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THE UNIFORMIZING FUNCTION FOR CERTAIN SIMPLY CONNECTED RIEMANN SURFACES

HOWARD BENTON CURTIS, JR.

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This paper contains a definition of a class of simply connected Riemann surfaces, the determination of the type of a surface from this class, and a representation of the uniformizing function and its derivative as infinite products of quotients as well as quotients of infinite products.

Definition of the class of surfaces. Let $\{a_{2n-1}\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that for $n \geq 1$,

$$0 < a_{2n-1} < b_{2n-1} < b_{2n}$$

and $b_{2n+1} < b_{2n}$. A surface F of the class to be discussed consists of sheets S_n , $n = 1, 2, 3, \dots$, over the w -sphere, where for S_n a copy of the w -sphere,

- (a) S_1 is slit along the real axis from a_1 to b_1 .
- (b) For $n \geq 1$, S_{2n} is slit along the real axis from a_{2n-1} to b_{2n-1} and from b_{2n} to $+\infty$.
- (c) For $n \geq 1$, S_{2n+1} is slit along the real axis from a_{2n+1} to b_{2n+1} and from b_{2n} to $+\infty$.
- (d) For $n \geq 1$, S_n is joined to S_{n+1} along the slits to make the b_n coincide and to form first order branch points at the end-points of the slits.

The uniformizing function. Because F is simply connected and noncompact, there exists a unique function g which maps F schlichtly and conformally onto $\{|z| < R \leq \infty\}$, where for $f(z) = g^{-1}(z)$, $f(0) = 0 \in S_1$ and $f'(0) = 1$. Two surfaces of hyperbolic type are obtained by slitting each sheet of F along the uncut parts of the real axis, and an application of the reflection principle to the uniformizing function of one of these surfaces shows that $f(z)$ is real for real z . Let $f(\alpha_{2k-1}) = a_{2k-1}$, $f(-\beta_k) = b_k$, $f(\gamma_{2k}) = \infty \in S_{2k}$ and S_{2k+1} , $f(-\gamma_1) = \infty \in S_1$, and $f(\delta_k) = 0 \in S_k$. The image of F in the z -plane satisfies the following properties. The image of S_n is a region which is symmetric about the real axis. S_1 is mapped onto a domain containing the origin and bounded by a simple closed curve C_1 which intersects the real axis at $-\beta_1$ and α_1 . For $n \geq 2$, S_n is mapped onto an annular region about the origin and bounded by two simple closed curves C_{n-1} and C_n , which

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are images of cuts. For n odd, C_n intersects the real axis at $-\beta_n$ and α_n , while for n even, C_n intersects the real axis at $-\beta_n$ and γ_n . Furthermore, for $k \geq 1$,

$$-\beta_{k+1} < -\beta_k < -\gamma_1 < 0 < \alpha_{2k-1} < \delta_{2k} < \gamma_{2k} < \delta_{2k+1} < \alpha_{2k+1} .$$

The approximating closed surfaces. Let F_n be the surface formed from the first $2n + 2$ sheets of F with the slit in S_{2n+2} from b_{2n+2} to ∞ deleted, so that F_n is a compact, simply connected surface.

NOTATION. $\alpha_\varphi^* = 1 - z/\alpha_\varphi$, $\beta_\varphi^* = 1 + z/\beta_\varphi$,
 $\gamma_\varphi^* = 1 - z/\gamma_\varphi$, $\delta_\varphi^* = 1 - z/\delta_\varphi$.

