NORM DECREASING HOMOMORPHISMS OF GROUP ALGEBRAS

Frederick Paul Greenleaf
NORM DECREASING HOMOMORPHISMS
OF GROUP ALGEBRAS

FREDERICK P. GREENLEAF

The homomorphisms \( \varphi \) of the group algebra \( L'(F) \) into the algebra \( M(G) \) of measures, where \( F \) and \( G \) are locally compact groups, has been completely determined when both groups are abelian by P. J. Cohen, and when \( G \) is compact and the homomorphism is norm decreasing and order-preserving by Glicksberg. In this paper the structure of norm decreasing homomorphisms \( \varphi \) is determined for arbitrary locally compact \( F \) and \( G \). As an application the special structure of all norm decreasing monomorphisms is determined, along with the rather elegant structure of all norm decreasing homomorphisms mapping \( L'(F) \) onto \( L'(G) \).

The analysis is effected by finding all multiplicative subgroups of the unit ball of measures on a locally compact group for, as we show, each \( \varphi \) extends to a norm decreasing homomorphism \( \varphi : M(F) \rightarrow M(G) \), and is determined by the image under \( \varphi \) of the group of point masses on \( G \), a multiplicative subgroup of the unit ball in \( M(G) \).

This paper completes a study of norm decreasing homomorphisms on group algebras initiated by Glicksberg in [4] and [5]. If \( G \) is a locally compact group we will denote its group algebra by \( L'(G) \) and its convolution algebra of bounded regular Borel measures by \( M(G) \). We present a complete structural analysis of the subgroups of the unit ball in \( M(G) \), and a structure theory classifying all norm decreasing homomorphisms \( \varphi : L'(F) \rightarrow M(G) \) where \( F \) and \( G \) are locally compact groups. As an application we determine the special structure of all monomorphisms \( \varphi \) mapping \( L'(F) \) into \( M(G) \) and all norm decreasing homomorphisms which map \( L'(F) \) onto \( L'(G) \).

Let \( C_0(G) \) be the sup norm algebra of all continuous complex valued functions on \( G \) which vanish at infinity, and recall that \( C_0(G)^* = M(G) \). If \( \mu \in M(G) \) its support \( s(\mu) \) is defined so that \( x \in s(\mu) \iff \exists \psi \in C_0(G) \) vanishing outside of \( U \), with \( \langle \mu, \psi \rangle \neq 0 \). Then \( s(\mu) \) is a Borel set. If \( I' \) is a subset in \( M(G) \) we define \( \text{supp}(I') = \bigcup \{ s(\mu) : \mu \in I' \} \). The convolution of \( \mu, \lambda \in M(G) \) is given as an element of \( C_0(G)^* \) by defining

\[
\langle \mu * \lambda, \psi \rangle = \left[ \int \int \psi(st) d\mu(s) d\lambda(t) \right]
\]

Received April 24, 1964 and in revised form August 1, 1964. This research was carried out under an Air force grant (AFOSR-407-63).
for all $\varphi \in C_b(G)$. If $M(G)$ is given the total variation norm it becomes a Banach algebra under this multiplication.

We first show that if $\mu, \lambda \in M(G)$ then

1. $\|\mu * \lambda\| = \|\mu\| \cdot \|\lambda\| \Rightarrow s(\mu * \lambda) = (s(\mu)s(\lambda))^{-}$

2. $\|\mu * \lambda\| = \|\mu\| \cdot \|\lambda\| \Rightarrow |\mu * \lambda| = |\mu|*|\lambda|.$

These facts were first pointed out, in somewhat less general form, by Wendel [11] and Glicksberg [4]. These results suffice for the analysis of the subgroups of the unit ball in $M(G)$.

In order to determine the norm decreasing homomorphisms $\varphi: L^1(F) \to M(G)$ we use an important observation that such a map always extends to a norm decreasing homomorphism $\bar{\varphi}: M(F) \to M(G)$ which is continuous on norm bounded sets as a mapping of $(M(F), (so))$ into $(M(G), (\sigma))$. Here $(\sigma)$ is the usual weak * topology on $M(G)$, and $(so)$ is the strong operator topology on $M(F)$ gotten by letting $M(F)$ act by left convolution on the ideal $L^1(F) \subset M(F)$.

The author is greatly indebted to the earlier work of Glicksberg presented in [4], [5]. He is also pleased to acknowledge Professor Glicksberg’s helpful commentary in private correspondence. It will be clear to the reader familiar with [4] that the proof of the fundamental relation $\|\mu * \lambda\| = \|\mu\| \|\lambda\| \Rightarrow |\mu * \lambda| = |\mu|*|\lambda|$ is a simple adaptation of a Glicksberg theorem dealing with compact groups. The simpler proof given here was suggested by Glicksberg.

1. Preliminaries. Throughout this paper we will find it convenient to write convergence of a net $\{x_j\}$ to a point $x$ in a topological space $(X, \tau)$ as $x_j \xrightarrow{\tau} x$ or $x \xleftarrow{\tau} x_j$, interchangeably. To avoid confusion in discussing homomorphisms we will use the terms homomorphism (epimorphism, monomorphism) for into (onto, 1:1) homomorphisms; we reserve the term isomorphism for 1:1 onto homomorphisms.

Most measure theoretic notions are taken from Halmos [3], including definition of Baire and Borel sets. In the following discussion let $B = B(G)$ ($B_0 = B_0(G)$) be the collection of Borel (Baire) sets in $G$. If a function $f$ is defined on $G$ and if $H$ is a $\sigma$-ring of sets in $G$, we say that $f$ is $H$ measurable on all $H$ sets of $G$ if $\chi_E f$ is $H$ measurable for each set $E \in H$ ($\chi_E = \text{characteristic function of } E$). It is clear that $B$ measurability on $B_0$ sets implies $B$ measurability on $B$ sets in $G$.

If $\mu \in M(G)$ define its Baire contraction $\mu'$ by restricting its domain of definition to be $B_0(G)$. A regular Borel measure is uniquely determined by its Baire contraction (see [3], 54. D). If $E \in B(G)$ it must be $\sigma$-bounded, and hence there is a Baire set $A \supset E$; this applies in particular to $s(\mu)$ where $\mu \in M(G)$. If $E \in B$ and $f$ is $B$ measurable on $G$, we let $\int_E f \, d\mu$ denote the integral $\int_0 \chi_E f \, d\mu$.

In applying the Fubini theorem, Borel functions have rather
pathological properties when compared to those of Baire functions. These difficulties arise from the fact that the product $\sigma$-ring $B_e \times B_e = B_o(G \times G)$, while we only know $B_o \times B_o \subset B \times B \subset \mathcal{B}(G \times G)$ for the corresponding Borel sets. If $f$ is $B_o$ measurable on $G$ and if $A, B \in B_o$, then the function $\chi_{A \times B}(s, t)f(st)$ is $B_e \times B_e$ measurable on $G \times G$ (hence $B \times B$ measurable) and we can apply Fubini to the convolution-like integral

$$(*) \quad \int_{\sigma \times \sigma} \chi_{A \times B}(s, t)f(st)d\mu \times \lambda(s, t).$$

If $f$ is $B$ measurable, the best we can say is that $\chi_{A \times B}(s, t)f(st)$ is $B(G \times G)$ measurable, but this does not give the $B \times B$ measurability required to make $(*)$ well defined. To avoid these difficulties we will rely on the following well known observations.

(R1) If $f$ is bounded and $B_o$ measurable, and if $E, F \in B_o$, then $\chi_{E \times F}(s, t)f(st)$ is $B_e \times B_e$ measurable on $G \times G$.

(R2) If $f$ is bounded and $B_o$ measurable, and if $\mu, \lambda \in M(G)$, let us choose any sets $E, F \in B_o$ such that $E \supset s(\mu)$, $F \supset s(\lambda)$. Then

$$\int_{\sigma} f(x)d\mu \ast \lambda(x) = \int_{\sigma \times \sigma} \chi_{E \times F}(s, t)f(st)d\mu \times \lambda(s, t).$$

(R3) If $\mu \in M(G)$ there exists a unimodular function $f_\mu$ which is $B_o$ measurable on $B_o$ sets in $G$, such that $\mu = f_\mu \mid \mu \mid$. Thus if $E \in B$,

$$\mu(E) = \int_{\sigma} \chi_{\delta_E}(s)d\mu(s) = \int_{\sigma} \chi_{\delta_E}(s)f_\mu(s)d \mid \mu \mid (s).$$

Notice that $\Psi(t) = \int \psi(st)d\mu(s)$ is in $\mathcal{C}_0(G)$ if $\psi \in \mathcal{C}_0(G)$, and all $\mathcal{C}_0(G)$ functions are $B_o$ measurable on $G$, so the definition of convolution is meaningful for $\mu, \lambda \in M(G)$. Convolution is actually independent of the order of iteration of the integrals used to define it, in fact the above remarks show that

$$\int_{\sigma} \int_{\sigma} \psi(st)d\mu(s)d\lambda(t) = \int_{\sigma \times \sigma} \chi_{E \times F}(s, t)\psi(st)d\mu \times \lambda(s, t) = \int_{\sigma} [\int_{\sigma} \psi(st)d\lambda(t)]d\mu(s),$$

if $E, F \in B_o$ are such that $E \supset s(\mu)$, $F \supset s(\lambda)$.

If $G$ is a locally compact group and if $Q \subset G$ we let $\mathcal{E}_q = \{\delta_x : x \in Q\}$ where $\delta_x$ is the point mass at $x$ for $x \in G$. Let $\mathcal{C}_0[X]$ be the convex span of a set $X \subset M(G)$ and if $\gamma$ is a vector space topology on $M(G)$, denote the $(\gamma)$-closed convex span of $X$ as $\mathcal{C}_0[X : \gamma]$. We will need the
following lemmas about the (σ) and (so) topologies on $M(G)$ (see introduction).

**Lemma 1.1.1.** If $\mu_j \overset{(so)}{\to} \mu$ with $\|\mu_j\| \leq M < \infty$ in $M(F)$, and if $\psi$ is right uniformly continuous and bounded on $F$, then we have $\langle \mu_j, \psi \rangle \to \langle \mu, \psi \rangle$.

**Proof.** Since $\psi$ is uniformly continuous, there exists $f \in L'(F)$ corresponding to $\epsilon > 0$ such that $\|f\| = 1$ and $\left| \int_F \psi(st)f(t)dt - \psi(s) \right| < \epsilon/M$ for all $s \in F$. Then we have $|\langle \mu_j*f, \psi \rangle - \langle \mu_j, \psi \rangle| < \epsilon$ for $j \in J$ and likewise for $\mu$, so that $|\langle \mu_j, \psi \rangle - \langle \mu, \psi \rangle| < 3\epsilon$ for $j \geq J$.

**Lemma 1.1.2.** If $Q$ is a compact set in locally compact group $G$, and if $S$ is the circle group, then $\text{co}[S^Q : so]$ is compact in the (so) topology, as is $\text{co}[S^Q : \sigma]$ in the (σ) topology. On the unit ball the identity $j : (M(G), so) \to (M(G), \sigma)$ is continuous by 1.1.1, so $\text{co}[S^Q : so]$ is (σ) compact and hence must contain $\text{co}[S^Q : \sigma]$. Since $Q$ is compact it is known that $\text{co}[S^Q : \sigma] = \{\mu \in M(G) : \|\mu\| \leq 1, s(\mu) \subset Q\}$, and on these sets the (σ) and (so) topologies coincide.

**Proof.** Clearly $S^Q$ is both (σ) and (so) compact, thus from [2, p. 511] we see that $\text{co}[S^Q : so]$ is compact in the (so) topology, as is $\text{co}[S^Q : \sigma]$ in the (σ) topology. On the unit ball the identity $j : (M(G), so) \to (M(G), \sigma)$ is continuous by 1.1.1, so $\text{co}[S^Q : so]$ is (σ) compact and hence must contain $\text{co}[S^Q : \sigma]$. Since $Q$ is compact it is known that $\text{co}[S^Q : \sigma] = \{\mu \in M(G) : \|\mu\| \leq 1, s(\mu) \subset Q\}$, and $\mu_n \overset{\text{norm}}{\to} \mu$. Thus $\mu$ is in the norm closure of $\bigcup_{n=1}^{\infty} \text{co}[S^{K_n} : so]$, which lies within $\text{co}[S^Q : so]$.

**Lemma 1.1.3.** If $G$ is a locally compact group, $\text{co}[S^Q : so]$ is the unit ball in $M(G)$ if $S$ is the circle group.

**Proof.** Let $\mu \in M(G)$, $\|\mu\| = 1$, and let $K_n$ be compacta such that $K_{n+1} \supset K_n$ and $\bigcup_{n=1}^{\infty} K_n \supset s(\mu)$. Then $\mu_n = \mu|K_n \in M(G)$ is such that $\|\mu_n\| \leq 1$, $\mu_n \in \text{co}[S^{K_n} : so]$, and $\mu_n \overset{\text{norm}}{\to} \mu$. Thus $\mu$ is in the norm closure of $\bigcup_{n=1}^{\infty} \text{co}[S^{K_n} : so]$, which lies within $\text{co}[S^Q : so]$.

**Lemma 1.1.4.** On the unit ball in $M(G)$, convolution is a jointly strong operator continuous operation.

**Proof.** Let $\mu_j \overset{(so)}{\to} \mu$ and $\lambda_k \overset{(so)}{\to} \lambda$ in the unit ball. If $f \in L'(G)$, then because $\|\mu_j\| \leq 1$ for all $j \in J$ we have $\|\mu_j * \lambda_k * f - \mu * \lambda * f\| \leq \|\mu_j * (\lambda_k * f) - \mu_j * (\lambda * f)\| + \|\mu_j * (\lambda * f) - \mu * (\lambda * f)\| \leq \|\lambda_k * f - \lambda * f\| + \|\mu_j * (\lambda * f) - \mu * (\lambda * f)\| \to 0$.

**Lemma 1.1.5.** If $G$ is a locally compact group, the unit ball in
$L'(G) \subset M(G)$ is (so) dense in the unit ball in $M(G)$; in particular, $L'(G)$ is (so) dense in $M(G)$.

Proof. Clearly there exists a left approximate identity $\{e_i\}$ of norm one in $L'(G)$. If $\mu \in M(G)$ then $\mu * e_i \in L'(G)$ and $\|\mu * e_i\| \leq \|\mu\|$; furthermore, if $f \in L'(G)$ we have

$$\|\mu * f - (\mu * e_i) * f\| = \|\mu * f - \mu * (e_i * f)\| \to 0.$$

2. Idempotent measures of norm one. If $G$ is a locally compact group and $K \subset G$ is a compact subgroup, define $m_K \in M(G)$ to be the normalized Haar measure on $K$, so that

$$\langle m_K, \psi \rangle = \int_K \psi(s) dm_K(s)$$

for $\psi \in C_c(G)$. Let $K^*$ be the set of all continuous unimodular multiplicative complex valued functions on $K$, and if $\beta \in K^*$ let $\beta m_K$ denote the Haar measure on $K$ weighted with the function $\beta$. Then $\beta m_K$ is an idempotent of norm one in $M(G)$; it is our purpose to show that these are the only idempotent measures of norm one in $M(G)$.

**Theorem 2.1.1.** Let $G$ be a locally compact group. Then if $\mu, \lambda \in M(G)$ are such that $\|\mu * \lambda\| = \|\mu\| \cdot \|\lambda\|$ it follows that $s(\mu * \lambda) = (s(\mu)s(\lambda))^{-}$, the closure in $G$ of $s(\mu)s(\lambda)$.

