

# Pacific Journal of Mathematics

**A SET OF NONNORMAL NUMBERS**

MICHEL MENDÈS FRANCE

## A SET OF NONNORMAL NUMBERS

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Let  $P$  be the set of real polynomials and let  $E(P)$  be the set of real numbers whose  $n$ th binary digit from a certain point on is 0 or 1 according as  $[\varphi(n)]$  is even or odd for some  $\varphi \in P$ . We prove that no number in  $E(P)$  is normal in the binary system and that  $E(P)$  has Hausdorff dimension 0.

Some notations and definitions. It is well known that every real number  $x$  of the unit interval which is not a binary fraction can be expanded in the binary system

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{2^n}$$

where  $(\varepsilon_n(x))_{n \in \mathbb{N}}$  is a uniquely determined sequence of functions taking values 0 or 1. The functions  $r_n(x) = 1 - 2\varepsilon_n(x)$  are known as the Rademacher functions.

We shall say that  $x$  is a normal number (in the binary system) if for every positive integer  $s$  and every sequence of positive, strictly increasing integers  $k_1, k_2, \dots, k_s$  one has:

$$(1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N r_{n+k_1}(x) \cdots r_{n+k_s}(x) = 0.$$

One can prove that this definition is equivalent to the other usual ones [3], [4], [6].

If  $t$  is a real number,  $[t]$  will denote the greatest integer not greater than  $t$  and  $\{t\} = t - [t]$  the fractional part of  $t$ .

Let  $P$  be the set of real polynomials and let  $E(P)$  be the set of points  $x$  such that for some  $\varphi \in P$  and for some  $n_0 \geq 0$ ,  $r_n(x) = \exp i\pi[\varphi(n)]$  for all integers  $n > n_0$ .

We wish to prove first the following theorem:

**THEOREM 1.**  $E(P)$  contains only nonnormal numbers.

This result shows that the measure of  $E(P)$  is null, since almost all numbers are normal. Now the question arises if  $E(P)$  contains "almost all" (in a sense soon to be made precise) nonnormal numbers or not. We answer this question by stating the known result:

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The Hausdorff dimension of the set of nonnormal numbers is 1, (see for example [1]),

and by proving our second theorem :

**THEOREM 2.** *The Hausdorff dimension of  $E(P)$  is 0.*

**2. Proof of Theorem 1.** Let  $x$  be an element of  $E(P)$ . We show that for a certain sequence of increasing positive integers  $k_1, k_2, \dots, k_s$  the equation (1) does not hold.

Let  $\varphi$  be a polynomial such that  $r_n(x) = \exp i\pi[\varphi(n)]$  for all sufficiently large integers  $n$ . Without loss of generality we may suppose that this relation holds for all positive integers, for normality or non-normality are asymptotic properties. Let the expansion of  $\varphi$  be

$$(2) \quad \varphi(n) = \alpha_\nu n^\nu + \alpha_{\nu-1} n^{\nu-1} + \dots + \alpha_1 n + \alpha_0, \quad \nu \geq 1.$$

If all the numbers  $\alpha_j (1 \leq j \leq \nu)$  are rational, then  $x$  is clearly rational, hence nonnormal. If one of the numbers  $\alpha_j (1 \leq j \leq \nu)$  is irrational, we can without loss of generality suppose that the leading coefficient  $\alpha_\nu$  is irrational. Indeed, suppose that  $\alpha_\mu (1 \leq \mu < \nu)$  is irrational and that  $\alpha_{\mu+1}, \dots, \alpha_\nu$  are rational. Let  $q$  be the least common denominator of the  $\nu - \mu$  fractions  $\alpha_{\mu+1}, \dots, \alpha_\nu$ . If  $x$  is normal, then so is the number  $y$  defined by  $r_n(y) = \exp i\pi[\varphi(2qn)]$  for all integers  $n$ . But clearly  $[\varphi(2qn)] \equiv [\psi(n)] \pmod{2}$  where  $\psi(n) = \alpha_\mu (2q)^\mu n^\mu + \dots + \alpha_0$ . This shows that we can now deal with  $\psi$ , the leading coefficient of which is irrational.

From now on in this section,  $\varphi$  is defined by equation (2) where  $\alpha_\nu$  is an irrational number.