LEMMA 1. Let R_n be the unique rational function which maps the z -sphere one-to-one onto the simply connected compact surface F_n with $R_n(0) = 0 \in S_1$, $R'_n(0) = 1$, and $R_n(\infty) = \infty \in S_{2n+2}$. Then

$$R_n(z) = [z/(1 + z/\gamma_{1,n})] \left[\prod_{k=2}^{2n+2} \delta_{k,n}^* \right] / \left[\prod_{k=1}^n (\gamma_{2k,n}^*)^2 \right]$$

and

$$R'_n(z) = [1/(1 + z/\gamma_{1,n})^2] \left[\prod_{k=0}^n \alpha_{2k+1,n}^* \right] \left[\prod_{k=1}^{2n+1} \beta_{k,n}^* \right] / \left[\prod_{k=1}^n (\gamma_{2k,n}^*)^3 \right] .$$

Proof. The representations of R_n and R'_n must contain factors shown and can contain no more. The $\alpha_{2k+1,n}$, $-\beta_{k,n}$, $\gamma_{2k,n}$, and $\delta_{k,n}$, which are ordered in the same manner as the α_{2k+1} , $-\beta_k$, γ_{2k} , and δ_k , are images of a_{2k+1} , b_k , ∞ , and 0 , respectively, under R_n^{-1} .

LEMMA 2. F is parabolic.

Proof. Suppose that F is hyperbolic, and thus g maps F onto $\{|z| < R < \infty\}$. If D_n is the z -plane slit along the real axis from $-\beta_{2n+1,n}$ to $-\infty$, then $\zeta = \psi_n(z) = g[R_n(z)]$ defines a Schlicht mapping of D_n onto a simply connected region Δ_n of the ζ -plane bounded by C_{2n+2} and the segment $(-\beta_{2n+2}, -\beta_{2n+1})$. If $T_n(z) = z(1 - z/4\beta_{2n+1,n})^{-2}$, then $\zeta = \psi_n[T_n(z)]$ defines a properly normalized, Schlicht mapping of $\{|z| < 4\beta_{2n+1,n}\}$ onto Δ_n such that if the Koebe Distortion Theorem is applied to this map, then $\beta_{2n+1,n} \leq d(0, C_{2n+2}) \leq R < \infty$, where $d(0, C_{2n+2})$ is the distance from $\zeta = 0$ to the curve C_{2n+2} . Thus there exists a subsequence $\{\beta_{2n_j+1,n_j}\}$ such that $\beta_{2n_j+1,n_j} \rightarrow A \leq R$ as $j \rightarrow \infty$, and ψ_{n_j} is a Schlicht mapping of D_{n_j} onto Δ_{n_j} . If D is the z -plane slit along the negative real axis from $-A$ to $-\infty$, then $\{\psi_{n_j}\}$ forms a family of functions which is normal in D , and hence there exists a subsequence $\{\psi_i\}$ such that as $i \rightarrow \infty$, $\psi_i(z) \rightarrow \psi(z)$ uniformly on any compact sub-

set of D . Because $D_i \rightarrow D$ and $\psi_i(z) \rightarrow \psi(z)$ as $i \rightarrow \infty$, then $\Delta_i \rightarrow \{|z| < R\}$ and ψ maps D onto $\{|\zeta| < R\}$ in a one-to-one manner. ([1], p. 18). Then $R_i(z) = f[\psi_i(z)] \rightarrow f[\psi(z)] = H(z)$ uniformly on any compact subset of D as $i \rightarrow \infty$, where H is meromorphic in D , while $H(z) \not\equiv \infty$ because $R_i(0) = 0$. H maps D onto F .

Now let D^* be the z -plane slit along the real axis from $-A$ to $+\infty$. For i sufficiently large, $R_i(z)$ assumes no negative real values in any compact subset of D^* , and thus $\{R_i\}$ is a family of functions which is normal in D^* . Therefore, there exists a subsequence $\{R_m\}$ of $\{R_i\}$ such that as $m \rightarrow \infty$, $R_m(z) \rightarrow G(z)$ uniformly on any compact subset of D^* . H and G have a common domain of convergence, so that G is the analytic continuation of H . Then $w = G(z)$ defines a mapping of the z -plane punched at $z = A$ and ∞ one-to-one and conformally onto an open doubly connected Riemann surface F^* of which F is a subsurface obtained by inserting some slits in F^* over the real axis. This is impossible, as is clear from the definition of F . Hence $R = \infty$.