Proof. It is sufficient to consider the case $\|\mu\| = \|\lambda\| = 1$. Clearly $(s(\mu)s(\lambda))^{-} \supset s(\mu * \lambda)$. If this inclusion is proper we can find a compact Baire set $U$ which is such that $(\text{int} U) \cap (s(\mu)s(\lambda))^{-} \neq \emptyset$, while at the same time $U \cap s(\mu * \lambda) = \emptyset$. Let $E, F \in B_c(G)$ be such that $E \supset s(\mu)$, $F \supset s(\lambda)$, and define $V = \{(s, s^{-1}u) : u \in U, s \in E \subset G \times G$; notice that $V \in B_c \times B_c$ and is such that $\chi_V(s, t) = \chi_E(st)$ for $s \in E$, $t \in G$, thus

$$\int_{(E \times F) \cap V} \psi(st)d\mu \times \lambda(s, t)$$

$$= \int_{E \times F} \chi_E(s, t) \psi(st)d\mu \times \lambda(s, t)$$

$$= \int_{E \times F} \chi_F(st) \psi(st)d\mu \times \lambda(s, t)$$

$$= \int_G \chi_F(x) \psi(x)d\mu \times \lambda(x).$$

Given $\varepsilon > 0$, there is a function $\psi \in C_c(G)$ such that $\|\psi\|_\infty = 1$ and $|\langle \mu * \lambda, \psi \rangle| > 1 - \varepsilon$. If $V$ is any Baire set in $G \times G$, then
1 - \varepsilon \langle \mu * \lambda, \psi \rangle = \left| \int_{G \times G} \chi_{E \times F}(s, t) \psi(st) d\mu \times \lambda(s, t) \right|

(*)

= \left| \int_{E \times F} \psi(st) d\mu \times \lambda(s, t) \right|

= \left| \int_{(E \times F) \setminus V} \psi(st) d\mu \times \lambda(s, t) \right|

+ \left| \int_{(E \times F) \cap V} \psi(st) d\mu \times \lambda(s, t) \right|.

For \( V \) as above, the right hand side of (*) consists of the single term

\[
\left| \int_{(E \times F) \cap V} \psi(st) d\mu \times \lambda(s, t) \right| \leq \int_{(E \times F) \cap V} \left| \psi(st) \right| d\mu \times \left| \lambda \right|(s, t)
\]

\[
\leq \int_{E \times F} d\mu \times \left| \lambda \right| - \int_{(E \times F) \cap V} d\mu \times \left| \lambda \right|
\]

= 1 - \int_{(E \times F) \cap V} d\mu \times \left| \lambda \right|.

But from our definition of \( U \) it is clear that \( \int_{(E \times F) \cap V} d\mu \times \left| \lambda \right| = \delta > 0 \), and thus for all \( \varepsilon > 0 \) we get \( 1 - \varepsilon \leq 1 - \delta \), a contradiction.

**Theorem 2.1.2.** If \( G \) is a locally compact group and if \( \mu, \lambda \in M(G) \) are measures such that \( \| \mu \| = \| \lambda \| = 1 \), then \( \| \mu * \lambda \| = \| \mu \| \| \lambda \| \).

**Proof.** Again it suffices to consider the case \( \| \mu \| = \| \lambda \| = 1 \). If \( F \supset s(\mu) \) is a Baire set, it is \( \sigma \)-bounded and from the Radon-Nikodym theorem we know that there is a Baire measurable function \( f_\mu \) on \( F \) such that \( \mu(E) = \int \chi_E(x) f_\mu(x) d\mu \) for all \( E \in B(G) \) such that \( E \subset F \). Clearly \( f_\mu(x) = 1 \) \( \mu \)-a.e. on \( F \); we define a new function

\[
\rho_\mu(x) = \begin{cases} f_\mu(x) & \text{if } x \in F \text{ and } |f_\mu(x)| = 1 \\ 1 & \text{for all other } x \in G. \end{cases}
\]

Then \( \rho_\mu \) is a unimodular function on \( G \) Baire measurable on Baire sets in \( G \).

We will show \( |\mu * \lambda| \leq |\mu| \| \lambda \| \). Since these are positive measures, both of norm one (since \( \| \mu * \lambda \| = \| \mu \| \| \lambda \| \)) for any positive measures \( \mu, \lambda \in M(G) \), our result must follow. If \( \psi \in C_c(G) \) and \( \psi \geq 0 \), then

\[
\langle \mu * \lambda, \psi \rangle = \int \frac{\psi(x)}{\rho_{\mu \lambda}(x)} d\mu * \lambda(x)
\]

\[
= \int \left[ \int \frac{\psi(st)}{\rho_{\mu \lambda}(st)} d\mu(s) \right] d\lambda(t)
\]

\[
= \int \left[ \int \psi(st) \frac{\rho_\mu(s) \rho_\lambda(t)}{\rho_{\mu \lambda}(st)} d\mu(s) \right] d\lambda(t).
\]
Now the last integral is positive and the integrand is a unimodular multiple of \( \psi(st) \), so it must be less than or equal to
\[
\int_G \left[ \int_G \psi(st) d\mu(s) \right] d\lambda(t) = \langle |\mu| * |\lambda|, \psi \rangle.
\]
The following lemma is given in Loynes [6] and Pym [8], and is also a simple consequence of 2.1.1 and 2.1.2.

**Proposition 2.1.3.** If \( G \) is a locally compact group and if \( \mu \in\text{M}(G) \) is a positive idempotent of norm one, then there is a compact subgroup \( K \subset G \) such that \( \mu = \kappa m_K \).

We can now prove the main assertion of this section.

**Theorem 2.1.4.** If \( G \) is a locally compact group and \( \mu \in\text{M}(G) \) is an idempotent of norm one, then there is a compact subgroup \( K \subset G \) and a function \( \rho \in K^\circ \) such that \( \mu = \rho m_K \).

**Proof.** Write \( \mu = \rho |\mu| \) where \( \rho \) is a unimodular function on \( G \), Baire measurable on Baire sets in \( G \). From 2.1.2 we see that \(|\mu|\) is a positive idempotent of norm one, so that \(|\mu| = m_K\) for some compact subgroup \( K \subset G \) from 2.1.3.

Now \( \rho \) is a bounded Borel measurable function on \( K \) since \( B(K) = \{E \cap K : E \in B(G)\} \), so the function \( \rho \ast \rho(t) = \int_K \rho(s^{-1}t)\rho(s)dm_K(s) \) is continuous on \( K \) (we are taking \( \ast \) as the convolution of two functions on \( K \) here). If \( \psi \in C_0(G) \) and if \( F \in B_0(G) \) is such that \( F \supset s(\mu) = K \), we have
\[
\int_K \psi(x)\rho(x) dm_K(x) = \langle \mu, \psi \rangle = \langle \mu \ast \mu, \psi \rangle
\]
\[
= \int_G \left[ \int_G \psi(st) \rho(t) dm_K(t) \right] \rho(s) dm_K(s)
\]
\[
= \int_G \left[ \int_K \psi(t) \rho(s^{-1}t) dm_K(t) \right] \rho(s) dm_K(s)
\]
\[
= \int_{G \times G} \chi_{F \times F}(s, t) \rho(s^{-1}t) \cdot \psi(t) \rho(s) \chi_{K \times K}(s, t) dm_K \times m_K(s, t)
\]
Clearly \( \psi(t) \rho(s) \chi_{K \times K}(s, t) \) is \( B \times B \) measurable and a slight modification of \( R1 \) gives the \( B_1 \times B_1 \) measurability of \( \rho(s^{-1}t) \chi_{F \times F}(s, t) \) on \( G \times G \). Thus Fubini applies and we get
\[
= \int \chi_{K \times K}(s, t) \psi(t) \rho(s) \rho(s^{-1}t) dm_K \times m_K(s, t)
\]
\[
= \int_K \left[ \int_K \rho(s) \rho(s^{-1}t) dm_K(s) \right] \psi(t) dm_K(t)
\]
\[
= \int_K \psi(t) \rho \ast \rho(t) dm_K(t)
\]
so $\rho$ is $|\mu|$-a.e.
identical to a continuous function on $K$. Taking $\rho$ to be
continuous on $K$, it is clear that $(s, t) \rightarrow \rho(st)$ is continuous on
$K \times K$. But we can apply the argument of 2.1.2:

$$
1 = \| \mu \star \mu \| = \int \int_{K} \frac{\rho_{\mu}(s)\rho_{\mu}(t)}{\rho_{\mu}(st)} \, d \mu |(s) \, d \mu |(t) \\
= \int \int_{K} \frac{\rho(s)\rho(t)}{\rho(st)} \, d\mu_{K}(s) \, d\mu_{K}(t),
$$

which means that $\rho$ is a multiplicative function on $K$.

3. Subgroups of the unit ball in a measure algebra. In this
section we consider a locally compact group $G$ and let $\Gamma$ be a subgroup
in the unit ball of $M(G)$. We will denote this unit ball by $\Sigma_{M(G)}$ and
refer to the weak * topology on $M(G)$ as the $(\sigma)$ topology. Given $\Gamma$
we denote $H_{0} = \text{supp}(\Gamma) = \bigcup \{s(\mu) : \mu \in \Gamma\}$.

**Lemma 3.1.1.** Both $H_{0}$ and its closure in $G$ are subgroups of $G$,
and if the unit of $\Gamma$ is denoted $i$ ($i = \rho m_{K}$ for some
compact subgroup $K \subset G$ and some $\rho \in K^{\ast}$), then $K$ is a normal
subgroup of both $H_{0}$ and its closure. Furthermore, if $\mu \in \Gamma$
then $s(\mu)$ is a single coset of the group $K$ in $H_{0}$.

**Proof.** If $\mu \in \Gamma$ then $s(\mu)$ is a union of right (or of left)
cosets of $K$ because $i \ast \mu = \mu \ast i = \mu \Rightarrow (s(\mu)s(i))^{-1} = s(\mu) \cdot K = K \cdot s(\mu) = s(\mu)$
from 2.1.1. If $\mu \in \Gamma$ then $s(\mu^{-1}) = s(\mu)^{-1}$. In fact, if $x \in s(\mu)$, $y \in s(\mu^{-1})$
then $xy = k \in (s(\mu)s(\mu^{-1}))^{-1} = s(\mu \ast \mu^{-1}) = K$, so that $x^{-1} = yk^{-1} \in s(\mu^{-1})K = \text{s}(\mu^{-1})$, and vice versa.

If $g_{1} \in s(\mu)$, $g_{2} \in s(\mu^{-1})$ we have the relations

$$(\ast) \quad K = s(i) = (s(\mu)s(\mu^{-1}))^{-1} \supset s(\mu)s(\mu^{-1}) \supset g_{1}Kg_{2} \supset g_{1}g_{2}K$$

$$(\ast\ast) \quad K = s(i) = (s(\mu)s(\mu^{-1}))^{-1} \supset s(\mu)s(\mu^{-1}) \supset Kg_{1}g_{2}K \supset \{g_{1}, g_{2}\}.$$ 

Thus $s(\mu)$ is a single coset of $K$; otherwise we could find $g_{1}, g_{2} \in s(\mu)$
with $g_{1} \in Kg_{2}$, and this would $\Rightarrow g_{1}g_{2}^{-1} \in K$. But $g_{2}^{-1} \in s(\mu^{-1}) = s(\mu^{-1})$ and

$$(\ast\ast) \Rightarrow g_{1}g_{2}^{-1} \in K,$$ 

a contradiction. We see now that all supports are compact and hence $s(\mu)s(\lambda) = s(\mu \ast \lambda)$ for all $\mu, \lambda \in \Gamma$.

Clearly $H_{0}$ is a subgroup of $G$ since $s(\mu \ast \lambda) = s(\mu)s(\lambda)$ and $s(\mu^{-1}) = s(\mu^{-1})$; hence its closure is also a subgroup in $G$. We get normality
of $K$ by considering $g \in H_{0}$ and taking any $\mu \in \Gamma$ such that $g \in s(\mu)$. Then if we take $g_{1} = g$, $g_{2} = g^{-1} \in s(\mu^{-1})$ in $(\ast)$, we get $K \supset gKg^{-1}$.

The following theorem gives the structure of a subgroup $\Gamma$; it
gives only a necessary condition on the structure of a collection of
measures \( \Gamma \) in the unit ball of \( M(G) \) in order that \( \Gamma \) be a subgroup. Necessary and sufficient conditions will be given later.

**Proposition 3.1.2.** If \( \Gamma \) is a subgroup of \( \Sigma_{M(G)} \) for locally compact group \( G \), let \( i = \rho m_K \) be its unit and let \( H_0 = \text{supp}(\Gamma) \). Then there exists a subgroup \( \Omega \subset S \times G \), with the property

\[
H_0 = \{ g \in G : (\alpha, g) \in \Omega \text{ for some } |\alpha| = 1 \},
\]

such that \( \Gamma = \{ \alpha \delta_s \ast \rho m_K : (\alpha, g) \in \Omega \} \).

**Remark.** Here \( S \) is the circle group and \( S \times G \) is the usual product group. In 2.1.4 we have already shown that the unit is \( i = \rho m_K \) where \( K \) is normal in \( H_0 \) and \( \rho \in K^* \).

**Proof.** Let \( \rho_\mu \) be a unimodular function on \( G \), Baire measurable on Baire sets in \( G \), such that \( \mu = \rho_\mu |\mu| \) for \( \mu \in \Gamma \). If \( \mu \in \mathcal{S}(\mu) \) we have shown that \( s(\mu) = gK \) and we know that \( \rho_\mu \) is determined \( |\mu|-\text{a.e.} \) on \( s(\mu) \). But if \( s(\mu) = gK \), then \( |\mu| = \delta_\mu * m_K \); in fact, we have \( \mu * i = \mu \), which is \( = |\mu| * i = |\mu| * m_K = |\mu| \), and this gives \( |\mu| = \delta_\mu * m_K \) since \( ||\mu|| = 1 \). We first show that \( \rho_\mu \) is \( |\mu|-\text{a.e.} \) identical to a continuous function on \( s(\mu) \), or equivalently that \( \rho_\mu s(x) = \rho_\mu (gx) \) is \( m_K\)-a.e. equal to a continuous function on \( K \). We have

\[
\int G \psi d\mu = \int_K \psi(gt) \rho_\mu s(t) dm_K(t)
\]

for \( \psi \in C_0(G) \), while \( \mu = \rho \equiv

\[
\int G \psi d\mu = \int_G \left[ \int_K \psi(st) \rho(s) dm_K(s) \right] dm_K(t) = \int_K \left[ \int_K \psi(sgt) \rho(s) \rho_\mu(gt) dm_K(s) \right] dm_K(t).
\]

For \( g \in H_0 \), the map \( \pi_s : s \rightarrow gsg^{-1} \) is an automorphism of \( K \) such that \( m_K(\pi_s E) = m_K(E) \) for Borel sets \( E \subset K \); thus if we define \( \pi_s \ast \beta(s) = \beta(gsg^{-1}) \) for \( s \in K \), \( g \in H_0 \), and \( \beta \in K^* \), then \( \pi_s \ast \beta \in K^* \) and the last expression above is

\[
\int_K \left[ \int_K \psi(gst) \pi_s^* \rho(s) \rho_\mu(gt) dm_K(s) \right] dm_K(t)
\]

\[
= \int_K \left[ \int_K \psi(gt) \pi_s^* \rho(s) dm_K(s) \right] dm_K(t) = \int_K \psi(gt)[\pi_s^* \rho \ast \rho_\mu](t) dm_K(t)
\]

where \( \ast \) gives the convolution of two functions on \( K \) rather than
functions on $G$. Since $\pi_g*$ and $\rho_\mu^*$ are bounded and $B(K)$ measurable functions on $K$, their convolution on $K$ is a continuous function, and the above equalities $\rho_\mu^* = \pi_g* \rho \ast \rho_\mu^*$ hold a.e. on $K$.

Take each function $\rho_\mu$ to be continuous on $s(\mu)$ for $\mu \in \Gamma$. Then we have $\rho_\mu(x)\rho_\lambda(y) = \rho_{\mu\lambda}(xy)$ for all $(x, y) \in s(\mu) \times s(\lambda)$ in $G \times G$, because

$$1 = \int_G d |\mu \ast \lambda| = \int_G \frac{1}{\rho_{\mu\lambda}(z)} d(\mu \ast \lambda)(z)$$

$$= \int_{s(\mu)} \left( \int_{s(\lambda)} \frac{\rho_\mu(s)\rho_\lambda(t)}{\rho_{\mu\lambda}(st)} d|\mu|(s) d|\mu|(t) \right),$$

and since the last integrand is continuous and unimodular. If $s \in K$ and $g \in s(\mu)$ for $\mu \in \Gamma$, then we have

$$\rho_\mu^*(s) = \rho_\mu(gs) = \rho_{\mu^*}(gs) = \rho_{\lambda}(g)\rho(s)$$

$$= \rho_{\mu}(g)\rho(s) = c_g \cdot \rho(s)$$

which means that $\rho_\mu^* = c_g \cdot \rho$ on $K$ where $c_g = \rho_\mu(g)$ is a scalar of modulus one. Clearly $\mu = \rho_\mu(g) \cdot (\delta_g \ast \rho m_K)$ if $g \in s(\mu)$; i.e. if $g \in s(\mu)$, then for some scalar $\alpha$ with $|\alpha| = 1$ we have $\mu = \alpha \delta_g \ast \rho m_K$.

