNECTED RIEMANN SURFACES

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This paper contains a definition of a class of simply connected Riemann surfaces, the determination of the type of a surface from this class, and a representation of the uniformizing function and its derivative as infinite products of quotients as well as quotients of infinite products.

Definition of the class of surfaces. Let \( \{a_{2n-1}\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be two sequences of real numbers such that for \( n \geq 1 \),

\[
0 < a_{2n-1} < b_{2n-1} < b_{2n}
\]

and \( b_{2n+1} < b_{2n} \). A surface \( F \) of the class to be discussed consists of sheets \( S_n, n = 1, 2, 3, \ldots \), over the \( w \)-sphere, where for \( S_n \) a copy of the \( w \)-sphere,

(a) \( S_1 \) is slit along the real axis from \( a_1 \) to \( b_1 \).
(b) For \( n \geq 1 \), \( S_{2n} \) is slit along the real axis from \( a_{2n-1} \) to \( b_{2n-1} \) and from \( b_{2n} \) to \( +\infty \).
(c) For \( n \geq 1 \), \( S_{2n+1} \) is slit along the real axis from \( a_{2n+1} \) to \( b_{2n+1} \) and from \( b_{2n} \) to \( +\infty \).
(d) For \( n \geq 1 \), \( S_n \) is joined to \( S_{n+1} \) along the slits to make the \( b_n \) coincide and to form first order branch points at the endpoints of the slits.

The uniformizing function. Because \( F \) is simply connected and noncompact, there exists a unique function \( g \) which maps \( F \) schlichtly and conformally onto \( \{|z| < R \leq \infty\} \), where for \( f(z) = g^{-1}(z) \), \( f(0) = 0 \in S_1 \) and \( f'(0) = 1 \). Two surfaces of hyperbolic type are obtained by slitting each sheet of \( F \) along the uncut parts of the real axis, and an application of the reflection principle to the uniformizing function of one of these surfaces shows that \( f(z) \) is real for real \( z \). Let

\[
\begin{align*}
f(\alpha_{2k-1}) &= a_{2k-1}, 
f(-\beta_k) &= b_k, 
f(\gamma_{2k}) &= \infty \in S_{2k} \quad \text{and} \quad S_{2k+1}, 
f(-\gamma_k) &= \infty \in S_1, 
\end{align*}
\]

and \( f(\delta_k) = 0 \in S_k \). The image of \( F \) in the \( z \)-plane satisfies the following properties. The image of \( S_1 \) is a region which is symmetric about the real axis. \( S_1 \) is mapped onto a domain containing the origin and bounded by a simple closed curve \( C_1 \) which intersects the real axis at \( -\beta_1 \) and \( \alpha_1 \). For \( n \geq 2, S_n \) is mapped onto an annular region about the origin and bounded by two simple closed curves \( C_{n-1} \) and \( C_n \), which

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are images of cuts. For \( n \) odd, \( C_n \) intersects the real axis at \(-\beta_n\) and \( \alpha_n \), while for \( n \) even, \( C_n \) intersects the real axis at \(-\beta_n\) and \( \gamma_n \). Furthermore, for \( k \geq 1 \),
\[
-\beta_{k+1} < -\beta_k < -\gamma_1 < 0 < \alpha_{2k-1} < \delta_{2k} < \gamma_{2k} < \delta_{2k+1} < \alpha_{2k+1}.
\]

The approximating closed surfaces. Let \( F_n \) be the surface formed from the first \( 2n + 2 \) sheets of \( F \) with the slit in \( S_{2n+2} \) from \( b_{2n+2} \) to \( \infty \) deleted, so that \( F_n \) is a compact, simply connected surface.

**Notation.**
\[
\alpha^*_\phi = 1 - z/\alpha^* \quad , \quad \beta^*_\phi = 1 + z/\beta^* \quad ,
\gamma^*_\phi = 1 - z/\gamma^* \quad , \quad \delta^*_\phi = 1 - z/\delta^* .
\]

**Lemma 1.** Let \( R_n \) be the unique rational function which maps the \( z \)-sphere one-to-one onto the simply connected compact surface \( F_n \) with \( R_n(0) = 0 \in S \), \( R_n'(0) = 1 \), and \( R_n(\infty) = \infty \in S_{2n+2} \). Then
\[
R_n(z) = [z/(1 + z/\gamma_{1,n})]\left[\prod_{k=2}^{2n+2} \delta^*_k \right]/\left[\prod_{k=1}^{n} (\gamma^*_{2k,n})^2 \right]
\]
and
\[
R'_n(z) = [1/(1 + z/\gamma_{1,n})^2]\left[\prod_{k=0}^{n} \alpha^*_{2k+1,n} \prod_{k=1}^{2n+1} \beta^*_k \right]/\left[\prod_{k=1}^{n} (\gamma^*_{2k,n})^2 \right].
\]