LEMMA 3. $R_n(z) \rightarrow f(z)$ uniformly on any compact subset of the z -plane as $n \rightarrow \infty$.

Proof. Because $\Delta_n \rightarrow \{|\zeta| < \infty\}$ as $n \rightarrow \infty$, it follows ([1], p. 18) that $z = R_n^{-1}[f(\zeta)] \rightarrow \zeta = g[R_n(z)]$ uniformly on any compact subset of the ζ -plane as $n \rightarrow \infty$. Also, $D_n \rightarrow \{|z| < \infty\}$ and $R_n(z) \rightarrow f(z)$ uniformly on any compact subset of the z -plane as $n \rightarrow \infty$.

LEMMA 4. $\alpha_{2k-1,n} \rightarrow \alpha_{2k-1}$, $\beta_{k,n} \rightarrow \beta_k$, $\gamma_{2k,n} \rightarrow \gamma_{2k}$, and $\delta_{k,n} \rightarrow \delta_k$ as $n \rightarrow \infty$.

Proof. This is a consequence of Hurwitz's Theorem.

LEMMA 5. The infinite product

$$\pi(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^* \delta_{2k+1}^* / (\gamma_{2k}^*)^2]$$

converges uniformly on any compact subset of the z -plane.

Proof. Since $\gamma_{2k} \rightarrow \infty$ and $\delta_k \rightarrow \infty$ as $k \rightarrow \infty$, then for any $R > 0$, there exists $n_0 = n_0(R)$ such that for $k \geq n_0$, $\delta_k > R$ and $\gamma_{2k} > R$. Then consider

$$M_p(z) = \prod_{k=n_0}^{n_0+p} [\delta_{2k}^* \delta_{2k+1}^* / (\gamma_{2k}^*)^2].$$

M_p is holomorphic for $|z| \leq R$ and $M_p(z) \neq 0$ for $|z| \leq R$. A sufficient

condition for the uniform convergence of $M_p(z)$ in $E = \{|z| \leq R\}$ as $p \rightarrow \infty$ is the uniform convergence in E of

$$\sum_{k=n_0}^{n_0+p} \log [\delta_{2k}^* \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \text{ as } p \rightarrow \infty ,$$

where each logarithm is the principal value. By the Cauchy criterion, this last sequence converges uniformly in E provided for $z \in E$ and for any $\epsilon > 0$, there exists $N(\epsilon) > 0$ such that for $n > N(\epsilon)$ and $p > 0$,

$$\left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \right| < \epsilon .$$

Now since $\delta_{2k} < \delta_{2k+1}$ and since $\gamma_{2k} < \delta_{2k+2} < \delta_{2k+3}$, then for $m \geq 1$ and $p > 0$,

$$0 < \sum_{k=n_0+n}^{n_0+n+p} [1/(\delta_{2k})^m + 1/(\delta_{2k+1})^m - 2/(\gamma_{2k})^m] < 2/(\delta_{2n_0+2n})^m .$$

Then for all $p > 0$ and $z \in E$,

$$\begin{aligned} \left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \right| &= \left| -\sum_{m=1}^{\infty} [z^m/m] \sum_{k=n_0+n}^{n_0+n+p} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right| \\ &\leq \sum_{m=1}^{\infty} [R^m/m] [2/(\delta_{2n_0+2n})^m] \leq 2 \sum_{m=1}^{\infty} [R/(\delta_{2n_0+2n})]^m = 2R/(\delta_{2n_0+2n} - R) . \end{aligned}$$

Since $\delta_{2n_0+2n} \rightarrow \infty$ as $n \rightarrow \infty$, the Cauchy criterion is satisfied and M_p converges uniformly in E . Thus $\Pi(z)$ converges uniformly in any compact subset of the z -plane.

LEMMA 6. $\pi(z) = f(z)$.