Let $\Omega = \{(\alpha, g) \in S \times G : \alpha \delta_g \ast \rho m_K \in \Gamma\}$. We have shown that for each $g \in H_0$ we can find a scalar $|\alpha| = 1$ such that $(\alpha, g) \in \Omega$, so we only have to show that

$$(\alpha_1 \delta_{v_1} \ast \rho m_K) \ast (\alpha_2 \delta_{v_2} \ast \rho m_K) = \alpha_1 \alpha_2 \delta_{\alpha_1 v_1 + \alpha_2 v_2} \ast \rho m_K.$$
let $H_0 = \text{supp}(\Gamma)$. Then $K_0 = \{x \in K : \rho(x) = 1\}$ is a compact subgroup of $G$ which is normal in both $K$ and $H_0$.

Proof. If $\mu \in \Gamma$ and $\rho_\mu$ is the unimodular function, Baire measurable on Baire sets in $G$, such that $\mu = \rho_\mu | \mu |$, then we know that $\rho_\mu$ is a translate of $\rho$ to $s(\mu)$, and we also know that $\rho_\mu(x)\rho_\lambda(y) = \rho_{\mu\lambda}(xy)$ for all $x \in s(\mu), y \in s(\lambda)$, from 3.1.2. Obviously $K_0$ is normal in $K$; normality in $H_0$ is more troublesome.

If $y \in K_0, x \in H_0$, then $xyx^{-1} \in K$ and if $x \in s(\mu)$ we get

\[
\rho(xyx^{-1}) = \rho_\mu(x)\rho_\lambda(y)\rho_\mu^{-1}(x^{-1}) = \rho_\mu(x) \cdot 1 \cdot \rho_\mu^{-1}(x^{-1})
\]

\[
= \rho_\mu^{-1}(xx^{-1}) = \rho(e) = 1,
\]

which $\Rightarrow xyx^{-1} \in K_0$.

**PROPOSITION 3.1.6.** Let $G$ be a locally compact group, let $H_0$ be an arbitrary subgroup, let $K \supset K_0$ be a pair of compact subgroups of $G$ which lie within $H_0$ and are normal therein, and assume that $\rho \in K^*$ is a function such that $K_0 = \text{Ker} \rho$. Then we have $\rho m_K \delta_\gamma \rho m_K = \delta_\delta \rho m_K$ for all $g \in H_0 \Rightarrow K$ is central in $H_0 \text{ mod } K_0$ (i.e. $K/K_0$ is a central subgroup of $H_0/K_0$).

Proof. If $K$ is central in $H_0 \text{ mod } K_0$ and $\varphi \in C_c(G)$, then

\[
\langle \rho m_K \delta_\gamma \rho m_K, \varphi \rangle = \int \left[ \int \left[ \int \psi(sxt)\rho(s)\rho(t)dm_K(s) \right]d[\delta_\gamma](x) \right]dm_K(t)
\]

\[
= \int \left[ \int \varphi(sgt)\rho(st)dm_K(s) \right]dm_K(t).
\]

But $sg = gs \text{ mod } K_0$, so that $sg = gs k$ for some $k \in K_0$, and the last expression becomes

\[
= \int \left[ \int \varphi(gskt)\rho(st)dm_K(s) \right]dm_K(t)
\]

\[
= \int \left[ \int \varphi(gst)\rho(st)dm_K(s) \right]dm_K(t)
\]

\[
= \int \psi(gs)\rho(s)dm_K(s) = \langle \delta_\gamma \rho m_K, \varphi \rangle,
\]

since $(\rho m_K)^* = \rho m_K$.

If, conversely, $\rho m_K \delta_\gamma \rho m_K = \delta_\delta \rho m_K$ for all $g \in H_0$, we show that $K$ is central in $H_0 \text{ mod } K_0$ as follows. Let

\[
\psi_j = \frac{\chi_{\sigma U_j}}{m_K(U_j \cap K)}
\]

where $\{U_j : j \in J\}$ is a basis of compact symmetric neighborhoods of
the unit in $G$, and make $\{\psi_j : j \in J\}$ a net of functions in $L'(G)$ under the obvious partial ordering. Then we have

$$1 = \rho(e) \leftarrow \int_K \chi_{\nu_j}(s) \frac{\chi_{\nu_j}(U_j)}{m_K(U_j)} \rho(s) dm_K(s)$$

$$= \int_K \chi_{\nu_j}(gs) \frac{\chi_{\nu_j}(U_j)}{m_K(U_j)} \rho(s) dm_K(s)$$

$$= \int_K \psi_j(gs) \rho(s) dm_K(s)$$

$$= \int_K \psi_j d[\delta_s \ast \rho m_K] = \int_K \psi_j d[\rho m_K \ast \delta_s \ast \rho m_K]$$

$$= \int_K \left[ \int_K \psi_j(st) \rho(s) \rho(t) dm_K(s) \right] dm_K(t)$$

$$= \int_K \left[ \int_K \chi_{\nu_j}(gst) \frac{\chi_{\nu_j}(U_j)}{m_K(U_j)} \rho(gsg^{-1}) \rho(t) dm_K(s) \right] dm_K(t)$$

$$= \int_K \left[ \int_K \chi_{\nu_j}(st) \frac{\chi_{\nu_j}(U_j)}{m_K(U_j)} \rho(gsg^{-1}) \rho(t) dm_K(s) \right] dm_K(t) .$$

But $\rho$ is uniformly continuous on $K$ and hence, given $\varepsilon > 0$ there is an index $j(\varepsilon)$ such that $j > j(\varepsilon)$ in the partial ordering of $J \Rightarrow \varepsilon$ if $t \in t' U_j$. Hence if $j > j(\varepsilon)$ we get: $\chi_{\nu_j}(st) \neq 0 \Rightarrow t \in s^{-1} U_j$, which $\Rightarrow |\rho(t) - \rho(s^{-1})| < \varepsilon$. Some trivial computations then show that the last integral is always within $\varepsilon$ of the following expression if $j > j(\varepsilon)$.

$$\int_K \rho(gsg^{-1}) \rho(s^{-1}) dm_K(s) = \int_K \rho(gsg^{-1}) \rho(s) dm_K(s) .$$

But $s \rightarrow \rho(gsg^{-1})$ is a function in $K^*$, and from the known orthogonality of one dimensional representations of $K$, this integral can be nonzero $\Rightarrow \rho(gsg^{-1}) = \rho(s)$ for all $s \in K$. This means that $gs = sg \mod K_0$ for all $s \in K$, $g \in H_0$.

**Corollary 3.1.7.** If $G$ is a locally compact group and $\Gamma'$ is a subgroup of $\Sigma_{M(G)}$, let $H_0 = \text{supp}(\Gamma)$, and let us write the unit of $\Gamma'$ as $i = \rho m_K$ as in 2.1.4, where $K \subset G$ is a compact subgroup and $\rho \in K^*$. Then if $K_0 = \text{Ker} \, \rho$, $K$ must be central in $H_0 \mod K_0$.

**Proof.** From 3.1.5 we know that $K_0$ must be normal in $H_0$. Furthermore, $i \ast \mu = \mu \ast \rho m_K \ast \delta_0 \ast \rho m_K = \delta_0 \ast \rho m_K$ for all $g \in H_0$ (see 3.1.2).

**Theorem 3.1.8.** (Structure Theorem for Subgroups). Let $G$ be locally compact group and let $\Gamma$ be a subgroup of $\Sigma_{M(G)}$ with unit $i$. Then we have
(1) \( H_\circ = \cup \{ s(\mu) : \mu \in \Gamma \} \) is a subgroup of \( G \).
(2) \( i = \rho m_K \) where \( K\subset G \) is a compact subgroup and \( \rho \in K^\circ \).
(3) \( K \) and \( K_\circ = \text{Ker} \rho \) lie within \( H_\circ \) and are normal in \( H_\circ \).
(4) \( K \) is central in \( H_\circ \) mod \( K_\circ \).
(5) \( \Omega = \{(\alpha, g) \in S \times G : \alpha \delta_g \ast \rho m_K \in \Gamma \} \) is a subgroup of \( S \times G \) with \( H_\circ = \{ g \in G : (\alpha, g) \in \Omega \) for some \( |\alpha| = 1 \} \).

and we have \( \Gamma = \{ \alpha \delta_g \ast \rho m_K : (\alpha, g) \in \Omega \} \).

Conversely, let \( H_\circ \) be a subgroup in \( G \), let \( K\subset G \) be a compact subgroup lying within \( H_\circ \), and let \( \rho \in K^\circ \) be chosen such that 
(1) \( K \) and \( K_\circ = \text{Ker} \rho \) are both normal in \( H_\circ \).
(2) \( K \) is central in \( H_\circ \) mod \( K_\circ \).
and let \( \Omega \) be any subgroup of \( S \times G \) with \( H_\circ = \{ g \in G : (\alpha, g) \in \Omega \) for some \( |\alpha| = 1 \} \). Then \( \Gamma = \{ \alpha \delta_g \ast \rho m_K : (\alpha, g) \in \Omega \} \) is a subgroup of \( \Sigma_{M, \langle \alpha \rangle} \) with \( H_\circ = \cup \{ s(\mu) : \mu \in \Gamma \} \), with \( i = \rho m_K \) as a unit, and with \( \Omega \subset \{(\alpha, g) \in S \times G : \alpha \delta_g \ast \rho m_K \in \Gamma \} = \Omega \cdot \{(\rho(k), k) : k \in K \} \).

Proof. The first part follows from 3.1.2, 3.1.5, and 3.1.7. Conversely, if \( K \) is central in \( H_\circ \) mod \( K_\circ = \text{Ker} \rho \) in a scheme of this sort we must have \( \rho m_K \ast \delta_g \ast \rho m_K = \delta_g \ast \rho m_K \) for all \( g \in H_\circ \). This means that \( \Gamma \) is a group, since the only difficulty in showing this lies in the verification that \( \Gamma \) is closed under convolution. It follows immediately that \( H_\circ = \cup \{ s(\mu) : \mu \in \Gamma \} \) and that \( \tau : (\alpha, g) \to \alpha \delta_g \ast \rho m_K \) is a homomorphism of \( \Omega \) onto \( \Gamma \) with kernel \( \Omega \cap \{(\rho(k), k) : k \in K \} \). Notice that \( \Omega \) and \( \Omega \cdot \{(\rho(k), k) : k \in K \} \) give rise to the same group of measures \( \Gamma \).

The classical example of a subgroup in \( \Sigma_{M, \langle \alpha \rangle} \) is a group of translates of normalized Haar measure \( \Gamma = \{ \delta_x \ast m_0 : x \in G_0 \} \), where \( Q \subset G \) is a compact subgroup, normal in the subgroup \( G_0 \). Theorem 3.1.8 can be stated in a form which shows that every subgroup \( \Gamma \subset \Sigma_{M, \langle \alpha \rangle} \) corresponds to a subgroup of this type in \( \Sigma_{M_0(S \times G)} \) rather than \( \Sigma_{M_0(G)} \).

Let \( \pi_0, \pi_\alpha \) be the projection homomorphisms in \( S \times G \) and let \( \Omega \supset \Omega_0 \) be subgroups in \( S \times G \) satisfying the conditions 
(1) \( \Omega_0 \) is a compact subgroup of \( S \times G \) normal in \( \Omega \).
(2) \( S \cap \Omega_0 = (1, e) \), so \( \pi_\alpha^{-1}(x) \cap \Omega_0 \) is a single point if \( x \in \pi_\alpha(\Omega_0) \).

If we are given a compact subgroup \( K \subset G \) and a function \( \rho \in K^\circ \), we define the mappings 
\[
\begin{align*}
\tau_\alpha : S \times G &\longrightarrow M(G) \\
\tau^\ast : C_\alpha(G) &\longrightarrow C_\alpha(S \times G) \\
\tau** : M(S \times G) &\longrightarrow M(G)
\end{align*}
\]
such that \( \tau_\alpha(\alpha, g) = \alpha \delta_g \ast \rho m_K \), \( \tau^\ast \psi(\alpha, g) = \langle \tau_\alpha(\alpha, g), \psi \rangle \), and \( \langle \tau** \mu, \psi \rangle = \langle \mu, \tau^\ast \psi \rangle \). Clearly \( \tau^\ast \psi \in C_\alpha(S \times G) \) since \( K \) is compact, and \( \tau** \delta_{(\alpha, g)} = \langle \delta_{(\alpha, g)}, \psi \rangle \).
For \((a, g) \in S \times G\). Furthermore, \(\tau^{**} : (M(S \times G), (\sigma)) \rightarrow (M(G), (\sigma))\) is a linear map which is continuous on norm bounded sets since \(\mu_j \rightarrow \mu\) in \(\Sigma_{M(S \times G)}\), \(\psi \in C_d(G)\):

\[
\langle \tau^{**} \mu_j, \psi \rangle = \langle \mu_j, \tau^* \psi \rangle \rightarrow \langle \mu, \tau^* \psi \rangle = \langle \tau^{**} \mu, \psi \rangle.
\]

Also, \(\tau^{**}\) is a norm decreasing linear map.

Now take \(K = \pi_0(\Omega)\), where \(\Omega \supset \Omega_0\) satisfy (1) and (2) above, and define the function \(\rho(k)\) on \(K\) such that \((\rho(k), k) \in \Omega_0\) for each \(k \in K\). It is clear that \(\rho \in K^\times\) since \(\Omega_0\) is compact. Let us also define \(H_0 = \pi_0(\Omega), K_0 = \pi_0(\Omega \cap G)\). Then \(K_0 = \text{Ker} \rho\) and it is easily verified that \(K_0\) is normal in both \(H_0\) and \(K\), and that \(K\) is central in \(H_0\) mod \(K_0\), from conditions (1) and (2). Thus \(\Gamma = \{\alpha \delta_s * \rho m_K : (\alpha, g) \in \Omega\}\) is a subgroup in \(\Sigma_{M(G)}\) since 3.1.8 applies to the system of objects \(H_0, K, K_0, \rho\).

The mapping \(\tau^{**} : (\alpha, g) \rightarrow \alpha \delta_s * \rho m_K\) is a homomorphism on \(\Omega\) since

\[
\tau^{**} \delta_{(\alpha_1, g_1, \delta_1)} \ast \rho m_K = \alpha_1 \alpha \delta_{g_1} * \rho m_K = \alpha_1 \alpha \delta_{g_1} * \rho m_K = \alpha \delta_{g_1} * \rho m_K = \alpha \delta_{g_1} * \rho m_K = \tau^{**} \delta_{(\alpha_1, g_1, \delta_1)} \cdot \tau^{**} \delta_{(\alpha_2, g_2)} \cdot \tau^{**} \delta_{(\alpha_3, g_3)}.
\]

From normality of \(\Omega_0\) in \(\Omega\) it follows that \(\Gamma^\sim = \{\delta_x * m_{\Omega_0} : x \in \Omega\}\) is a subgroup of \(\Sigma_{M(S \times G)}\).

**Lemma 3.1.9.** If \(\Omega \supset \Omega_0\) satisfy conditions (1), (2) above, and if \(\Gamma = \{\alpha \delta_s * \rho m_K : (\alpha, g) \in \Omega\}\) then \(\tau_0 : \Omega \rightarrow \Gamma\) is an epimorphism with kernel \(\Omega_0\).

**Proof.** We have indicated that \(\tau_0\) is an epimorphism. If \(\tau_0(\alpha, g) = \rho m_K\), then \(g \in K\) and we have \(\alpha \delta_s * \rho m_K = \alpha \rho(g) \rho m_K = \rho m_K\); hence \(\alpha = \rho(g)\) and \((\alpha, g) = (\rho(g), g)\) with \(g \in K\), so \((\alpha, g) \in \Omega_0\) by definition of \(\rho\).