We need the known identity for polynomials of degree  $\nu$  :

$$(3) \quad \varphi(n + \nu) \equiv \binom{\nu}{1} \varphi(n + \nu - 1) - \binom{\nu}{2} \varphi(n + \nu - 2) + \dots + (-1)^{\nu-1} \binom{\nu}{\nu} \varphi(n) + \nu! \alpha_\nu$$

and the lemma :

**LEMMA 1.** *If  $F(x_1, x_2, \dots, x_\nu)$  is a Riemann integrable function which is of period 1 in each variable and if  $\varphi$  is a real polynomial of degree  $\nu$ , the leading coefficient of which is irrational, then the following equality holds :*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N F(\varphi(n), \varphi(n + 1), \dots, \varphi(n + \nu - 1)) = \int_{T^\nu} F(x_1, x_2, \dots, x_\nu) dx_1, \dots, dx_\nu.$$

This is a very well known corollary of Weyl's theorems on uniform distribution (see for example [2]).

Combining equality (3) and Lemma 1, one can write :

$$\begin{aligned}
 L &= \lim \frac{1}{N} \sum_{n=1}^N \exp i\pi([\varphi(n + \nu)] \\
 &\quad - \binom{\nu}{1}[\varphi(n + \nu - 1)] + \dots + (-1)^\nu \binom{\nu}{\nu}[\varphi(n)]) \\
 &= \lim \frac{1}{N} \sum_{n=1}^N \exp i\pi\left(\left[\binom{\nu}{1}\varphi(n + \nu - 1)\right.\right. \\
 &\quad \left.\left. - \binom{\nu}{2}\varphi(n + \nu - 2) + \dots + \nu! \alpha_\nu\right] \right. \\
 &\quad \left. - \binom{\nu}{1}[\varphi(n + \nu - 1)] + \dots + (-1)^\nu[\varphi(n)]\right) \\
 &= \int_{T^\nu} \exp i\pi\left(\left[\binom{\nu}{1}2x_\nu - \binom{\nu}{2}2x_{\nu-1} + \dots + \nu! \alpha_\nu\right] \right. \\
 &\quad \left. - \binom{\nu}{1}[2x_\nu] + \dots + (-1)^\nu[2x_1]\right) dx_1 \dots dx_\nu .
 \end{aligned}$$

By putting  $2x_j = y_j, j = 1, 2, \dots, \nu$ , the integral becomes

$$\begin{aligned}
 L &= \frac{1}{2^\nu} \int_{(0,2)^\nu} \exp i\pi \left( \left[ \binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \dots + \nu! \alpha_\nu \right] \right. \\
 &\quad \left. - \binom{\nu}{1}[y_\nu] + \dots + (-1)^\nu[y_1] \right) dy_1 \dots dy_\nu .
 \end{aligned}$$

Now the identity  $[x + \varepsilon y] = [x] + \varepsilon[y] + [\{x\} + \varepsilon\{y\}]$ ,  $\varepsilon = \pm 1$  shows that one has :

$$\begin{aligned}
 \left[ \binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \dots + \nu! \alpha_\nu \right] &= \binom{\nu}{1}[y_\nu] - \binom{\nu}{2}[y_{\nu-1}] + \dots + [\nu! \alpha_\nu] \\
 &\quad + \left[ \binom{\nu}{1}\{y_\nu\} - \dots + \{\nu! \alpha_\nu\} \right]
 \end{aligned}$$

so that :

$$\begin{aligned}
 L &= \frac{\pm 1}{2^\nu} \int_{(0,2)^\nu} \exp i\pi \left[ \binom{\nu}{1}\{y_\nu\} - \binom{\nu}{2}\{y_{\nu-1}\} + \dots + \{\nu! \alpha_\nu\} \right] dy_1 \dots dy_\nu \\
 &= \pm \int_{T^\nu} \exp i\pi \left[ \binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \dots + \{\nu! \alpha_\nu\} \right] dy_1 \dots dy_\nu .
 \end{aligned}$$

Consider the hyperplane  $\binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \dots + (-1)^{\nu-1}y_1 = -\{\nu! \alpha_\nu\}$  in the euclidean space  $R^\nu$ . It has rational coefficients except for the constant term, which is irrational. Hence it cannot split the unit cube  $(0, 1)^\nu$  into two regions of equal volume. Therefore the integral  $L$

cannot be 0. Finally we notice that  $L$  may be written

$$L = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N r_{n+\nu}(x) (r_{n+\nu-1}(x))^{(1)} \cdots r_n(x);$$

this completes the demonstration.