Proof. As a consequence of Lemma 4, there exists $r > 0$ such that $R_n(z)/z \neq 0$ and $\pi(z)/z \neq 0$ for $|z| < r$, while each of these quotients defines a function which is holomorphic for $|z| < r$ and takes the value 1 at $z = 0$. Thus using the principal value of the logarithm, for $|z| < r$,

$$\begin{aligned} \log [R_n(z)/z] - \log [\pi(z)/z] &= \log [R_n(z)/\pi(z)] = \log [(1 + z/\gamma_1)/(1 + z/\gamma_{1,n})] \\ &= \sum_{m=1}^{\infty} \{z^m/m\} \left\{ \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^n (1/\delta_{2k+1,n}^m) - \sum_{k=1}^n (2/\gamma_{2k,n}^m) \right. \\ &\quad \left. - \sum_{k=1}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right\} . \end{aligned}$$

Therefore, for $n_0 > 2$, as $n \rightarrow \infty$,

$$0 \leq \lim \sup \left| \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^n (1/\delta_{2k+1,n}^m) \right|$$

$$\begin{aligned}
 & - \sum_{k=1}^n (2/\gamma_{2k,n}) - \sum_{k=1}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \\
 & \cong \limsup \left| \sum_{k=n_0}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=n_0}^n (1/\delta_{2k+1,n}^m) - \sum_{k=n_0}^n (2/\gamma_{2k,n}^m) \right. \\
 & \quad \left. - \sum_{k=n_0}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right| \\
 & \cong \limsup |(1/\delta_{2n_0,n}^m) + (1/\delta_{2n_0+1,n}^m) + (1/\delta_{2n_0}^m) + (1/\delta_{2n_0+1}^m)| \\
 & = (2/\delta_{2n_0}^m) + (2/\delta_{2n_0+1}^m) .
 \end{aligned}$$

Since $\delta_{2n_0} \rightarrow \infty$ and $\delta_{2n_0+1} \rightarrow \infty$ as $n_0 \rightarrow \infty$, it follows that the limit as $n \rightarrow \infty$ of each coefficient of the preceding expansion of $\log [R_n(z)/\pi(z)]$ is zero. Furthermore, because as $n \rightarrow \infty$, $\{\log [R_n(z)/\pi(z)]\}_{n=1}^{\infty}$ converges uniformly on $\{|z| < r\}$, then $\log [R_n(z)/\pi(z)] \rightarrow 0$ as $n \rightarrow \infty$. Thus $\pi(z) = \lim_{n \rightarrow \infty} R_n(z) = f(z)$.

LEMMA 7. $\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$, $\sum_{k=1}^{\infty} 1/\beta_k < \infty$, $\sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty$, and $\sum_{k=2}^{\infty} 1/\delta_k < \infty$.

Proof. Again by Lemma 4, there exists $r > 0$ such that $f'(z) \neq 0$ and $R'_n(z) \neq 0$ for $|z| < r$. Since $R_n(z) \rightarrow f(z)$, it follows that $R'_n(z) \rightarrow f'(z)$ and thus $\log R'_n(z) \rightarrow \log f'(z)$ uniformly in $\{|z| < r\}$ as $n \rightarrow \infty$. Thus for $|z| < r$, $\log R'_n(z)$

$$\begin{aligned}
 & = \sum_{m=1}^{\infty} [z^m/m] \left[- \sum_{k=0}^n 1/\alpha_{2k+1,n}^m \right. \\
 & \quad \left. + \sum_{k=1}^{2n+1} (-1)^{m+1}/\beta_{k,n}^m + 2(-1)^m/\gamma_{1,n}^m + \sum_{k=1}^n 3/\gamma_{2k,n}^m \right] .
 \end{aligned}$$

Hence, for $m = 1$,

$$\lim_{n \rightarrow \infty} \left| - \sum_{k=0}^n 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n} + \sum_{k=1}^n 3/\gamma_{2k,n} \right| < \infty .$$