**Theorem 3.1.10.** Given subgroups \(\Omega \supset \Omega_0\) in \(S \times G\) satisfying (1) and (2) let \(K = \pi_0(\Omega)\), define \(\rho = \pi_s \circ (\pi_0 | \Omega_0)^{-1}\) on \(K\), and define \(\tau^{**} : M(S \times G) \rightarrow M(G)\) as above. Then \(\rho \in K^\times\) (so \(\tau_0\) and \(\tau^{**}\) are well defined), \(\Gamma = \{\alpha \delta_s * \rho m_K : (\alpha, g) \in \Omega\}\) and \(\Gamma^\sim = \{\delta_x * m_{\Omega_0} : x \in \Omega\}\) are subgroups in \(\Sigma_{M(G)}\) and \(\Sigma_{M(S \times G)}\) respectively, and \(\tau^{**}\) is an isomorphism between \(\Gamma^\sim\) and \(\Gamma\). Conversely, if \(\Gamma \subset \Sigma_{M(G)}\) is a subgroup with unit \(i = \rho m_K\), it arises from a pair of subgroups \(\Omega \supset \Omega_0\) in \(S \times G\) which satisfy conditions (1) and (2) by means of the above construction if we take \(\Omega = \{(\alpha, g) \in S \times G : \alpha \delta_s * \rho m_K \in \Gamma\}\) and \(\Omega_0 = \{(\alpha, g) \in S \times G : \alpha \delta_s * \rho m_K = \rho m_K\}\).

**Proof.** To establish the first part we will show that \(\tau^{**}(\delta_x * m_{\Omega_0}) = \tau^{**}(\delta_x) = \tau_0(x)\) for any \(x \in \Omega\); then from 3.1.9 it is clear that \(\tau^{**}\) is an isomorphism between \(\Gamma^\sim\) and \(\Gamma\). But \(\Omega_0\) is compact, so there exists a net \(\{\lambda_j\}\) in \(\text{co} [\mathcal{E}_{\Omega_0}]\) with \(\lambda_j \rightarrow m_{\Omega_0}\); hence \(\delta_x * \lambda_j \rightarrow \delta_x * m_{\Omega_0}\) and
NORM DECREASING HOMOMORPHISMS OF GROUP ALGEBRAS

Let $G$ be a locally compact group and let $\Gamma \subset \Sigma_{M(G)}$ be a subgroup with unit $i = \rho m_K$. Form the pair of subgroups in $S \times G$:

$$\Omega = \{ (\alpha, g) \in S \times G : \alpha \delta_\alpha \ast \rho m_K \in \Gamma \} \supset$$

$$\Omega_0 = \{ (\alpha, g) \in S \times G : \alpha \delta_\alpha \ast \rho m_K = \rho m_K \}.$$

Then we have $\Omega_0 = \{ (\rho(k), k) : k \in K \}$ and this is a compact subgroup of $S \times G$, normal in $\Omega$. If we define the map $\tau_0 : (\alpha, g) \rightarrow \alpha \delta_\alpha \ast \rho m_K$ for $(\alpha, g) \in S \times G$, then $\tau_0 : S \times G \rightarrow (M(G), (\sigma))$ is continuous and $\tau_0 : \Omega \rightarrow \Gamma$ is an epimorphism with kernel $\Omega_0$.

**Proof.** If $\tau_0(\alpha, g) = i$ then $g \in K$ and we have $\alpha \delta_\alpha \ast \rho m_K = \alpha \cdot \bar{\rho(g)} \cdot \rho m_K$.

Hence $\alpha = \rho(g)$ and $(\alpha, g) = (\rho(g), g)$ with $g \in K$. Since $\rho \in K^\ast$, $\Omega_0$ is a compact subgroup of $S \times G$. Let $H_0 = \text{supp}(\Gamma)$, $K_0 = \text{Ker} \rho$; from 3.1.8, $K$ is central in $H_0 \mod K_0$, so $\delta_\alpha \ast \rho m_K = \rho m_K \ast \delta_\alpha \ast \rho m_K$ for $g \in H_0$ (see 3.1.6). Thus $\tau_0$ is a homomorphism on $\Omega$ (that it is onto is clear from 3.1.2) since

$$\tau_0(\alpha, \alpha, g, g_1) = \alpha, \alpha \delta_{\alpha} \ast \delta_{\alpha} \ast \rho m_K$$

$$= \alpha, \alpha \delta_{\alpha} \ast \rho m_K \ast \delta_{\alpha} \ast \rho m_K = \tau_0(\alpha, g) \ast \tau_0(\alpha, g_1)$$

if $g, g_1 \in H_0$. Obviously $\Omega_0 = \text{Ker} \tau_0 | \Omega$, so $\Omega_0$ is normal in $\Omega$. The continuity of $\tau_0$ is clear.

4. Norm decreasing homomorphisms on locally compact groups. Let $G$ be a locally compact group and consider on $M(G)$ the $(\sigma)$ and $(so)$ topologies defined in §1. Every norm decreasing homomorphism on $L'(G)$ extends naturally to a norm decreasing homomorphism on $M(G)$. To appreciate the usefulness of this extension theorem it is helpful to recall 1.1.3.

**Theorem 4.1.1.** Let $F, G$ be locally compact groups and let $\varphi : L'(F) \rightarrow M(G)$ be any norm decreasing homomorphism. Then $\varphi$ extends uniquely to a norm decreasing homomorphism $\overline{\varphi} : M(F) \rightarrow M(G)$ which is continuous on norm bounded sets as a map of $(M(F), (so))$. 

---

In the first part we showed that $\tau^\ast$ must be an isomorphism of $\{ \delta_x \ast m_{\Omega} : x \in \Omega \}$ onto $\{ \alpha \delta_x \ast \rho m_K : (\alpha, g) \in \Omega \}$ and we know that $\Gamma$ coincides with the latter subgroup of $\Sigma_{M(G)}$ from 3.1.2.
If \( \{e_j : j \in J\} \) is a left approximate identity of norm one in \( L^1(F) \) then the extension is given explicitly by the formula
\[
\varphi(\mu) = \lim_{n \to \infty} \{ \varphi(e_j \ast \mu) : j \in J \} \quad \text{all } \mu \in M(F),
\]
where the limit is in the \((\sigma)\) topology. A similar result holds for right approximate identities.

**Proof.** Let \( B = \varphi(L^1(F)) \) and let \( A \) be the \((\sigma)\) closure of \( B \) in \( M(G) \), so that \( A \) and \( B \) are subalgebras of \( M(G) \).

**Lemma 4.1.2.** Let \( \{e_j^l : j \in J\} \) be a left approximate identity of norm one and let \( \{e_k^r : k \in K\} \) be a right approximate identity of norm one in \( L^1(F) \). Then in \( M(G) \) the \((\sigma)\) limit points of the nets \( \{\varphi(e_j^l)\} \) and \( \{\varphi(e_k^r)\} \) all coincide in a single idempotent \( \iota \in M(G) \), so we must have convergence
\[
\varphi(e_j^l) \xrightarrow{(\sigma)} \iota \quad \varphi(e_k^r) \xrightarrow{(\sigma)} \iota
\]
in the \((\sigma)\) topology. If \( \varphi \neq 0 \) then \( \iota \neq 0 \) and we have \( a = \iota \ast a = a \ast \iota \) for all \( a \in A \).

**Proof.** Since \( \| \varphi(e_j^l) \| \leq 1 \) there is at least one \((\sigma)\) limit point \( \lambda \) for this net, and for an appropriate subnet we get \( \varphi(e_{j_{(n)}}^l) \xrightarrow{(\sigma)} \lambda \). Thus if \( f \in L^1(F) \) we have
\[
\lambda \ast \varphi f \xrightarrow{(\sigma)} \varphi(e_{j_{(n)}}^l) \ast \varphi f = \varphi(e_{j_{(n)}}^l \ast f) \xrightarrow{\text{norm}} \varphi(f).
\]
Hence if \( \{\mu_i : i \in I\} \) is a net in \( B \) with \( \mu_i = \varphi(f_i) \) and \( \mu_i \xrightarrow{(\sigma)} \mu \) in \( A \), we have
\[
\lambda \ast \mu_i \xrightarrow{(\sigma)} \lambda \ast \mu_i = \lambda \ast \varphi f_i = \varphi f_i \xrightarrow{(\sigma)} \mu_i \xrightarrow{(\sigma)} \mu
\]
so that \( \lambda \ast a = a \) for all \( a \in A \). In particular we have \( \lambda \in A \) so \( \lambda \ast \lambda = \lambda \). Similarly if \( \nu \) is a \((\sigma)\) limit point of \( \{\varphi(e_k^r)\} \) then \( \nu \ast \nu = \nu \) and \( a \ast \nu = a \) for all \( a \in A \).

If \( \lambda, \nu \) are \((\sigma)\) limit points as above, we have \( \lambda, \nu \in A \), which \( \Rightarrow \lambda = \lambda \ast \nu = \nu \); hence \( \lambda = \nu \) and all limit points (left or right) coincide in a single idempotent \( \iota \) such that \( \iota \ast a = a \ast \iota = a \) if \( a \in A \). If \( \varphi \neq 0 \), clearly \( \iota \neq 0 \).

The main step in our proof is to show that, if \( \{f_j : j \in J\} \) is a
norm bounded net in $L^i(F)$ which is (so) convergent to some $\mu \in M(F)$, then the net $\{\varphi(f_j)\}$ converges to a limit $\lambda_\mu$ in the (σ) topology, and this limit depends only on $\mu$, rather than on the particular choice of the net $\{f_j\}$. This can be done for any $\mu \in M(F)$, in view of 1.1.5.

First consider any $f \in L^i(F)$ and notice that $\|f \ast f - \mu \ast f\| \to 0$, which $\iff \|\varphi f \ast \varphi f - \varphi(\mu \ast f)\| \to 0$. Let $\lambda$ be any (σ) limit point of the norm bounded net $\{\varphi(f_j)\}$; there exists a convergent subnet $\varphi(f_{j(k)}) \xrightarrow{(\sigma)} \lambda$. Then

$$\varphi(\mu \ast f) \xrightarrow{\text{norm}} \varphi(f_{j(k)} \ast f) = \varphi f \xrightarrow{(\sigma)} \lambda \ast \varphi f$$

for all $f \in L^i(F)$, which means that $\varphi(\mu \ast f) = \lambda \ast \varphi f$ for all $f \in L^i(F)$. Clearly $\lambda \in A$ since each $\varphi(f_j) \in B$, and this means that

$$\lambda = \lambda \ast e_k = \lambda \ast \varphi(e_k) = \varphi(\mu \ast e_k) \ast \varphi(\mu \ast e_k).$$

Thus we get

$$\lambda = \lim \{\varphi(\mu \ast e_k) : k \in K\}$$

in the (σ) topology, and this formula doesn’t depend on anything but the choice of $\mu \in M(F)$. Hence if $f_j \xrightarrow{(\omega)} \mu$, then $\lambda$ is the only possible limit point of $\{\varphi(f_j)\}$, so if we take $\lambda_\mu = \lim \{\varphi(\mu \ast e_k)\}$, we always have $\varphi f \xrightarrow{(\sigma)} \lambda_\mu$.

Notice that if $f \in L^i(F)$ we have $\|f \ast e_k - f\| \to 0$, which gives

$$\lambda_f \xrightarrow{(\sigma)} \varphi(f \ast e_k) \xrightarrow{\text{norm}} \varphi(f)$$

so that $\varphi f = \lambda_f$ for all $f \in L^i(F)$. Now define $\tilde{\varphi}(\mu) = \lambda_\mu$ for $\mu \in M(F)$, and verify the properties required. Clearly $\tilde{\varphi}(f) = \varphi(f)$ for all $f \in L^i(F)$, so $\tilde{\varphi}$ extends $\varphi$.

If (γ) is a locally convex topology on $M(F)$ we define the bounded (γ) topology (bγ) by taking as a basis of neighborhoods about zero all sets $X \cap Y$ where $X$ is a (γ) neighborhood of zero, and $Y$ is a fixed norm neighborhood of zero. From the discussion above we know that if $\mu \in M(F)$ and if $W$ is a (σ) neighborhood of zero in $M(G)$, then there is an open (bso) neighborhood $V$ of zero in $M(F)$ such that $\tilde{\varphi}((\mu + V) \cap L^i(F)) \subset \tilde{\varphi} \mu + W$. Now let $W' \subset W$ be a (σ) neighborhood of zero such that $W' = -W'$ and $W' + W' \subset W$, and let $U$ be an open (bso) neighborhood of zero in $M(F)$ such that

$$\tilde{\varphi}((\mu + U) \cap L^i(F)) \subset \tilde{\varphi} \mu + W'.$$

If $\lambda \in \mu + U$ we can find a (bso) neighborhood $U_\lambda$ of zero such that

1. $\tilde{\varphi}((\lambda + U_\lambda) \cap L^i(F)) \subset \tilde{\varphi} \lambda + W'$
2. $\lambda + U_\lambda \subset \mu + U$. 

Then we have
\[ \phi((\lambda + U) \cap L(F)) \subset \phi((\mu + U) \cap L(F)) \subset \phi(\mu + W) \]
and \( \phi((\lambda + U) \cap L(F)) \subset \phi(\lambda + W) \), which together imply that
\[ (\phi(\lambda + W) \cap (\phi(\mu + W)) \neq \emptyset \]
(see 1.1.5), which means that \( \phi(\lambda) \in \phi(\mu + W) \). Hence \( \phi(\mu + U) \subset \phi(\mu + W) \)
as required for continuity.

Clearly \( \| \phi(\mu) \| \leq \sup \{ \| \phi(\mu * e_n) \| \} \leq \| \mu \| \) for \( \mu \in M(F) \), and if \( \mu, \lambda \in M(F) \) we have
\[ \phi(\mu * \lambda) \overset{(s)}{=} \phi(\mu * (\lambda * e_n)) = \phi(\mu * \phi(\lambda * (e_n * \phi(\lambda * e_n)))) \overset{(s)}{=} \phi(\mu * \phi(\lambda) \overset{(s)}{=} \lambda. \]
Hence \( \phi \) is a norm decreasing homomorphism.

**Example.** In 4.1.1 we cannot replace the (so) topology with the (s) topology in \( M(F) \). Indeed, if \( Z = \text{integers}, S = \text{circle} \), and if \( \beta \) is some irrational number, then
\[ \phi(\sum_{n=1}^{\infty} \alpha_n \delta_{e_n}) = \sum_{n=1}^{\infty} \alpha_n \delta_{e_n \beta} \]
gives a norm decreasing homomorphism \( \phi : L^*(Z) \to M(S) \). This map coincides with its extension \( \bar{\phi} \). The sequence \( \{ \mu_n : \delta_{e_n} \} : n = 1, 2, \cdots \) is (s) convergent to zero, while \( \phi(\mu_n) \) is not (s) convergent in \( M(S) \).

**Remark.** The proof of 4.1.1 is also valid for any bounded homomorphism \( \phi : L(F) \to M(G) \), which means that the structure of a bounded homomorphism is determined once we know the structure of the bounded group of measures \( \bar{\phi}(\mathcal{E}_F) \); however, the structure of the bounded subgroups in \( M(G) \) is generally not known unless \( G \) is abelian or the subgroup lies within \( \Sigma M(G) \).

**4.2. The structure of norm decreasing homomorphisms.** If \( \bar{\phi} \) extends the norm decreasing homomorphism \( \phi : L(F) \to M(G) \), as in 4.1.1, then \( \Gamma = \bar{\phi}(\mathcal{E}_F) \) is a subgroup of the unit ball in \( M(G) \). Using the continuity properties of \( \bar{\phi} \) demonstrated in 4.1.1 and our knowledge of the structure of \( \Gamma \) we can determine \( \phi \) completely (see 1.1.3).

Let \( i = \rho m_0 \) be the unit of \( \Gamma \) and denote \( H_0 = \text{supp}(\Gamma), \Omega = \{(\alpha, g) \in S \times G : \alpha \delta_{e_n} * \rho m_0 \in \Omega \}, \Omega_0 = \{(\alpha, g) \in S \times G : \alpha \delta_{e_n} * \rho m_0 \in \Omega \}, \) and \( K_0 = \text{Ker} \rho \). Let \( \pi : S \times G \to (S \times G/\Omega_0) \) be the canonical map onto the space of right cosets of \( \Omega_0 \), so \( \pi \) is a homomorphism when restricted to \( S \times H_0 \), and let \( \tau_0 : (\alpha, g) \to \alpha \delta_{e_n} * \rho m_0 \) for \( (\alpha, g) \in S \times G \). Then define \( \theta : F \to (\Omega/\Omega_0) \subset (S \times G/\Omega_0) \), to be \( \theta = \pi * \tau_0^{-1} \circ \bar{\phi} \circ \delta \), so that \( \theta(\alpha) = \pi(\alpha, g) \) if and only if \( \bar{\phi}(\delta_{e_n}) = \alpha \delta_{e_n} * \rho m_0 \) in \( M(G) \). The mappings involved are shown in the following (commutative) diagram.

\[
\begin{array}{ccc}
S \times G \ni \Omega & \overset{\bar{\phi}}{\longrightarrow} & \Gamma \\
\downarrow \pi & & \downarrow \phi \circ \delta \\
(S \times G/\Omega_0) \ni \Omega/\Omega_0 & \overset{\theta}{\longrightarrow} & F
\end{array}
\]

**Figure 1**
Proposition 4.2.1. The map \( \theta : F \to \Omega / \Omega_0 \) is an epimorphism and is continuous as a mapping \( \theta : F \to (S \times G / \Omega_0)_r \).