**Proof.** The representations of \( R_n \) and \( R'_n \) must contain factors shown and can contain no more. The \( \alpha_{2k+1,n}, -\beta_k,n, \gamma_{2k,n}, \) and \( \delta_k,n \), which are ordered in the same manner as the \( \alpha_{2k+1}, -\beta_k, \gamma_{2k}, \) and \( \delta_k \), are images of \( a_{2k+1}, b_k, \infty, \) and \( 0, \) respectively, under \( R^{-1}_n \).

**Lemma 2.** \( F \) is parabolic.

**Proof.** Suppose that \( F \) is hyperbolic, and thus \( g \) maps \( F \) onto \( \{ |z| < R < \infty \} \). If \( D_n \) is the \( z \)-plane slit along the real axis from \(-\beta_{2n+1,n} \) to \( -\infty \), then \( \zeta = \psi_n(z) = g[R_n(z)] \) defines a Schlicht mapping of \( D_n \) onto a simply connected region \( \Delta_n \) of the \( \zeta \)-plane bounded by \( C_{2n+2} \) and the segment \((-\beta_{2n+2}, -\beta_{2n+1}) \). If \( T_n(z) = z(1 - z/4\beta_{2n+1,n})^{-2} \), then \( \zeta = \psi_n(T_n(z)) \) defines a properly normalized, Schlicht mapping of \( \{ |z| < 4\beta_{2n+1,n} \} \) onto \( \Delta_n \) such that if the Koebe Distortion Theorem is applied to this map, then \( \beta_{2n+1,n} \leq d(0, C_{2n+2}) \leq R < \infty \), where \( d(0, C_{2n+2}) \) is the distance from \( \zeta = 0 \) to the curve \( C_{2n+2} \). Thus there exists a subsequence \( \{ \beta_{2n+1,n,j} \} \) such that \( \beta_{2n+1,n,j} \to A \leq R \) as \( j \to \infty \), and \( \psi_{n,j} \) is a Schlicht mapping of \( D_{n,j} \) onto \( \Delta_{n,j} \). If \( D \) is the \( z \)-plane slit along the negative real axis from \(-A \) to \( -\infty \), then \( \{ \psi_{n,j} \} \) forms a family of functions which is normal in \( D \), and hence there exists a subsequence \( \{ \psi_i \} \) such that as \( i \to \infty \), \( \psi_i(z) \to \psi(z) \) uniformly on any compact sub-
set of $D$. Because $D_i \to D$ and $\psi_i(z) \to \psi(z)$ as $i \to \infty$, then $D_i \to \{|z| < R\}$ and $\psi$ maps $D$ onto $\{|\zeta| < R\}$ in a one-to-one manner. ([1], p. 18). Then $R_i(z) = f[\psi_i(z)] \to f[\psi(z)] = H(z)$ uniformly on any compact subset of $D$ as $i \to \infty$, where $H$ is meromorphic in $D$, while $H(z) \neq \infty$ because $R_i(0) = 0$. $H$ maps $D$ onto $F$.

Now let $D^*$ be the $z$-plane slit along the real axis from $-A$ to $+\infty$. For $i$ sufficiently large, $R_i(z)$ assumes no negative real values in any compact subset of $D^*$, and thus $\{R_i\}$ is a family of functions which is normal in $D^*$. Therefore, there exists a subsequence $\{R_m\}$ of $\{R_i\}$ such that as $m \to \infty$, $R_m(z) \to G(z)$ uniformly on any compact subset of $D^*$. $H$ and $G$ have a common domain of convergence, so that $G$ is the analytic continuation of $H$. Then $w = G(z)$ defines a mapping of the $z$-plane punched at $z = A$ and $\infty$ one-to-one and conformally onto an open doubly connected Riemann surface $F^*$ of which $F$ is a subsurface obtained by inserting some slits in $F^*$ over the real axis. This is impossible, as is clear from the definition of $F$. Hence $R = \infty$.