Because $0 < \gamma_{1,n} < \beta_{1,n}$ and $0 < \gamma_{2k,n} < \alpha_{2k+1,n}$, then

$$\begin{aligned}
 0 & < \sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^n 2/\gamma_{2k,n} \\
 & < - \sum_{k=0}^n 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n} \\
 & \quad + \sum_{k=1}^n 3/\gamma_{2k,n} + 1/\alpha_{1,n} + 2/\gamma_{1,n} .
 \end{aligned}$$

Therefore, as $n \rightarrow \infty$,

$$0 \leq \limsup \left[\sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^n 2/\gamma_{2k,n} \right] < \infty ,$$

$$\limsup \sum_{k=1}^{2n+1} 1/\beta_{k,n} < \infty, \text{ and } \limsup \sum_{k=1}^n 1/\gamma_{2k,n} < \infty .$$

Furthermore, because for

$$k \geq 1, \gamma_{2k,n} < \delta_{2k+1,n} < \alpha_{2k+1,n} < \delta_{2k+2,n} ,$$

$$\limsup_{n \rightarrow \infty} \sum_{k=3}^{2n+2} 1/\delta_{k,n} \leq \limsup_{n \rightarrow \infty} \sum_{k=1}^n 2/\gamma_{2k,n} < \infty$$

and

$$\limsup_{n \rightarrow \infty} \sum_{k=1}^n 1/\alpha_{2k+1,n} \leq \limsup_{n \rightarrow \infty} \sum_{k=1}^n 1/\gamma_{2k,n} < \infty .$$

Hence

$$\limsup_{n \rightarrow \infty} \sum_{k=0}^n 1/\alpha_{2k+1,n} < \infty \text{ and } \limsup_{n \rightarrow \infty} \sum_{k=2}^{2n+2} 1/\delta_{k,n} < \infty .$$

For all $N > 0$, as $n \rightarrow \infty$,

$$\sum_{k=0}^N 1/\alpha_{2k+1} = \sum_{k=0}^N \lim 1/\alpha_{2k+1,n} \leq \limsup \sum_{k=0}^n 1/\alpha_{2k+1,n} < \infty ,$$

and thus $\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$. The convergence of the other series is established in a similar manner.

LEMMA 8. *Each of the three infinite products in*

$$P(z) = [1/(1 + z/\gamma_1)^2] \left[\prod_{k=0}^{\infty} \alpha_{2k+1}^* \prod_{k=1}^{\infty} \beta_k^* / \prod_{k=1}^{\infty} (\gamma_{2k}^*)^3 \right]$$

converges uniformly on any compact subset of the z -plane.

Proof. This is a consequence of Lemma 7.

LEMMA 9. $f'(z) = [\exp(\delta z)][P(z)]$ where δ is real.

Proof. By Lemma 4, there exists $r > 0$ such that for $|z| < r$, $R'_n(z) \neq 0$ and $f'(z) \neq 0$. For $m \geq 1$, consider the coefficient of z^m/m in the Taylor expansion of $\log [R'_n(z)/P(z)]$ about $z = 0$ for $|z| < r$. Because of Lemma 7, there exists $M > 0$ such that for all $n \geq 1$,

$$\sum_{k=1}^n 1/\gamma_{2k,n} < M \text{ and } \sum_{k=1}^{\infty} 1/\gamma_{2k} < M .$$

Then because of the ordering of the $\gamma_{k,n}$ and γ_k , for each $k < n$, $k/\gamma_{2k,n} < M$ and $k/\gamma_{2k} < M$. Thus for each $N > 1$, as $n \rightarrow \infty$,

$$\begin{aligned} & \limsup \left| \sum_{k=1}^n 1/\gamma_{2k,n}^m - \sum_{k=1}^{\infty} 1/\gamma_{2k}^m \right| \\ & \leq \limsup \left| \sum_{k=N}^n 1/\gamma_{2k,n}^m - \sum_{k=N}^{\infty} 1/\gamma_{2k}^m \right| \leq 2M^m \sum_{k=N}^{\infty} 1/k^m, \end{aligned}$$

which implies for $m \geq 2$, as $n \rightarrow \infty$

$$\lim \left[\sum_{k=1}^n 1/\gamma_{2k,n}^m - \sum_{k=1}^{\infty} 1/\gamma_{2k}^m \right] = 0 .$$