Proof. Let \( x_1, x_2 \in F \) and let \((\alpha_1, g_1), (\alpha_2, g_2) \in \Omega\) be chosen such that \( \pi(\alpha_i, g_i) = \theta(x_i) \), and let \((\alpha, g) \in \Omega\) be chosen such that \( \pi(\alpha, g) = \theta(x_1, x_2) \). Thus \( \bar{\varphi}(\delta_{x_1}) = \alpha_1 \delta_{g_1} \ast \rho m_K \). We have \( \theta(x_1, x_2) = \theta(x_1) \theta(x_2) \) if \( \pi(\alpha, g) = \pi(\alpha_1, g) \pi(\alpha_2, g) \), which happens if \((\alpha, \alpha_1, g, g_2) \in (\alpha, g) \Omega_0 \).

This follows since
\[
\bar{\varphi}(\delta_{x_1 x_2}) = \bar{\varphi}(\delta_{x_1}) \ast \bar{\varphi}(\delta_{x_2}) = \rho m_K .
\]

We want to show \( \theta : F \to (S \times G / \Omega_0)_r \) is continuous. Because \( \bar{\varphi} \circ \delta \) and \( \pi \) are continuous it suffices to show that \( \tau_\delta : S \times G \to N = \tau_0(S \times G) \) is an open map when \( N \) has the restricted \((\sigma)\) topology. Let \((\alpha_0, g_0) \in S \times G\) and let \((\alpha, g) \in \Omega_0 \) be chosen such that \( \pi(\alpha_0, g_0) = \bar{\varphi}(\delta_{x_1}) = \rho m_K \) and \( \pi(\alpha, g) = \rho m_K \) if \( \pi(\alpha, g) = \rho m_K \).

Let \( \tau_\delta(\alpha_0, g_0) = \rho m_K \) and \( \tau_\delta(\alpha, g) = \rho m_K \) if \( \pi(\alpha, g) = \rho m_K \).

\[
\tau_\delta(\alpha_0, g_0) = \rho m_K ,
\]

Since \( g_0 \to g_0 \), we must have \( \alpha_0 \to \alpha_0 \) and \( \alpha_0 \) is eventually in \( U \); hence \( \tau_\delta(\alpha_0, g_0) = \tau_\delta(\alpha, g_0) \) is eventually in \( \tau_\delta(U \times V) \), a contradiction.

Let \( \tau^* \psi(\alpha, g) = \langle \alpha \delta_g \ast \rho m_K, \psi \rangle \) for \( \psi \in C_0(G) \). Then \( \tau^* \psi \in C_0(S \times G) \) since \( K \) is compact, and in fact \( \tau^* \psi \) is constant on right cosets of \( \Omega_0 \) in \( S \times G \) since \( \Omega_0 = \{(\rho(k), k) : k \in K\} \). If \( \mathcal{T} \in C_0(S \times G) \) and is constant on right cosets of \( \Omega_0 \), let us identify it with a function
\[
\pi^* \mathcal{T} \in C_0(S \times G / \Omega_0)_r .
\]

This function vanishes at infinity since \( \Omega_0 \) is compact. We can give an integral representation for norm decreasing homomorphisms as follows.

Theorem 4.2.2. Let \( F, G \) be locally compact groups and let \( \varphi : L^1(F) \to M(G) \) be a nonzero norm decreasing homomorphism with extension \( \bar{\varphi} \) to \( M(F) \), as in 4.1.1. Denote
\[
(1) \quad \Gamma = \bar{\varphi}(\mathbb{Z}_p),
(2) \quad i = \rho m_K \text{ the unit of } \Gamma,
(3) \quad \Omega = \{(\alpha, g) \in S \times G : \alpha \delta_g \ast \rho m_K \in \Gamma\}.
\]
\( \Omega_0 = \{ (\alpha, g) \in S \times G : \alpha \delta_s \ast \rho m_K = i \} \).

Define the maps
\[
\begin{align*}
\tau_0 : S \times G & \longrightarrow \Gamma \\
\tau^* : C_0(G) & \longrightarrow C_0(S \times G) \\
\pi^* \tau^* : C_0(G) & \longrightarrow C_0(S \times G/\Omega_0)
\end{align*}
\]
\[\theta : F \longrightarrow \Omega/\Omega_0\]
as indicated above. Then we have the representation
\[
(\ast) \quad \langle \bar{\theta} \mu, \psi \rangle = \langle \mu, (\pi^* \tau^* \psi) \circ \theta \rangle
\]
for all \( \mu \in M(F) \) and \( \psi \in C_0(G) \).

**Remark.** Since \( \theta : F \longrightarrow \Omega/\Omega_0 \) is a continuous homomorphism and \( \pi^* \tau^* \psi \in C_0((S \times G/\Omega_0)) \), it follows that \( (\pi^* \tau^* \psi) \circ \theta \) is a uniformly continuous function on \( F \). Thus the right hand side of (\ast\) is uniquely determined. We will want to make use of 1.1.1 in the following discussion.

**Proof.** We have \( \langle \bar{\theta} (\delta_s), \psi \rangle = \langle \delta_s, \pi^* \tau^* \psi \rangle = \langle \delta_s, (\pi^* \tau^* \psi) \circ \theta \rangle \) if \( x \in F \). If \( \mu \in M(F) \) is of norm one then there exists a net \( \{ \sigma_j : j \in J \} \) in the convex span of the extreme points of \( \Sigma_{M(F)} \) such that \( ||\sigma_j|| = 1 \) and \( \sigma_j \xrightarrow{(s0)} \mu \) (see 1.1.3). If we write \( \sigma_j = \sum \lambda(j, x) \delta_s \) (finite sum), we can apply 1.1.1 to get
\[
\langle \bar{\theta} \mu, \psi \rangle = \sum \lambda(j, x) \langle \bar{\theta} (\delta_s), \psi \rangle = \sum \lambda(j, x) \langle \delta_s, (\pi^* \tau^* \psi) \circ \theta \rangle = \langle \sigma_j, (\pi^* \tau^* \psi) \circ \theta \rangle = \langle \mu, (\pi^* \tau^* \psi) \circ \theta \rangle.
\]
Thus \( \langle \bar{\theta} \mu, \psi \rangle = \langle \mu, (\pi^* \tau^* \psi) \circ \theta \rangle \).

As a converse we have the following theorem which classifies all norm decreasing homomorphisms.

**Theorem 4.2.3.** Let \( F, G \) be locally compact groups and let \( \Gamma \) be a subgroup of \( \Sigma_{M(\Omega)} \) with unit \( i = \rho m_K \) and with
\[
\Omega = \{ (\alpha, g) \in S \times G : \alpha \delta_s \ast \rho m_K \in \Gamma \},
\]
\( \Omega_0 = \{ (\alpha, g) \in S \times G : \alpha \delta_s \ast \rho m_K = i \} \). Then if \( \theta : F \longrightarrow \Omega/\Omega_0 \) is any continuous epimorphism \( (\Omega/\Omega_0 \) is given the restricted topology from \( (S \times G/\Omega_0) \), the relation
\[
(\ast) \quad \langle \bar{\theta} \mu, \psi \rangle = \langle \mu, (\pi^* \tau^* \psi) \circ \theta \rangle
\]
for \( \mu \in M(F), \psi \in C_0(G) \) defines a norm decreasing homomorphism...
φ : (M(F), (so)) \rightarrow (M(G), (σ)) which is continuous on norm bounded sets, and we have \( \bar{φ} (\mathbb{E}_φ) = Γ \).

**Remark.** If \( \bar{φ} \) has the above continuity properties it is clear that \( \bar{φ} \) is obtained, as in 4.1.1, by extending the norm decreasing homomorphism \( \bar{φ} | L^i(F) \).

**Proof.** From 3.1.11 we see that \( Ω_o \) is a compact subgroup of \( S × G \) which is normal in \( Ω \), so \( Ω/Ω_o \) is well defined. We have also noted that \( \tau^* φ (\alpha, g) = \langle αρ_ρ^* ρm_κ, ψ \rangle \) is in \( C_δ(S × G) \) and

\[
π^* τ^* φ ∈ C_δ((S × G/Ω_o)),
\]
so \( (π^* τ^* φ) \circ θ \) is bounded and uniformly continuous on \( F \). Hence \( (*) \) is always well defined.

Clearly \( \bar{φ} : (M(F), (so)) \rightarrow (M(G), (σ)) \) is a norm decreasing linear map, and continuity on norm bounded sets follows from 1.1.1. Now \( \bar{φ} (\tilde{δ}_x) = \alpha δ_φ^* ρm_κ \) for all \( (\alpha, g) ∈ π^{-1}θ(x) \), so \( \bar{φ} \circ δ = \tau_0^* π^{-1} θ \); thus, \( \bar{φ} (\mathbb{E}_φ) = Γ \) and \( \bar{φ} \) is a continuous homomorphism of \( (\mathbb{E}_φ, (so)) \) into \( (M(G), (σ)) \). Convolution is a jointly \( (so) \) continuous operation in \( Σ_{M(F)} \), so \( \bar{φ} \) is a norm decreasing homomorphism of \( M(F) \) in view of the density theorems 1.1.3, 1.1.4.

A norm decreasing homomorphism \( φ : L^i(F) \rightarrow M(G) \) is order preserving if \( \mu \succeq 0 \Rightarrow φ(μ) \succeq 0 \). From the continuity properties given in 4.1.1 and the structure theorem 3.1.8 it follows that \( φ \) is order preserving \( ⇔ \bar{φ} (\mathbb{E}_φ) \) is a group of translates of Haar measure \( \{ δ_x * m_χ : x ∈ G \} \), where \( Q ⊆ G \) is a compact subgroup, normal in the subgroup \( G_o \). Every norm decreasing homomorphism \( φ \) is closely related to an order preserving norm decreasing homomorphism of \( L^i(F) \) into \( M(S × G) \).

If \( Ω ⊆ Ω_o \) are two subgroups in \( S × G \) satisfying conditions (1) and (2) in the discussion following 3.1.8, define the maps \( τ_0, \cdots , τ^{**} \) as indicated there.

**Theorem 4.2.4.** If \( \bar{φ} : (M(F), (so)) \rightarrow (M(G), (σ)) \) is a norm decreasing homomorphism, continuous on norm bounded sets, and if \( Γ = \bar{φ} (\mathbb{E}_φ) \) has unit \( i = ρm_κ \), then the subgroups

\[
Ω = \{(α, g) ∈ S × G : αδ_φ^* ρm_κ ∈ Γ\} ⊇ Ω_o = \{(α, g) ∈ S × G : αδ_φ^* ρm_κ = ρm_κ\}
\]
satisfy conditions (1) and (2) of 3.1.9 and we can factor \( \bar{φ} = τ^{**} \circ Φ \) where \( Φ \) is some order preserving norm decreasing homomorphism of \( M(F) \) into \( M(S × G) \). Here \( Φ \) maps \( \mathbb{E}_φ \) to the group of measures \( \{ δ_x * m_χ : x ∈ Ω \} \) and \( τ^{**} \) is a homomorphism on the range of \( Φ \).
Conversely, if $\Phi : M(F) \to M(S \times G)$ is any order preserving norm decreasing homomorphism, let $\Omega = \text{supp}(\Phi(\mathcal{E}_x)) \supseteq \Omega_0 = s(\Phi(\mathcal{E}_x))$. If $\Omega, \Omega_0$ satisfy conditions (1), (2) in 3.1.9, then $\phi = \tau^{**} \circ \Phi : (M(F), (s_o)) \to (M(G), (\sigma))$ is a norm decreasing homomorphism, continuous on norm bounded sets.

Proof. Let $\Omega \supseteq \Omega_0$ satisfy (1) and (2) of 3.1.9 and define $M_\alpha$ to be the subspace of measures in $M(S \times G)$ whose intersection with $\Sigma_{M(S \times G)}$ is co-$\Omega_0$. We assert that

(*) $M_\alpha$ is a subalgebra in $M(S \times G)$, $\tau** = \tau^{**}(\delta_a * m_\alpha) = \tau^{**}(\delta_a) = \tau_a(x)$ for $x \in \Omega$, and $\tau**$ is a norm decreasing homomorphism on $M_\alpha$.

Clearly $M_\alpha = \{ \mu : s(\mu) \subseteq \Omega \}$, and is a subalgebra. We have already shown (in discussing 3.1.10) that $m_\alpha \in M_\alpha$ and $\tau**(\delta_a * m_\alpha) = \tau^{**}(\delta_a) = \tau_a(x)$ for all $x \in \Omega$. Thus $\tau**$ is multiplicative on $\mathcal{E}_\alpha$, and since convolution is separately $(\sigma)$ continuous we can show that $\tau^{**}(\delta_a * \mu) = \tau^{**}(\delta_a) * \tau^{**}(\mu)$ for $\mu \in M_\alpha, x \in \Omega$. Then if $\lambda_j \overset{(a)}{\to} \lambda$ for $|| \lambda || = 1$, where $\lambda_j \in \text{co}[S \times G]$ we use the same idea once more to get

$$\tau**(\lambda * \mu) \overset{(a)}{\to} \tau**(\lambda_j * \mu) = \tau**(\lambda_j) * \tau**(\mu) \overset{(a)}{\to} \tau**(\lambda) * \tau**(\mu),$$

so $\tau** | M_\alpha$ is a homomorphism.

Now $\Phi$ maps $\mathcal{E}_x$ into $M_\alpha$, and if $\mu \in M(F), || \mu || = 1$, there exists a net $\{ \mu_j \} \subseteq \text{co}[\mathcal{E}_x]$ such that $\mu_j \overset{(s_0)}{\to} \mu$. This means $\Phi \mu_j \overset{(a)}{\to} \Phi \mu$ while $\Phi \mu_j \in \Sigma_{M(S \times G)} \cap M_\alpha$, so $\Phi \mu \in M_\alpha$ and $\Phi$ maps $M(F)$ into $M_\alpha$. Thus $\tau^{**} \Phi$ is well defined and is a norm decreasing homomorphism with the desired continuity properties ($\tau^{**}$ is $(\sigma)$ continuous on $M(S \times G)$).

Conversely, let $\widetilde{\phi}$ be given; then $\Omega, \Omega_0$ defined above satisfy (1) and (2), as shown in 3.1.11. The homomorphism $\theta : F \to \Omega/\Omega_0$, associated with $\widetilde{\phi}$ as in 4.2.2, is continuous, so $\theta^{**} \psi = \psi \circ \theta$ is uniformly continuous and bounded (UCB) on $F$ and we can consider the dual maps.

$$\theta^{**} : M(F) \to M(\Omega/\Omega_0),$$

For $\psi \in C_0(S \times G)$ define $\pi^* \psi \in C_0((S \times G)/\Omega_0)$ by lifting the function $\pi^* \psi(x) = \int \psi(\alpha t) dm_\alpha (t)$ (constant on right cosets of $\Omega_0$) over to the coset space $(S \times G)/\Omega_0$.

The desired map $\Phi$ is given by

$$\langle \Phi \mu, \psi \rangle = \langle \theta^{**} \mu, \pi^* \psi \rangle = \langle \mu, (\pi^* \psi) \circ \theta \rangle$$

for $\psi \in C_0(S \times G)$. It is easy to verify that $\Phi(\delta_a) = \delta_{(a, \alpha)} * m_\alpha$ for all $(\alpha, g) \in \pi^{-1}(\theta(x))$; therefore, as indicated in 3.1.10, we have $\langle \tau^{**} \Phi(\delta_a), \psi \rangle = \langle \tau^{**}(\delta_{(a, \alpha)} * m_\alpha), \psi \rangle = \langle \tau^{**}(\delta_{(a, \alpha)}), \psi \rangle = \langle \alpha \delta_a * \rho m, \psi \rangle = \langle \tilde{\phi}(\delta_a) \psi \rangle$ for all $\psi \in F$. Thus $\widetilde{\phi} = \tau^{**} \Phi$ on $\mathcal{E}_x$. But from 1.1.1 we see that $\Phi$
defined above is continuous on norm bounded sets mapping from the 
\((\mathcal{S} \circ)\) to the \((\sigma)\) topology; clearly, then \(\tau \circ \Phi : (M(F), (\mathcal{S} \circ)) \to (M(G), (\sigma))\) is continuous on norm bounded sets. Now \(\bar{\phi}\) and \(\tau \circ \Phi\) both enjoy 
this continuity property and coincide on \(S \circ F\); from 1.1.3 it follows 
that they coincide on all of \(M(F)\), and this is the desired factorization 
of \(\bar{\phi}\).