**Lemma 3.** $R_n(z) \to f(z)$ uniformly on any compact subset of the $z$-plane as $n \to \infty$.

**Proof.** Because $D_n \to \{|\zeta| < \infty\}$ as $n \to \infty$, it follows ([1], p. 18) that $z = R_n^{-1}[f(\zeta)] \to \zeta = g[R_n(z)]$ uniformly on any compact subset of the $\zeta$-plane as $n \to \infty$. Also, $D_n \to \{|z| < \infty\}$ and $R_n(z) \to f(z)$ uniformly on any compact subset of the $z$-plane as $n \to \infty$.

**Lemma 4.** $\alpha_{2k-1,n} \to \alpha_{2k-1}$, $\beta_{k,n} \to \beta_k$, $\gamma_{2k,n} \to \gamma_{2k}$, and $\delta_{k,n} \to \delta_k$ as $n \to \infty$.

**Proof.** This is a consequence of Hurwitz's Theorem.

**Lemma 5.** The infinite product

$$\prod (1 + z/\gamma_i) \prod_k \frac{\delta_{2k}}{\delta_{2k+1}/(\gamma_{2k})^2}$$

converges uniformly on any compact subset of the $z$-plane.

**Proof.** Since $\gamma_{2k} \to \infty$ and $\delta_k \to \infty$ as $k \to \infty$, then for any $R > 0$, there exists $n_0 = n_0(R)$ such that for $k \geq n_0$, $\delta_k > R$ and $\gamma_{2k} > R$. Then consider

$$M_p(z) = \prod_{k=n_0}^{n_0+p} \frac{\delta_{2k} \delta_{2k+1}/(\gamma_{2k})^2}.$$  

$M_p$ is holomorphic for $|z| \leq R$ and $M_p(z) \neq 0$ for $|z| \leq R$. A sufficient
condition for the uniform convergence of $M_p(z)$ in $E = \{|z| \leq R\}$ as $p \to \infty$ is the uniform convergence in $E$ of

$$
\sum_{k=n_0}^{n_0+p} \log \left[ \frac{\delta_{2k}^\ast \delta_{2k+1}^\ast / (\gamma_{2k}^\ast)^2}{\delta_{2k+1}^\ast / (\gamma_{2k}^\ast)^2} \right] \text{ as } p \to \infty,
$$

where each logarithm is the principal value. By the Cauchy criterion, this last sequence converges uniformly in $E$ provided for $z \in E$ and for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for $n > N(\varepsilon)$ and $p > 0$,

$$
\left| \sum_{k=n_0}^{n_0+p} \log \left[ \frac{\delta_{2k}^\ast \delta_{2k+1}^\ast / (\gamma_{2k}^\ast)^2}{\delta_{2k+1}^\ast / (\gamma_{2k}^\ast)^2} \right] \right| < \varepsilon.
$$

Now since $\delta_{2k} < \delta_{2k+1}$ and since $\gamma_{2k} < \delta_{2k+2} < \delta_{2k+3}$, then for $m \geq 1$ and $p > 0$,

$$
0 < \sum_{k=n_0}^{n_0+p} \left[ 1/(\delta_{2k})^m + 1/(\delta_{2k+1})^m - 2/(\gamma_{2k})^m \right] < 2/(\delta_{2n_0+2n})^m.
$$

Then for all $p > 0$ and $z \in E$,

$$
\left| \sum_{k=n_0}^{n_0+p} \log \left[ \frac{\delta_{2k}^\ast \delta_{2k+1}^\ast / (\gamma_{2k}^\ast)^2}{\delta_{2k+1}^\ast / (\gamma_{2k}^\ast)^2} \right] \right| = \left| \sum_{m=1}^{\infty} \left[ \frac{z^m}{m} \right] \sum_{k=n_0}^{n_0+p} \left[ (1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m \right] \right|
\leq \sum_{m=1}^{\infty} \left[ R^m/m \right] 2/(\delta_{2n_0+2n})^m \leq 2 \sum_{m=1}^{\infty} \left[ R/(\delta_{2n_0+2n}) \right]^m = 2R/(\delta_{2n_0+2n} - R).
$$

Since $\delta_{2n_0+2n} \to \infty$ as $n \to \infty$, the Cauchy criterion is satisfied and $M_p$ converges uniformly in $E$. Thus $\Pi(z)$ converges uniformly in any compact subset of the $z$-plane.