Similarly, the other terms in the coefficient of z^m/m have a limit of zero for $m \geq 2$, and the coefficient of z is real. Then as $n \rightarrow \infty$, $\log [R'_n(z)/P(z)] \rightarrow \log [f'(z)/P(z)] = \delta z$, and thus $f'(z) = [\exp(\delta z)][P(z)]$.

LEMMA 10. $\delta = 0$.

Proof. Because the factors of $P(z)$ are canonical products of genus zero with real zeros, for $\varepsilon > 0$ and $0 < \rho \leq |\arg z| \leq \pi - \rho$, $P(z) = 0[\exp(\varepsilon|z|)]$ and $1/P(z) = 0[\exp(\varepsilon|z|)]$. Then if $\arg z$ satisfies the preceding conditions and $|z|$ is sufficiently large, then

$$\exp[\delta \mathcal{R}(z) - \varepsilon|z|] \leq |f'(z)| \leq \exp[\delta \mathcal{R}(z) + \varepsilon|z|] .$$

Let $A_1 = \{z \mid \pi/4 \leq \arg z \leq \pi/3\}$ and $A_2 = \{z \mid 2\pi/3 \leq \arg z \leq 3\pi/4\}$. If $\delta > 0$, then there exists $\varphi_1 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geq \exp(\varphi_1|z|)$ when $z \in A_1$ and $|f'(z)| \leq \exp(-\varphi_1|z|)$ when $z \in A_2$. Thus as $z \rightarrow \infty$ in A_2 , $f'(z) \rightarrow 0$, and because $f(z) \geq b_{2n} > 0$ for z on the curve C_{2n} , $f(z) \rightarrow k \geq 0$ as $z \rightarrow \infty$ in A_2 . Thus for n sufficiently large, $b_{2n} < k + 1$. Since $f'(z)dz > 0$ in the positive sense on the part of the curve C_{2n+1} in A_1 , $b_{2n+1} - a_{2n+1} \rightarrow \infty$ as $n \rightarrow \infty$, where $a_{2n+1} > 0$ and thus $b_{2n+1} \rightarrow \infty$ as $n \rightarrow \infty$. Because $b_{2n+1} < b_{2n}$, a contradiction has been reached and $\delta \not> 0$. If $\delta < 0$, then there exists $\varphi_2 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geq \exp(\varphi_2|z|)$ when $z \in A_2$ and $|f'(z)| \leq \exp(-\varphi_2|z|)$ when $z \in A_1$. Similarly, $\delta \not< 0$.

THEOREM. A Riemann surface of the class defined is parabolic and its mapping function f is given by

$$f(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^* \delta_{2k+1}^*/(\gamma_{2k}^*)^2]$$

where

$$f'(z) = [1/(1 + z/\gamma_1)^2] \left[\prod_{k=0}^{\infty} \alpha_{2k+1}^* \prod_{k=1}^{\infty} \beta_k^* / \prod_{k=1}^{\infty} (\gamma_{2k}^*)^3 \right] .$$

Furthermore,

$$\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty, \sum_{k=1}^{\infty} 1/\beta_k < \infty, \sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty, \text{ and } \sum_{k=2}^{\infty} 1/\delta_k < \infty .$$

REMARKS. Lemmas 5 and 6 establish the representation of $f(z)$ as the product of quotients, while Lemmas 8 and 9 show a representation of $f'(z)$ as a quotient of products. However, Lemma 7 can be used to show that the representation of $f(z)$ can also be considered as the quotient of products.

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