5. Examples and applications. In 5.1 we analyze the special 
structure of norm decreasing monomorphisms \(\phi : L'(F) \to M(G)\) between 
locally compact groups \(F\) and \(G\); then in § 5.2 we give the structure 
of all norm decreasing homomorphisms \(\phi\) which map \(L'(F)\) onto \(L'(G)\). 
Maps in the latter class have very simple structure.

5.1. Norm decreasing monomorphisms. Let us denote \(\mathcal{S} = \{\alpha \delta_\alpha : |\alpha| = 1, x \in F\}\) and \(\mathcal{S} \circ = \mathcal{E}_F\) throughout this discussion.

**Lemma 5.1.1.** If \(\phi : L'(F) \to M(G)\) is a norm decreasing mono-

morphism, and if \(\bar{\phi}\) is its extension to \(M(F)\) as in 4.1.1, then \(\phi\) is 
a monomorphism of \(M(F)\) into \(M(G)\). Furthermore \(\bar{\phi}(\mathcal{S} \circ) \cap S_i = \{i\}\), 
where \(i = \bar{\phi}(\delta_\alpha)\), and \(\mu = \lambda\) in \(\bar{\phi}(\mathcal{S} \circ)\) whenever \(s(\mu) = s(\lambda)\).

*Proof.* If \(\mu, \lambda \in M(F)\) have \(\bar{\phi}(\mu) = \bar{\phi}(\lambda) = \xi\) and \(\mu \neq \lambda\), then there 
is some \(f \in L'(F)\) such that \(\mu * f \neq \lambda * f\) while \(\phi(\mu * f) = \phi(\lambda * f) = \xi * \phi f\), 
a contradiction. Hence \(\bar{\phi}(\mathcal{S} \circ) \cap S_i = \{i\}\) and the last property follows 
from 3.1.4.

We propose to study the structure of all norm decreasing homo-
morphisms \(\phi\) whose extensions \(\bar{\phi}\) have the special property \(\Gamma_0 \cap S_i = \{i\}\), 
where \(\Gamma_0 = \bar{\phi}(\mathcal{S} \circ)\) and \(i = \bar{\phi}(\delta_\alpha)\) is the unit in \(\Gamma_0\). This discussion 
will apply to norm decreasing monomorphisms as a particular case. 
Hereafter we will denote \(\Gamma' = \bar{\phi}(\mathcal{S})\), \(\Gamma_0 = \bar{\phi}(\mathcal{S} \circ)\) (writing the unit of 
these groups as \(i = \rho m_K\)), \(H_0 = \text{supp}(\Gamma')\), and 
\[
\Omega = \{ (\alpha, g) \in S \times G : \alpha \delta_\alpha * \rho m_K \in \Gamma' \}.
\]
Let \(\pi : G \to (G/K)\) be the canonical map onto the right coset space, 
so \(\pi : H_0 \to H_0/K\) is the corresponding canonical homomorphism. Let 
\(\gamma_F, \gamma_G\) be the topologies on \(F, G\) and, if \(\gamma\) is a group topology on \(G\), 
let \(\gamma/\pi\) denote the quotient space topology on \((G/K)\), (notice \(\gamma/\pi = \pi(\gamma)\)). 
The restriction of \(\gamma_G\) to a subset \(N \subset G\) is \(\gamma_G|N\). We will speak 
interchangeably of a topology \(\gamma\) and the collection of open sets it 
specifies.

The following lemma holds for all locally compact groups; notation 
is chosen so its meaning in the present context is clear.
LEMMA 5.1.2. Let $F, G$ be locally compact groups and consider any system of subgroups $K \subset H_0 \subset G$ with $K$ a compact subgroup in $G$ which is normal in $H_0$. Let $\pi : G \to (G/K)$, be the canonical map onto the right coset space. If $\xi : F \to (H_0/K, \gamma_0/\pi)$ is a continuous epimorphism then $\xi(\gamma)$ and $\pi^{-1}\circ \xi(\gamma)$ are topologies in $H_0/K$ and $H_0$ respectively; moreover, if $\gamma$ is the common refinement in $H_0$ of $(\gamma_0|H_0)$ and $\pi^{-1}\circ \xi(\gamma)$ then $(H_0, \gamma)$ is a locally compact topological group, $\gamma/\pi = \xi(\gamma)$, and $\xi : F \to (H_0/K, \gamma/\pi)$ is an open, continuous epimorphism.

REMARK. Unless $K$ is trivial, $\pi^{-1}\circ \xi(\gamma)$ will not be a Hausdorff topology, but in all other respects (homogeneity, joint continuity of multiplication, etc.) it is like a group topology.

Proof. The topology axioms for $\pi^{-1}\circ \xi(\gamma)$ follow if we can verify them for $\xi(\gamma)$. Only the finite intersection property is nontrivial. If $V_1, V_2 \in \xi(\gamma)$ let $U_i = V_i \cdot \ker \xi$ and notice that $\xi^{-1}(x) \cap U_i \neq \emptyset$ implies that $\xi^{-1}(x) \subset U_i$. Thus

$$
\xi(V_i) \cap \xi(V_2) = \xi(U_i) \cap \xi(U_2) = \{x : \xi^{-1}(x) \cap U_i \neq \emptyset, i = 1, 2\} = \xi(U_1 \cap U_2) \in \xi(\gamma).
$$

Now $(H_0, \gamma)$ is a Hausdorff space and the collection of sets $\mathcal{U} = \{U \cap V : U = W \cap H_0, W \in \gamma_0; V = \pi^{-1}\circ \xi(X), X \in \gamma\}$ is a base for $\gamma$. If $U \cap V \in \mathcal{U}$ then $(U \cap V)^{-1} = U^{-1} \cap V^{-1} \in \mathcal{U}$, so the inverse mapping is bicontinuous. It is quite easy to verify that $\gamma$ is homogeneous, in the sense that $\gamma = \{xU : U \in \gamma\}$ for any $x \in H_0$, so joint continuity of multiplication will only be proved at the identity $e \in H$. If $e$ lies within $U \cap V \in \mathcal{U}$ there exist $U_0 \in \gamma_0 | H_0$ and $V_0 \in \pi^{-1}\circ \xi(\gamma)$, which contain $e$, such that $U_0 \subset U$ and $V_0 \subset V$; hence $(U_0 \cap V_0) \times (U_0 \cap V_0)$ is an open neighborhood of $(e, e)$ in $(H_0, \gamma) \times (H_0, \gamma)$ which maps into $U \cap V$ under the product mapping.

Clearly $\gamma \supset \pi^{-1}\circ \xi(\gamma)$, so that $\gamma/\pi = \pi(\gamma) \supset \pi \circ \pi^{-1}(\xi(\gamma)) = \xi(\gamma)$. For the converse inclusion, we first make a few simple assertions:

1. If $A \subset H_0$ is a union of $K$-cosets and if $B$ is any subset of $H_0$, then $(A \cap B) \cdot K = A \cap (B \cdot K)$;

2. If $A_{\alpha} \subset H_0$ for indices $\alpha \in I$, then $(\bigcup_{\alpha \in I} A_{\alpha}) \cdot K = \bigcup_{\alpha \in I} (A_{\alpha} \cdot K)$.

Now a typical element in $\gamma$ has the form $X = \bigcup_{\alpha \in I} A_{\alpha} \cap B_{\alpha}$ where $A_{\alpha} = \pi^{-1}\circ \xi(U_{\alpha})$ for some $U_{\alpha} \in \gamma$, and $B_{\alpha} = V_{\alpha} \cap H_0$ for some $V_{\alpha} \in \gamma_0$. Evidently $A_{\alpha} = A_{\alpha} \cdot K$ and $B_{\alpha} \cdot K = \pi^{-1}\circ (\pi B_{\alpha})$, so we get

$$
\pi(X) = \pi(X \cdot K) = \pi(\bigcup_{\alpha \in I} (A_{\alpha} \cap B_{\alpha}) \cdot K) = \pi(\bigcup_{\alpha \in I} (A_{\alpha} \cap (B_{\alpha} \cdot K))) = \pi(\bigcup_{\alpha \in I} \pi^{-1}\xi(U_{\alpha}) \cap \pi^{-1}\pi(B_{\alpha})) = \bigcup_{\alpha \in I} \xi(U_{\alpha}) \cap \pi(B_{\alpha}) .
$$
But continuity of $\zeta$ implies that $\zeta(\gamma_F) \supset (\gamma_0/\pi \mid H_0/K)$ and it is easily verified that the latter collection of sets is just $\pi(\gamma_0 \mid H_o)$; hence the last term in (*) is in $\zeta(\gamma_F)$, giving $\pi(\gamma) \subset \zeta(\gamma_F)$. Clearly $\gamma/\pi = \zeta(\gamma_F) \Rightarrow \zeta: F \rightarrow (H_0/K, \gamma/\pi)$ is an open mapping; so $(H_0/K, \gamma/\pi)$ is topologically isomorphic to the locally compact quotient group $F/Ker(\zeta)$. To see the local compactness of $(H_0, \gamma)$, notice that $K \subset H_0$ is $\gamma$ compact because $\gamma \mid K = \gamma_0 \mid K$. A result due to Mackey gives local compactness (see Montgomery-Zippen [7], p. 52).

If $x \in F$ write $s(x) = s(\phi(d_x))$, a coset of $K$ in $H_o$. The map $\zeta = \pi \circ s: F \rightarrow ((G/K)_0, \gamma_0/\pi)$ carries $F$ onto $H_0/K$, is a homomorphism (see 2.1.1), and is continuous since $x_j \rightarrow x = \delta_{x_j} \rightarrow \overline{\varphi}(\delta_{x_j}) \rightarrow \overline{\varphi}(\delta_x) = \pi \circ s(x_j) \rightarrow \pi \circ s(x)$. If $\mu \in \Gamma_0$ and $g \in s(\mu)$, then we can write $\mu = \rho_\mu(\mu) = \rho_\mu(\delta_x \ast m_K)$ and we can take $\rho_\mu$ to be a unique continuous function on the coset $s(\mu) \subset H_0$. Assigning $\rho_\mu$ in this manner for each $\mu \in \Gamma_0$ we have $\rho_\mu(st) = \rho(s(t))$ for $s \in s(\mu) \subset s(\mu)$, as indicated in the proof of 3.1.2. Define $\rho$ on all of $H_0$ such that $\rho(x) = \rho_\mu(x)$ if $x \in s(\mu)$, $\mu \in \Gamma_0$; this is unambiguous since $s(\mu) = s(\lambda) \Rightarrow \mu = \lambda$ (we assume $\Gamma_0 \neq \{1\}$, so 3.1.4 applies).

Consider the group topology $\gamma$ on $H_0$ constructed as in 5.1.2 for the epimorphism $\zeta = \pi \circ s: F \rightarrow (H_0/K, \gamma_0/\pi)$.

PROPOSITION 5.1.3. $(H_0, \gamma)$ is a locally compact Hausdorff group and $\rho \in (H_0, \gamma)^*$. 

Proof. Clearly $\rho$ is a unimodular multiplicative function on $H_0$ which is continuous on cosets of $K$ (see proof of 3.1.2). The topological group properties of $(H_0, \gamma)$ were verified in 5.1.2. Given $\varepsilon > 0$ we can find a $\gamma_0$ neighborhood $V$ of the unit in $G$ such that $|\rho(g_1) - \rho(g_2)| < \varepsilon$ for all $g_1, g_2 \in H_0$ with $g_1 = g_2 \mod K$ and $g_1^{-1}g_2 \in V$. This is clear since $\rho_\mu(s) = \rho_\mu(gs) = \alpha s$ for some $|\alpha| = 1$, whenever $s \in K, g \in s(\mu), \mu \in \Gamma_0$ (see proof of 3.1.2), and we know $\rho$ is uniformly continuous on $K$.

Let $g \in H_0$; then $g_0 \in s(x_0)$ for some $x_0 \in F$, and if $U$ is a compact $\gamma$ neighborhood of $x_0$, $N = s(U)$ is a neighborhood of $g_0$ in $(H_0, \gamma)$. $N$ is compact since continuity of $\pi \circ s: F \rightarrow (H/K, \gamma/\pi) \rightarrow \pi \circ s(U)$ is compact, and since $K$ is a $\gamma$ compact subgroup in $H_0$. If $\rho$ fails to be $\gamma$ continuous at $g_0$ we can find a net $\{g_j\} \subset N$ such that $g_j \rightarrow g_0$ while $\rho(g_j) \rightarrow \beta \neq \beta_0 = \rho(g_0)$. For each index $j$ there exists an $x_j \in U$ such that $g_j \in s(x_j)$; we can assume that the net $\{x_j]\}$ is $\gamma$ convergent to some $x \in U$, which will $\Rightarrow \mu_j = \overline{\varphi}(\delta_{x_j}) \overset{(\sigma)}{\rightarrow} \overline{\varphi}(\delta_x) = \mu$. But this $\Rightarrow s(\mu) = g_0K = s(\mu_0)$, since $g_j \overset{(\gamma_0)}{\rightarrow} g_0$, so $\mu_j = \mu_0 = \overline{\varphi}(\delta_x)$. Recall that $\mu_j = \rho(\delta_{x_j} \ast m_K)$ and $\mu_0 = \rho(\delta_{x_0} \ast m_K)$ from the definition of $\rho$.

If $\overline{\varphi} \in C(G)$ has sup norm one and $\langle \mu_0, \overline{\varphi} \rangle = 0$, then $\mathcal{T}(\varphi) =$
\[
\int \psi(ts)\rho(t)dm_K(t) \text{ is in } C_0(G), \quad ||\psi||_\infty \leq 1, \quad \text{and } \psi(s) = \alpha \bar{\rho}(s) \text{ for all } s \in g_oK, \text{ where } \alpha = \langle \mu_0, \psi \rangle \text{ (a constant } \neq 0). \quad \text{Furthermore,}
\]
\[
\langle \mu_0, \psi \rangle = \left[ \int \left[ \int \psi(ts)\rho(t)dm_K(t) \right] \rho(s) d[\delta_{g_0} * m_K](s) \right] \psi(x) \rho(x) d[\delta_{g_0} * m_K](x) \psi(x) \rho(x) d[\delta_{g_0} * m_K](x) = \langle \mu_0, \psi \rangle.
\]

If \( \varepsilon > 0 \), we can insure that \( |\psi(g, s) - \psi(g, s)| < \varepsilon \) for all \( s \in K \) and \( j \geq j \), since \( g_j \overset{\gamma_j}{\rightarrow} g_o \), and this means that \( \alpha = \langle \mu_0, \psi \rangle = \int \psi(x) \rho(x) d[\delta_{g_0} * m_K](x) = \langle \mu_0, \psi \rangle \). The last integral is eventually within \( \varepsilon \) of
\[
\int \psi(g, s) \rho(g, s) dm_K(s) = \alpha \int \bar{\rho}(g, s) \rho(g, s) dm_K(s)
\]
\[
= \alpha \int \bar{\rho}(g_o) \rho(g_o) dm_K(s) \to \alpha \beta_0 \bar{\beta}.
\]

Since this is true for all \( \varepsilon > 0 \), and \( \beta_0 \neq \beta \), we have a contradiction.

**Corollary 5.1.4.** If \( F \) is a compact group and \( \varphi : L(F) \to M(G) \) is a norm decreasing monomorphism, then in 5.1.2 \( \Gamma = \varphi(\Gamma) \) is a \( (\sigma) \) compact subgroup of \( \sum_{\mu \in [G], H_0 = \text{supp}(\Gamma) \text{ is a compact subgroup in } G, \text{ and } \gamma = \gamma_0 | H_0 \in H_o. \text{ Thus if } \rho \text{ is defined as above, } \rho \in (H_o, \gamma_0)^\circ. \)

**Proof.** Clearly \( \Gamma \) is compact; \( H_o \) is then \( \gamma_o \) compact since \( (H_o/K, \gamma_o/\pi) \) is compact (recall \( \pi \circ s : F \to (H_o/K, \gamma_o/\pi) \) is a continuous epimorphism). By definition of \( \gamma \) the map \( \pi \circ s : F \to (H_o/K, \gamma/\pi) \) is continuous and we know that \( K \subset H_o \) is \( \gamma \) compact; thus \( H_o \) is \( \gamma \) compact as well as \( \gamma_o \) compact. Since \( \gamma \) is finer than \( \gamma_o \), these must be equivalent topologies on \( H_o. \)

Consider the following maps between measure algebras.