**Lemma 6.** $\pi(z) = f(z)$.

**Proof.** As a consequence of Lemma 4, there exists $r > 0$ such that $R_n(z)/z \neq 0$ and $\pi(z)/z \neq 0$ for $|z| < r$, while each of these quotients defines a function which is holomorphic for $|z| < r$ and takes the value $1$ at $z = 0$. Thus using the principal value of the logarithm, for $|z| < r$,

$$
\log \left[ R_n(z)/z \right] - \log \left[ \pi(z)/z \right] = \log \left[ R_n(z)/\pi(z) \right] = \log \left[ (1 + z/\gamma_1)/(1 + z/\gamma_{1,n}) \right]
- \sum_{m=1}^{\infty} \left[ \frac{z^m}{m} \right] \left\{ \sum_{k=1}^{n+1} (1/\delta_{2k}^m) + \sum_{k=1}^{n} (1/\delta_{2k+1}^m) - \sum_{k=1}^{n} (2/\gamma_{2k}^m) \right\}
- \sum_{k=1}^{\infty} \left[ (1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m \right].
$$

Therefore, for $n_0 > 2$, as $n \to \infty$,

$$
0 \leq \lim sup \left| \sum_{k=1}^{n+1} (1/\delta_{2k}^m) + \sum_{k=1}^{n} (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m \right|.
$$
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\[ - \sum_{k=1}^{n} \left( \frac{2}{\gamma_{2k}} \right) - \sum_{k=1}^{\infty} \left[ \left( \frac{1}{\delta_{2k}} \right) + \left( \frac{1}{\delta_{2k+1}} \right) - \frac{2}{\gamma_{2k}} \right] \]

\[ \leq \limsup \left\{ \sum_{k=n_0}^{n+1} \left( \frac{1}{\delta_{2k}} \right) + \sum_{k=n_0}^{n} \left( \frac{1}{\delta_{2k+1}} \right) - \sum_{k=n_0}^{n} \left( \frac{2}{\gamma_{2k}} \right) \right\} \]

\[ \leq \limsup \left\{ \left( \frac{1}{\delta_{2k}} \right) + \left( \frac{1}{\delta_{2k+1}} \right) - \frac{2}{\gamma_{2k}} \right\} \]

\[ \leq \limsup \left\{ \left( \frac{1}{\delta_{2k}} \right) + \left( \frac{1}{\delta_{2k+1}} \right) - \frac{2}{\gamma_{2k}} \right\} \]

Since \( \delta_{2n_0} \to \infty \) and \( \delta_{2n_0+1} \to \infty \) as \( n_0 \to \infty \), it follows that the limit as \( n \to \infty \) of each coefficient of the preceding expansion of \( \log \left[ \frac{R_n(z)}{\pi(z)} \right] \) is zero. Furthermore, because as \( n \to \infty \), \( \left\{ \log \left[ \frac{R_n(z)}{\pi(z)} \right] \right\}_{n=1} \) converges uniformly on \( \{ |z| < r \} \), then \( \log \left[ \frac{R_n(z)}{\pi(z)} \right] \to 0 \) as \( n \to \infty \). Thus \( \pi(z) = \lim_{n \to \infty} R_n(z) = f(z) \).

**Lemma 7.** \( \sum_{k=0}^{\infty} \frac{1}{\alpha_{2k+1}} < \infty, \sum_{k=1}^{\infty} \frac{1}{\beta_k} < \infty, \sum_{k=1}^{\infty} \frac{1}{\gamma_{2k}} < \infty \), and

\[ \sum_{k=2}^{\infty} \frac{1}{\delta_k} < \infty \).

**Proof.** Again by Lemma 4, there exists \( r > 0 \) such that \( f''(z) \neq 0 \) and \( R_n(z) \neq 0 \) for \( |z| < r \). Since \( R_n(z) \to f(z) \), it follows that \( R_n(z) \to f''(z) \) and thus \( \log R_n(z) \to \log f''(z) \) uniformly in \( \{ |z| < r \} \) as \( n \to \infty \). Thus for \( |z| < r \),

\[ \lim_{n \to \infty} \left| - \sum_{k=0}^{n} \frac{1}{\alpha_{2k+1}} + \sum_{k=1}^{2n+1} \frac{1}{\beta_{k,n}} - \frac{2}{\gamma_{1,n}} + \sum_{k=1}^{n} \frac{3}{\gamma_{2k,n}} \right| < \infty . \]

Hence, for \( m = 1 \),

\[ \lim_{n \to \infty} \left| - \sum_{k=0}^{n} \frac{1}{\alpha_{2k+1}} + \sum_{k=1}^{2n+1} \frac{1}{\beta_{k,n}} - \frac{2}{\gamma_{1,n}} + \sum_{k=1}^{n} \frac{3}{\gamma_{2k,n}} \right| < \infty . \]