**3. Proof of Theorem 2.** Let  $P_\nu$  denote the set of real polynomials of degree  $\nu$ , the coefficients of which are all in the interval  $[0, 2[$ . It is easily seen that to prove Theorem 2, it is sufficient to prove the lemma:

**LEMMA 2.** *Let  $E^\nu$  be the set of numbers  $x$  such that for some  $\varphi \in P_\nu$ ,  $r_n(x) = \exp i\pi[\varphi(n)]$  for all integers  $n$ . Then the Hausdorff dimension of  $E^\nu$  is 0.*

$$\text{Let } \varphi(n) = \alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0, \quad \alpha_j \in [0, 2]$$

and let  $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_\nu)$  be a point in the space  $(0, 2)^{\nu+1}$ . We are going to estimate the number  $N_\nu(p)$  of regions in  $(0, 2)^{\nu+1}$  which have the following property: when  $\alpha$  ranges over one of these regions, the sequence  $[\varphi(1)], [\varphi(2)], \cdots, [\varphi(p)]$  stays invariant. First let us show:

**LEMMA 3.** *The  $h$ -dimensional measure ( $0 \leq h \leq 1$ ) of the set  $E_p^\nu = \{x \mid r_n(x) = \exp i\pi[\varphi(n)]; n = 1, 2, \cdots, p; \varphi \in P_\nu\}$  satisfies the inequality*

$$h\text{-meas } (E_p^\nu) \leq \frac{N_\nu(p)}{2^{ph}}.$$

Indeed, when  $\varphi$  runs through  $P_\nu$ ,  $\alpha$  ranges over  $(0, 2)^{\nu+1}$ . The set  $E_p^\nu$  is composed of at most  $N_\nu(p)$  intervals, each of which has  $h$ -length  $\left(\frac{1}{2^p}\right)^h$ .

Now, if one notices that  $E^\nu = \bigcap_{p=1}^\infty E_p^\nu$ , one gets the result that the Hausdorff dimension of  $E^\nu$  cannot be greater than

$$\delta = \liminf_{p \rightarrow \infty} \frac{\log N_\nu(p)}{p \log 2}.$$

We wish to show that  $\delta = 0$  and we shall do so by proving our last lemma:

**LEMMA 4.** *When  $p$  goes to infinity, one has*

$$N_\nu(p) = 0(p^{(\nu+1)^2}) .$$

*Proof.* Let  $q$  be an integer such that

$$0 \leq q \leq 2(n^\nu + n^{\nu-1} + \dots + n + 1) - 1$$

Consider the set  $R_{n,q}$  of the points  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_\nu)$  defined by

$$q \leq \alpha_\nu n^\nu + \dots + \alpha_1 n + \alpha_0 < q + 1 .$$

Clearly, when  $\alpha$  runs through the region  $R_{n,q}$ , the quantity  $[\varphi(n)] = [\alpha_\nu n^\nu + \dots + \alpha_1 n + \alpha_0]$  stays equal to  $q$ . Then let  $q_1, q_2, \dots, q_p$  be any sequence of integers such that  $0 \leq q_j < 2(j^\nu + \dots + j + 1)$ ,  $j = 1, 2, \dots, p$ . When  $\alpha$  ranges over the set  $\bigcap_{n=1}^p R_{n,q_n}$ , the sequence  $[\varphi(1)], [\varphi(2)], \dots, [\varphi(p)]$  does not change. But the number of these regions is at most the number of different regions one can obtain by dissecting the space  $\mathbb{R}^{\nu+1}$  by hyperplanes  $\alpha_\nu n^\nu + \dots + \alpha_1 n + \alpha_0 = q$ . These hyperplanes are at most  $M = M_\nu(p) = \sum_{j=1}^p 2(j^\nu + \dots + j + 1) = 0(p^{\nu+1})$ . Now, one can show that the space  $\mathbb{R}^{\nu+1}$  is dissected into  $0(M^{\nu+1})$  regions by  $M$  hyperplanes [5] and therefore:

$$N_\nu(p) = 0(p^{(\nu+1)^2}) .$$

REMARK 1. It is easy to generalize Theorem 2 and obtain the following result. Let  $(f_n)_{n \in \mathbb{N}}$  be a countable set of real functions such that

$$\lim_{p \rightarrow \infty} \frac{\log^+ |f_n(p)|}{p} = 0, \quad \forall n \in \mathbb{N} .$$

( $\log^+$  denotes the maximum of 0 and  $\log$ ). Let  $Q$  be the set of all real finite linear combinations of the family  $(f_n)$ . Then the Hausdorff dimension of the set  $E(Q)$  is 0.

REMARK 2. The proof of Theorem 2 shows that the set  $E^\nu$  is not dense on the unit interval  $(0, 1)$ . On the other hand,  $E^\nu$  is invariant under the mapping  $x \rightarrow \{2x\}$ . From these two remarks, one sees that  $E^\nu$  is a Rajchman  $H$ -set and that  $E(p)$  is therefore a set of uniqueness for trigonometric series. This result is to be compared with the following corollary of Pyatetski-Shapiro's theorem:

The set of nonnormal numbers is not a set of uniqueness.

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