1. Let \( H, G \) be locally compact groups and let \( j : H \to G \) be a continuous monomorphism. Define \( j^{**} : (M(H), (\sigma)) \to (M(G), (\sigma)) \) such that \( \langle j^{**} \mu, \psi \rangle = \langle \mu, \psi \circ j \rangle \) for \( \psi \in C_0(G) \).
2. Let \( H \) be a locally compact group and let \( \rho \in H^\circ. \) Define \( A_\rho : (M(H), (\sigma)) \to (M(H), (\sigma)) \) such that \( A_\rho(\mu) = \rho \mu, \) so \( \langle A_\rho \mu, \psi \rangle = \langle \mu, \rho \psi \rangle. \)
3. Let \( F, H \) be locally compact groups, let \( K \) be a compact
normal subgroup in \( H \), and let \( \zeta : F \to (H/K, \gamma_n/\pi) \) be an open continuous epimorphism, where \( \pi : H \to H/K \) is the canonical homomorphism. Then define \( \Phi : (M(F), (\sigma_0)) \to (M(H), (\sigma_0)) \) such that 
\[
<\Phi \mu, \psi> = <\mu, (\pi^* \psi) \circ \zeta>,
\]
where the function \( \pi^* \psi(x) = \int \psi(xt) dm_\kappa(t) \), constant on cosets of \( K \), is considered as a function in \( C_0(H/K) \).

We assert that the maps in (1) ... (3) are all norm decreasing homomorphisms, continuous on norm bounded sets with respect to the topologies indicated. Since \( (A_p f) \ast f = \rho(\mu \ast \rho f) \), this assertion is clear for (2), and follows easily from 1.1.1 for (1), because \( \psi_{ij} \) is uniformly continuous and bounded on \( H \); we momentarily put off verification of (3). Once this assertion has been checked we can prove the following structure theorem.

**THEOREM 5.1.5.** If we are given groups and maps as in (1) ... (3) then the map \( \varphi = j^{**} \circ A_p \circ \Phi : (M(F), (\sigma_0)) \to (M(G), (\sigma_0)) \) is a norm decreasing homomorphism, continuous on norm bounded sets, with the special property that \( \Gamma_0 \cap S_i = \{i\} \), where \( \Gamma_0 = \varphi(S_0) \) and \( i \in \Gamma \) is its unit. Conversely, let \( \varphi : L^i(F) \to M(G) \) be a norm decreasing homomorphism whose extension \( \bar{\varphi} \) (as described in 4.1.1) has the special property \( \Gamma_0 \cap S_i = \{i\} \), where \( \Gamma_0 = \bar{\varphi}(S_0) \) and \( i = \rho m_K \) is its unit. If \( H_0 = \text{supp}(\Gamma) \), then we get \( \bar{\varphi} = j^{**} \circ A_p \circ \Phi \) by taking groups \( H = (H_0, \gamma) \supset K = (K, \gamma) \) and maps \( \zeta = \pi \circ s : F \to (H_0/K, \gamma/\pi) \), \( j = \text{id} : (H_0, \gamma) \to (G, \gamma_0) \), where \( \rho \in (H_0, \gamma) \) is the unique function on \( H_0 \), continuous on cosets of \( K \), with the property \( \mu = \rho | \mu | \) for all \( \mu \in \Gamma \).

**REMARK.** In the first part, \( \bar{\varphi} \) is clearly the extension of \( \varphi = \bar{\varphi} \mid L^i(F) \). Furthermore, the unit of \( \Gamma \) will be \( i = \rho m_K \) and \( \text{supp}(\Gamma) = H \), when \( H \) and \( K \) are regarded as subgroups in \( G \). In the second part the \( \gamma \) topology in \( H_0 \) is defined as in 5.1.2.

**Proof.** In the first part consider \( H \) and \( K \) as subgroups of \( G \) (with new group topologies) and \( j \) as the identity injecting \( H \) into \( G \); \( H \) has a topology finer than \( \gamma_0 \mid H \), but since \( j \) is continuous, it is a homeomorphism on compacta and on cosets of \( K \) in particular. If \( x \in F \) it is easy to verify that \( \bar{\varphi}(\delta_x) = \rho(\delta_x \ast m_K) \) for any \( g \in \bar{\pi}_1 \circ \zeta(x) \). From this it is clear that \( \Gamma \) has unit \( i = \rho m_K \), and that \( \Gamma_0 \cap S_i = \{i\} \).

Conversely let \( \varphi : L^i(F) \to M(G) \) be given. If we take \( H = (H_0, \gamma) \), \( K = (K, \gamma) \) and let \( \zeta = \pi \circ s, j = \text{id} : (H_0, \gamma) \to (G, \gamma_0) \), we see that \( H \) is a locally compact group and that \( \zeta : F \to (H_0/K, \gamma/\pi) \) is an open, continuous homomorphism (5.1.2); thus, the maps \( j^{**}, A_p, \Phi \) are well defined. We know \( \rho \in (H_0, \gamma) \) from 5.1.3.

If \( x \in F \) then \( |\bar{\varphi}(\delta_x)| = \delta_x \ast m_K \) for any \( g \in s(x) \) and \( \bar{\varphi}(\delta_x) = \rho(\delta_x \ast m_K) \).
by definition of \( \rho \). It is a simple matter to verify that

\[
id** \circ A_\rho \circ \Phi(\delta_x) = \rho(\delta_x * m_K)
\]

for any \( g \in s(x) \), so that \( \Phi = id** \circ A_\rho \circ \Phi \) on \( \mathcal{F} \). Since the maps on each side of this identity are continuous on norm bounded sets, as maps of \((M(F), (so))\) into \((M(G), (\sigma))\), we get \( \Phi = id** \circ A_\rho \circ \Phi \) on all \( r \) of \( M(F) \) from 1.1.3.

**Corollary 5.1.6.** A norm decreasing homomorphism \( \varphi : L'(F) \rightarrow M(G) \) is a monomorphism \( \Longleftrightarrow \) its extension has the structure \( \Phi = id** \circ A_\rho \circ \Phi \), as in 5.1.5, where the map \( \zeta = \pi \circ s \) which induces \( \Phi \) is an isomorphism of \( F \) onto \( H_0/K \).

**Proof.** If \( \varphi \) is a monomorphism, so is \( \Phi | F \) (see 5.1.1); now 5.1.5 applies and it is clear that \( \zeta = \pi \circ s \) is an isomorphism, as required for \( \Rightarrow \). Notice that the maps \( A_\rho \) and \( id** \) are always monomorphisms in (2) and (3) above. Conversely, in (3) we have \( \Phi = \pi** \circ \zeta** \), where \( \langle \zeta** \mu, \psi \rangle = \langle \mu, \psi \circ \zeta \rangle \) and \( \langle \pi** \mu, \psi \rangle = \langle \mu, \pi^* \psi \rangle \) define maps

\[
M(F) \overset{\zeta**}{\longrightarrow} M(H_0/K, \gamma/\pi) \overset{\pi**}{\longrightarrow} M(H_0, \gamma).
\]

Since \( \zeta = \pi \circ s : F \rightarrow (H_0/K, \gamma/\pi) \) is a topological isomorphism if \( \zeta \) is 1:1, \( \zeta** \) is a monomorphism. It is easy to verify that \( \pi^* (C_0(H_0, \gamma)) \) is sup norm dense in \( C_0(H_0/K, \gamma/\pi) \); hence \( \pi** \) is always a monomorphism.

In the following paragraphs we digress to study the map defined in (3) and prove the assertions about it which were used to prove 5.1.5. Then in 5.2, we will use these observations to study the structure of special norm decreasing homomorphisms.

**Theorem 5.1.7.** Let \( F \) and \( H \) be locally compact groups, let \( K \subset H \) be a compact normal subgroup, and let \( \zeta : F \rightarrow H/K \) be an open, continuous epimorphism. Then the map \( \Phi : (M(F), (so)) \rightarrow (M(H), (so)) \), defined such that \( \langle \Phi \mu, \psi \rangle = \langle \mu, (\pi^* \psi) \circ \zeta \rangle \), is a norm decreasing homomorphism, continuous on norm bounded sets, if we identify \( \pi^* \psi(x) = \int \psi(xt) dm_K(t) \) (constant on cosets of \( K \)) with a function in \( C_0(H/K) \) for each \( \psi \in C_0(H) \).

**Proof.** Consider the maps shown in Figure 2,

\[
\begin{array}{ccc}
(M(F), (so)) & \xrightarrow{\Phi} & (M(H), (so)) \\
\downarrow \Phi_1 & & \downarrow \pi** \\
(M(F/F_\rho), (so)) & \xrightarrow{\Phi_2} & (M(H/K), (so))
\end{array}
\]

Figure 2
where $F_0 = \text{Ker} (\zeta)$, $\langle \Phi \mu, \psi \rangle = \langle \mu, \psi \circ \pi_0 \rangle$ ($\pi_0 : F \to F/F_0$ is the canonical homomorphism), $\langle \Phi \mu, \psi \rangle = \langle \mu, \psi \circ \zeta \circ \pi_0^{-1} \rangle$, and where $\langle \pi^* \Phi \mu, \psi \rangle = \langle \mu, \pi^* \psi \rangle$. Clearly $\Phi_2$ is bicontinuous with respect to the topologies in Figure 2. Continuity of $\Phi$ follows from the lemmas below, since we can verify by direct computation that $\Phi = \pi^* \circ \phi_2 \circ \Phi_1$ on $M(F)$.

**Lemma 5.1.8.** Let $Q$ be a locally compact group with $Q_0 \subset Q$ a closed, normal subgroup, and let $\pi_0 : Q \to Q/Q_0$ be the canonical homomorphism. Define $\Phi : (M(Q), (\sigma)) \to (M(Q/Q_0), (\sigma))$ such that $\langle \Phi \mu, \psi \rangle = \langle \mu, \psi \circ \pi_0 \rangle$ for $\psi \in C_0(Q/Q_0)$. Then $\Phi$ is a norm decreasing homomorphism, continuous on norm bounded sets.

**Proof.** It is easy to verify that $\Phi$ is a norm decreasing homomorphism. We assert that $\Phi(M(Q)) = M(Q/Q_0)$, and in fact $\Phi(\Sigma_{\mu: \lVert \mu \rVert \leq 1, s(\mu) \subset X})$ for $X \subset Q$, we will show that $\Phi(\Sigma_K) = \Sigma_{\pi_0 K}$ for all compacta $K \subset Q$; since $\pi_0$ is open, this means that every $\mu$ with compact support in $M(Q/Q_0)$ is the (P-image of some $\mu \in M(Q)$ with $\lVert \mu \rVert = \lVert \lambda \rVert$. Clearly $\Phi(\Sigma_K) \subset \Sigma_{\pi_0 K}$, and $K$ compact $\rightarrow$ the map $\Phi : (\Sigma_K, (\sigma)) \to (M(Q/Q_0), (\sigma))$

is continuous, in fact if $\{\mu_j\} \subset \Sigma_K$ with $\mu_j \xrightarrow{(\sigma)} \mu$ and if $f \in C(Q)$ has $f = 1$ on $K$, then for any $\psi \in C(Q/Q_0)$ we get $\langle \Phi \mu_j, \psi \rangle = \langle \mu_j, \psi \circ \pi_0 \rangle = \langle \mu, f \cdot (\psi \circ \pi_0) \rangle = \langle \mu, f \cdot (\psi \circ \pi_0) \rangle = \langle \Phi \mu, \psi \rangle$. Now $\Sigma_K$ is precisely the $(\sigma)$-closed convex span of $\{a \delta_x : |x| = 1, x \in K\}$, so $\Phi(\Sigma_K)$ is $(\sigma)$-compact; since $\Phi(\delta_x) = \delta_{\pi_0 x}$ we have $\Phi(\Sigma_K) \supset \bigcup \{a \delta_{\pi_0 x} : |x| = 1, x \in K\}$, which gives the converse inclusion.

Now if $\lambda \in M(Q/Q_0)$ there are measures $\lambda_n$ with compact support such that $\lVert \lambda_n - \lambda \rVert \to 0$ and $\lVert \lambda \rVert = \lVert \lambda_1 \rVert + \Sigma_{n=1}^\infty \lVert \lambda_{n+1} - \lambda_n \rVert$ (restrict $\lambda$ to increasingly large compacta). Then there exist $\mu_n \in M(Q)$ with $\Phi(\mu_n) = \lambda_n$, $|\mu_n| = |\lambda_n|$ and $\Phi(\mu_{n+1}) = (\lambda_{n+1} - \lambda_n)$, $|\mu_{n+1}| = |\lambda_{n+1} - \lambda_n|$ for $n \geq 1$; hence $\mu = \Sigma_{n=1}^\infty \mu_n$ converges in $M(Q)$, $|\mu| = |\lambda|$, and $\Phi(\mu) = \lambda$ as required.

Next we show $\Phi(L'(Q)) \subset L'(Q/Q_0)$; in fact, if $\mu = \Phi f$ and $x \in Q/Q_0$, then given $\varepsilon > 0$ and $g \in \pi_0^{-1}(x)$ we can find a compact neighborhood $V$ of $g$ with $|\delta_g \ast f - f| < \varepsilon$ for all $h \in V$. Thus $\lVert \Phi(\delta_g) \ast \Phi f - \Phi f \rVert = |\delta_g \ast \mu - \mu| < \varepsilon$ for $y \in \pi_0 V$. Since $\pi_0$ is open and continuous, this means $\mu = \Phi f \in L'(Q/Q_0)$ (see Rudin [9], p. 230; the abelian hypothesis used there is superfluous). To prove 5.1.8 it is now sufficient to show that $\Phi(L'(Q))$ is norm dense in $L'(Q/Q_0)$. To prove density, let $\{e_{ij}\} \subset L'(Q)$ be a left approximate identity such that $e_{ij} > 0$, $\lVert e_{ij} \rVert = 1$, and $s(e_{ij})$ are compacta which are eventually within any fixed neighborhood of
the unit in \( Q \). Then \( s(\Phi e_j) \) are compacta shrinking to the unit in \( Q/Q_0 \).

But \( \langle \Phi e_j, \varphi \rangle \to \varphi(e) \) for all \( \varphi \in C_0(Q/Q_0) \), hence \( \Phi e_j \to \delta_e \) and \( || \Phi e_j || \leq 1 \); these facts together imply that \( \lim || \Phi e_j || = 1 \). Since \( \Phi e_j \geq 0 \) (clear), \( \{ \Phi e_j \} \subset L^1(Q/Q_0) \) is an approximate identity for \( L^1(Q/Q_0) \). Since \( \Phi(L^1(Q)) \) is an ideal in \( M(Q/Q_0) \), norm density of \( \Phi(L^1(Q)) \) in \( L^1(Q/Q_0) \) follows.

**Lemma 5.1.9.** If \( Q \) is a locally compact group, \( K \subset Q \) a compact normal subgroup with canonical homomorphism \( \pi: Q \to Q/K \), define \( \pi^{**}: (M(Q/K), (so)) \to (M(Q), (so)) \) such that \( \langle \pi^{**} \mu, \varphi \rangle = \langle \mu, \pi^{*} \varphi \rangle \) where \( \pi^{*} \varphi(x) = \int \varphi(xt)dm_K(t) \) (constant on cosets of \( K \)) is regarded as a function in \( C_0(Q/K) \). Then \( \pi^{**} \) is a norm decreasing homomorphism, continuous on norm bounded sets.

**Proof.** Normality of \( K \) in \( Q \) \( \Rightarrow \) \( h \circ m_K = m_K \circ h = m_K \circ h \circ m_K \) for all \( h \in L^1(Q) \). Define \( \xi: M(Q) \to M(Q/K) \) such that \( \langle \xi \mu, \varphi \rangle = \langle \mu, \varphi \circ \pi \rangle \).