Because \( 0 < \gamma_{1,n} < \beta_{1,n} \) and \( 0 < \gamma_{2k,n} < \alpha_{2k+1,n} \), then

\[ 0 < \sum_{k=1}^{2n+1} \frac{1}{\beta_{k,n}} + \sum_{k=1}^{n} \frac{2}{\gamma_{2k,n}} \]

\[ < - \sum_{k=0}^{n} \frac{1}{\alpha_{2k+1,n}} + \sum_{k=1}^{2n+1} \frac{1}{\beta_{k,n}} - \frac{2}{\gamma_{1,n}} + \sum_{k=1}^{n} \frac{3}{\gamma_{2k,n}} + \frac{1}{\alpha_{1,n}} + \frac{2}{\gamma_{1,n}} \]

Therefore, as \( n \to \infty \),

\[ 0 \leq \limsup \left[ \sum_{k=1}^{2n+1} \frac{1}{\beta_{k,n}} + \sum_{k=1}^{n} \frac{2}{\gamma_{2k,n}} \right] < \infty , \]
\[ \lim \sup_n \sum_{k=1}^{2n+1} \frac{1}{\beta_{k,n}} < \infty, \quad \text{and} \quad \lim \sup_n \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}} < \infty. \]

Furthermore, because for
\[ k \geq 1, \quad \gamma_{2k,n} < \delta_{2k+1,n} < \alpha_{2k+1,n} < \delta_{2k+3,n}, \]
\[ \lim \sup_{n \to \infty} \sum_{k=3}^{2n+2} \frac{1}{\delta_{k,n}} \leq \lim \sup_{n \to \infty} \sum_{k=1}^{n} \frac{2}{\gamma_{2k,n}} < \infty \]

and
\[ \lim \sup_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\alpha_{2k+1,n}} \leq \lim \sup_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}} < \infty. \]

Hence
\[ \lim \sup_{n \to \infty} \sum_{k=0}^{n} \frac{1}{\alpha_{2k+1,n}} < \infty \quad \text{and} \quad \lim \sup_{n \to \infty} \sum_{k=2}^{2n+2} \frac{1}{\delta_{k,n}} < \infty. \]

For all \( N > 0 \), as \( n \to \infty \),
\[ \sum_{k=0}^{N} \frac{1}{\alpha_{2k+1}} = \sum_{k=0}^{N} \lim \frac{1}{\alpha_{2k+1,n}} \leq \lim \sup_{k=0} \sum_{k=0}^{n} \frac{1}{\alpha_{2k+1,n}} < \infty, \]

and thus \( \sum_{k=0}^{\infty} \frac{1}{\alpha_{2k+1}} < \infty \). The convergence of the other series is established in a similar manner.

**Lemma 8.** Each of the three infinite products in
\[ P(z) = \frac{1}{(1 + z/\gamma_f)^2} \left[ \prod_{k=0}^{\infty} \alpha_{2k+1}^{\gamma_{2k+1}} \prod_{k=1}^{\infty} \beta_{k}^{\gamma_{2k}} / \prod_{k=1}^{\infty} (\gamma_{2k}^z)^3 \right] \]
converges uniformly on any compact subset of the \( z \)-plane.

**Proof.** This is a consequence of Lemma 7.

**Lemma 9.** \( f'(z) = [\exp(\delta z)] [P(z)] \) where \( \delta \) is real.

**Proof.** By Lemma 4, there exists \( r > 0 \) such that for \( |z| < r, R'_n(z) \neq 0 \) and \( f'(z) \neq 0 \). For \( m \geq 1 \), consider the coefficient of \( z^m/m \)
in the Taylor expansion of \( \log |R'_n(z)/P(z)| \) about \( z = 0 \) for \( |z| < r \). Because of Lemma 7, there exists \( M > 0 \) such that for all \( n \geq 1 \),
\[ \sum_{k=1}^{n} 1/\gamma_{2k,n} < M \quad \text{and} \quad \sum_{k=1}^{\infty} 1/\gamma_{2k} < M. \]

Then because of the ordering of the \( \gamma_{k,n} \) and \( \gamma_{k} \), for each \( k < n, k/\gamma_{2k,n} < M \) and \( k/\gamma_{2k} < M \). Thus for each \( N > 1 \), as \( n \to \infty \),
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\[ \lim \sup \left| \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}} - \sum_{k=1}^{\infty} \frac{1}{\gamma_{2k}} \right| \leq \lim \sup \left| \sum_{k=N}^{n} \frac{1}{\gamma_{2k,n}} - \sum_{k=N}^{\infty} \frac{1}{\gamma_{2k}} \right| \leq 2M^m \sum_{k=N}^{\infty} 1/k^m, \]

which implies for \( m \geq 2 \), as \( n \to \infty \)

\[ \lim \left[ \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}} - \sum_{k=1}^{\infty} \frac{1}{\gamma_{2k}} \right] = 0 . \]

Similarly, the other terms in the coefficient of \( z^m/m \) have a limit of zero for \( m \geq 2 \), and the coefficient of \( z \) is real. Then as \( n \to \infty \),

\[ \log \left[ R_n(z)/P(z) \right] \to \log \left[ f'(z)/P(z) \right] = \delta, \]

and thus \( f'(z) = [\exp (\delta z)]P(z) \).

**Lemma 10.** \( \delta = 0 \).

**Proof.** Because the factors of \( P(z) \) are canonical products of genus zero with real zeros, for \( \varepsilon > 0 \) and \( 0 < \rho \leq |\arg z| \leq \pi - \rho \), \( P(z) = 0[\exp (\varepsilon |z|)] \) and \( 1/P(z) = 0[\exp (\varepsilon |z|)] \). Then if \( \arg z \) satisfies the preceding conditions and \( |z| \) is sufficiently large, then

\[ \exp [\delta R(z) - \varepsilon |z|] \leq |f'(z)| \leq \exp [\delta R(z) + \varepsilon |z|] . \]

Let \( A_1 = \{ z | \pi/4 \leq \arg z \leq \pi/3 \} \) and \( A_2 = \{ z | 2\pi/3 \leq \arg z \leq 3\pi/4 \} \). If \( \delta > 0 \), then there exists \( \varphi_1 > 0 \) such that for \( |z| \) sufficiently large \( |f'(z)| \geq \exp (\varphi_1 |z|) \) when \( z \in A_1 \) and \( |f'(z)| \leq \exp (-\varphi_1 |z|) \) when \( z \in A_2 \). Thus as \( z \to \infty \) in \( A_1 \), \( f'(z) \to 0 \), and because \( f(z) \geq b_{2n} > 0 \) on the curve \( C_{2n}, f(z) \to k \geq 0 \) as \( z \to \infty \) in \( A_2 \). Thus for \( n \) sufficiently large, \( b_{2n} < k+1 \). Since \( f'(z)dz > 0 \) in the positive sense on the part of the curve \( C_{2n+1} \) in \( A_1 \), \( b_{2n+1} - a_{2n+1} \to \infty \) as \( n \to \infty \), where \( a_{2n+1} > 0 \) and thus \( b_{2n+1} \to \infty \) as \( n \to \infty \). Because \( b_{2n+1} < b_{2n} \), a contradiction has been reached and \( \delta \neq 0 \). If \( \delta < 0 \), then there exists \( \varphi_2 > 0 \) such that for \( |z| \) sufficiently large \( |f'(z)| \geq \exp (\varphi_2 |z|) \) when \( z \in A_2 \) and \( |f'(z)| \leq \exp (-\varphi_2 |z|) \) when \( z \in A_1 \). Similarly, \( \delta \neq 0 \).

**Theorem.** A Riemann surface of the class defined is parabolic and its mapping function \( f \) is given by

\[ f(z) = [z/(1 + z/\gamma_{1})] \prod_{k=1}^{\infty} \left[ \frac{\delta_{2k}^{\gamma_{2k}}}{\delta_{2k+1}^{\gamma_{2k+1}}/(\gamma_{2k}^{\gamma_{2k}})} \right] \]

where

\[ f'(z) = [1/(1 + z/\gamma_{1})^{2}] \left[ \prod_{k=0}^{\infty} \alpha_{2k+1}^{\beta_{k}} \prod_{k=1}^{\infty} \beta_{k}^{\gamma_{2k}} \prod_{k=1}^{\infty} (1/\gamma_{2k})^{3} \right] . \]

Furthermore,

\[ \sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty, \sum_{k=1}^{\infty} 1/\beta_{k} < \infty, \sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty, \text{ and } \sum_{k=2}^{\infty} 1/\delta_{k} < \infty . \]
Remarks. Lemmas 5 and 6 establish the representation of $f(z)$ as the product of quotients, while Lemmas 8 and 9 show a representation of $f''(z)$ as a quotient of products. However, Lemma 7 can be used to show that the representation of $f(z)$ can also be considered as the quotient of products.

Reference


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