It is a simple matter to verify that (1) \( \pi^{**} \xi(\mu) = \mu \circ m_K \) for all \( \mu \in M(Q) \), and (2) \( \pi^{**} \xi(\mu) = \mu \) for all \( \mu \in M(Q/K) \). One can also verify by direct computation that \( \pi^{**} \mu = (\pi^{**} \mu) \circ m_K \) for \( \mu \in M(Q/K) \). From (1) we see that \( \pi^{**}(M(Q/K)) = M(Q) \circ m_K \), so that \( \pi^{**}(L^1(Q/K)) \) is closed under right or left multiplication by elements of \( M(Q) \circ m_K \).

Finally, \( \pi^{**}(L^1(Q/K)) \subset L^1(Q) \); for \( \varepsilon > 0 \) and \( x \in Q \), and if \( f \in L^1(Q/K) \), we can find a neighborhood \( V \) of \( \pi(x) \) with \( || \delta_{\varepsilon} f - f || < \varepsilon \) whenever \( u \in V \). Thus, if \( W \) is a neighborhood of \( x \) such that \( \pi(W) \subset V \), we have \( || \delta_{\varepsilon} (\pi^{**} f) - \pi^{**} f || \leq || \xi(\delta_{\varepsilon}) \circ (\pi^{**} f) - \xi(\pi^{**} f) || = || \delta_{\varepsilon} f - f || < \varepsilon \) for all \( y \in W \). Thus \( \pi^{**} f \in L^1(Q) \) (again see Rudin [9], p. 230). If \( \{ e_j \} \) is a norm one approximate identity in \( L^1(Q/K) \), then \( e_j \to \delta_e \) and it is easy to show that \( \pi^{**} e_j \to \pi^{**} (\delta_e) \) from 1.1.1. We can arrange that the supports \( s(\pi^{**} e_j) \) shrink to \( s(m_K) = K \), a compact set; thus we get \( \pi^{**} e_j \to m_K \) by applying 1.1.2. Since \( \pi^{**}(L^1(Q/K)) \) is closed under right multiplication by elements of \( m_K \circ M(Q) = M(Q) \circ m_K \), we get (for any \( h \in L^1(Q) \)) \( || (\pi^{**} e_j) \circ m_K \circ h - m_K \circ m_K \circ h || \to 0 \), which \( \Rightarrow \pi^{**}(L^1(Q/K)) \) is norm dense in \( m_K \circ L^1(Q) \).

Consider \( \mu_j \to \mu \) in \( M(Q/K) \) with \( || \mu_j || \leq 1 \); if \( h \in L^1(Q) \) then \( (\pi^{**} \mu_j) \circ h = (\pi^{**} \mu_j) \circ m_K \circ h \). But we can approximate \( m_K \circ h \) in norm by some \( \pi^{**} f (f \in L^1(Q/K)) \) and we know that \( (\pi^{**} \mu_j) \circ (\pi^{**} f) = \pi^{**} (\mu_j \circ f) \to (\pi^{**} \mu) \circ (\pi^{**} f) \).

5.2. Norm decreasing homomorphisms which map \( L^1(F) \) to \( L^1(G) \). Suppose \( \phi \) actually maps \( L^1(F) \) onto \( L^1(G) \), then the structure of \( \phi \) is exceedingly simple. First recall that if \( \phi \) is a norm decreasing isomorphism of \( L^1(F) \) onto \( L^1(G) \) it is actually an isometry; furthermore, an isometric isomorphism has the special structure
\[ \langle \varphi f, \psi \rangle = \int_{F} \rho \circ s(x) \psi \circ s(x) df(x) \]

where \( s : F \to G \) is any topological isomorphism and \( \rho \in G^\times \), as was first proved by Wendel [10], [11]. Although the structure theorem 5.1.4 could be used as the basis for a direct proof of these results, it only gives conditions on the structure of \( \varphi \) which are necessary (but not sufficient) if we are to have \( \varphi(L^i(F)) = L^i(G) \). To identify these norm decreasing isomorphisms (or isometries) precisely we would have to retrace some of Wendel’s analysis rather than do this we use Wendel’s analysis as a starting point.

**Theorem 5.2.1.** Let \( \varphi : L^i(F) \to L^i(G) \) be a norm decreasing epimorphism. Then there exists a closed normal subgroup \( F_0 \subset F \), an isometric isomorphism \( \Lambda : L^i(F/F_0) \to L^i(G) \), and \( \beta \in F^\times \) with \( \text{Ker} \beta \subset F_0 \) such that \( \varphi = \Lambda \circ (\pi^* A_\beta) \), where \( A_\beta(\mu) = \beta \mu \), and the canonical homomorphism \( \pi : F \to F/F_0 \) gives \( \langle \pi^*(\mu), \psi \rangle = \langle \mu, \psi \circ \pi \rangle \) for \( \psi \in C_c(F/F_0) \).

**Proof.** First notice that, if \( s(x) = s(\varphi(\delta_x)) \), then \( s : F \to G \) is a continuous homomorphism; in fact, \( \varphi(\delta_x) = \rho m_K \) for compact subgroup \( K \subset G \) and \( \rho \in K^\times \), and if \( h \in L^i(G) \) we can write \( h = \varphi f \) for some \( f \in L^i(F) \). Thus \( h \ast \rho m_K = \varphi(f) \ast \varphi(\delta_x) = \varphi(f) = h \), which is impossible for all \( h \) unless \( K = \{ e \} \), so \( \varphi \) maps \( \mathcal{E}_{M(F)} \) into \( \mathcal{E}_{M(G)} \). For continuity of \( s \) see remarks preceding 5.1.3. Hence \( F_0 = \{ x \in F : s(x) = e \in G \} \) is a closed normal subgroup in \( F \). If we define \( \beta(x) = \alpha \in S \iff \varphi(\delta_x) = a\delta_x \), the continuity properties of \( \varphi \) (see 4.1.1) insure that \( \beta \in F^\times \); thus \( A_\beta : \mu \to \beta \mu \) is an isometric automorphism of \( M(F) \). The map \( \pi^* : M(F) \to M(F/F_0) \) has been discussed in 5.1.8; we assert that \( \pi^* \) has the following properties (which will be verified at the end of this proof):

1. \( \pi^* L^i(F) = L^i(F/F_0) \), and
2. \[ ||\pi^*(\mu)|| = \inf \{ ||\mu + n|| : n \in \text{Ker} (\pi^*) \}, \]

the quotient norm in \( M(F)/\text{Ker} (\pi^*) \).

Clearly \( \mu \in \text{Ker} (\pi^* A_\beta) \iff \langle \pi^* A_\beta(\mu), \psi \rangle = \int_{F} \beta(x) \psi(\pi x) d\mu(x) = 0 \) for all \( \psi \in C_c(F/F_0) \); it is not hard to show that \( \mu \in \text{Ker} \varphi \iff \langle \varphi(\mu), \psi \rangle = \int_{F} \varphi(\delta_x) \psi d\mu(x) = \int_{F} \beta(x) \varphi(\delta_x) \psi d\mu(x) = 0 \) for \( \psi \in C_c(G) \). The non-trivial first equality here can be seen from 4.2.2, or directly by looking at the action of \( \varphi \) on finite sums of point masses and using the (so) continuity of \( \varphi \). We assert that \( \text{Ker} \varphi \supset \text{Ker} (\pi^* A_\beta) \), so

\[ A = \varphi \circ (\pi^* A_\beta)^{-1} : M(F/F_0) \to M(G) \]

is a well defined homomorphism.
Lemma 5.2.2. If \( \{f_j\} \) is a net of bounded functions in \( C(F) \) with \( M = \sup \{||f_j||_\infty\} < \infty \) and \( f_j \to f \) uniformly on compacta, then \( \langle \mu, f_j \rangle \to \langle \mu, f \rangle \) for all \( \mu \in M(F) \).

Proof. As usual, for bounded \( f \in C(F) \) we define \( \langle \mu, f \rangle = \langle \mu, \chi_n f \rangle \) where \( E \in B(F) \) and \( E \supset s(\mu) \). For \( K \) compact in \( F \) we obviously have \( \langle \langle \mu|K, f_j \rangle \to \langle \langle \mu|K, f \rangle \) and for suitably chosen compacta \( K_n \subset s(\mu) \) we have \( ||\mu - (\mu|K_n)|| \to 0 \); hence \( \langle \mu, f_j \rangle \to \langle \mu, f \rangle \).

Each function \( \Psi(x) = \langle \delta_{s(x)}, \psi \rangle \) is continuous, bounded, and constant on cosets of \( F_0 \) in \( F \), if \( \psi \in C_0(G) \). But any bounded \( f \in C(F) \) which is constant on cosets of \( F_0 \) can be approximated uniformly on compacta by a uniformly bounded net of functions selected from \( \{\delta \in C_0(F/F_0)\} \); in fact, if \( K \subset F \) is compact so is \( \pi K \), and if \( U \) is a relatively compact open neighborhood of \( \pi K \), we can find continuous \( h \) such that \( h = 1 \) on \( \pi K \), \( h = 0 \) outside \( U \) and \( 0 \leq h \leq 1 \). Then \( f \circ h \circ \pi \) coincides with \( h_1 \circ \pi \) on \( F \), where \( h_1(x) = h(x) \cdot f(\pi^{-1}(x)) \in C_0(F/F_0) \); we have \( h_1 \circ \pi = f \) on \( K \) and \( ||h_1 \circ \pi||_\infty \leq ||f||_\infty \) as desired. Taking \( \Psi \) as the uniform on compacta limit of uniformly bounded net \( \{f_j \circ \pi\} \) we get

\[
\langle \bar{\varphi} \mu, \psi \rangle = \langle \beta \mu, \bar{\Psi} \rangle = \lim \langle \beta \mu, h_j \circ \pi \rangle = \lim \langle \pi^* \Delta_\rho(\mu), h_j \rangle = 0
\]

if \( \mu \in \ker (\pi^* \Delta_\rho) \), so \( \ker \bar{\varphi} \supset \ker (\pi^* \Delta_\rho) \).

Now \( ||A|| \leq 1 \) since (2) insures that

\[
||\pi^* \Delta_\rho(\mu)|| = \inf ||\mu + n|| : n \in \ker (\pi^* \Delta_\rho) \\
\geq \inf ||\mu + n|| : n \in \ker \varphi \\
\geq \inf ||\varphi \mu + \varphi n|| = ||\varphi \mu|| = ||\varphi \mu||
\]

for \( \mu \in M(F) \). Since \( \pi^* : (M(F), (so)) \to (M(F/F_0), (so)) \) is continuous on norm bounded sets (see 5.1.8), \( (\pi^*)^{-1} \) is open on \( \Sigma_{M(F/F_0)} \) relative to the \( (so) \) topologies; hence \( \Lambda : (M(F/F_0), (so)) \to (M(G), (\sigma)) \) is continuous on norm bounded sets. From (1) we see that \( \Lambda \) maps \( L^1(F/F_0) \) onto \( L^1(G) \), so \( \Lambda \) on \( M(F/F_0) \) coincides with the extension \( \tilde{\Lambda} \) from \( L^1(F/F_0) \) discussed in 4.1.1. Furthermore,

\[
\Lambda(\mathcal{E}_{F/F_0}) = \bar{\varphi} \circ (\pi^* \Delta_\rho)^{-1}(\mathcal{E}_{F/F_0}) = \bar{\varphi}([\beta(x)] \delta_x) : x \in F = \{\delta_{s(x)} : x \in F\}
\]

so \( \Lambda(\mathcal{E}_{F/F_0} \cap S(\delta)) = \{\delta_s\} \) in \( \mathcal{E}_{M(G)} \) and the analysis of 5.1 applies; i.e. we can write \( \Lambda = j^{**} \circ A \circ \zeta^* : M(F/F_0) \to M(G) \) as in 5.1.5. In our present context some of these maps are trivial since \( \Lambda(\delta_s) = \delta_{s(\pi^{-1}(s))} \), for \( x \in F/F_0 \); indeed, \( \rho \) and \( K \) are trivial, \( j \) is the injection of \( H = \{s(x) : x \in F\} \) into \( G \), and \( \zeta : F/F_0 \to H \) is given by \( \zeta(x) = s(\Lambda(\delta_s)) = s(\pi^{-1}(s)) \). But \( F_0 = \ker s = \ker \pi \) in \( F \), so \( \zeta \) is an isomorphism of \( F/F_0 \) onto \( H \); hence, as indicated in 5.1.6, \( \Lambda \) must be a monomorphism.
on $M(F/F_0)$. Thus $A$ is a norm decreasing isomorphism between $L^1(F/F_0)$ and $L^1(G)$, and Wendel’s analysis applies to $A$.

In proving 5.1.8 we showed that $\pi^*(M(F)) = M(F/F_0)$ and that $\pi^*(\Sigma_{M(F)}) = \Sigma_{M(F/F_0)}$. The latter identity proves assertion (2) above. Furthermore, we showed $\pi^*(L^1(F)) \subset L^1(F/F_0)$ is norm dense, and that a right approximate identity $\{e_j\}$ of norm one in $L^1(F)$ is mapped to the same sort of approximate identity $\{\pi^*e_j\}$ in $L^1(F/F_0)$. Let $f \in L^1(F/F_0)$, say with $\|f\| = 1$, and let $\mu \in M(F)$ be chosen with $\|\mu\| = 1$, $\pi^*\mu = \hat{f}$; then $\pi^*(\mu * e_j) = (\pi^*\mu) * (\pi^*e_j) = f * (\pi^*e_j) \rightarrow_f f$ and $\mu * e_j \in L^1(F)$ with $\|\mu * e_j\| \leq \|\mu\| = \|f\|$. Hence we see $\pi^*(\Sigma_{L^1(F)})$ is norm dense in $\Sigma_{L^1(F/F_0)}$. We can find $g_1 \in L^1(F)$ with $\|g_1\| \leq 1$ and $\|\pi^*g_1 - f\| \leq 1/2$. Since $\pi^*g_1 - f \in L^1(F/F_0)$, there exists $g_2 \in L^1(F)$ with $\|g_2\| \leq 1/2$ and $\|\pi^*g_2 - (f - \pi^*g_1)\| < (1/2)^2$. By continuing this selection we get $g_n \in L^1(F)$ with $\|g_n\| \leq (1/2)^{n-1}$ and $\|\pi^*g_n - (f - \sum_{j=1}^{n-1} \pi^*g_j)\| < (1/2)^n$. Thus $g = \sum_{n=1}^{\infty} g_n$ converges in $L^1(F)$ and $\pi^*g = \sum_{n=1}^{\infty} \pi^*g_n = f$, proving assertion (1) above.

Bibliography

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should by typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's résumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsu-sha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Basil Gordon, Acting Managing Editor until February 1, 1966.
Robert James Blatter, Group extension representations and the structure space .......................... 1101
Glen Eugene Bredon, On the continuous image of a singular chain complex ................................. 1115
David Hilding Carlson, On real eigenvalues of complex matrices .............................................. 1119
Hsin Chu, Fixed points in a transformation group .................................................................... 1131
Howard Benton Curtis, Jr., The uniformizing function for certain simply connected Riemann surfaces ............................................................................................................ 1137
George Wesley Day, Free complete extensions of Boolean algebras ........................................... 1145
Edward George Effros, The Borel space of von Neumann algebras on a separable Hilbert space ....................................................................................................................... 1153
Michel Mendès France, A set of nonnormal numbers .................................................................... 1165
Jack L. Goldberg, Polynomials orthogonal over a denumerable set ........................................... 1171
Frederick Paul Greenleaf, Norm decreasing homomorphisms of group algebras ......................... 1187
Fletcher Gross, The 2-length of a finite soluble group .................................................................... 1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, Algebras of bounded sequences ............. 1239
James Patrick Jans, Some aspects of torsion ............................................................................... 1249
Laura Ketchum Kodama, Boundary measures of analytic differentials and uniform approximation on a Riemann surface ................................................................. 1261
Alan G. Konheim and Benjamin Weiss, Functions which operate on characteristic functions ............................................................................................................................ 1279
Ronald John Larsen, Almost invariant measures .......................................................................... 1295
You-Feng Lin, Generalized character semigroups: The Schwarz decomposition ......................... 1307
Justin Thomas Lloyd, Representations of lattice-ordered groups having a basis .......................... 1313
Thomas Graham McLaughlin, On relative coinductivity .............................................................. 1319
Mitsuru Nakai, Φ-bounded harmonic functions and classification of Riemann surfaces .............. 1329
L. G. Novoa, On n-ordered sets and order completeness ............................................................ 1337
Fredos Papangelou, Some considerations on convergence in abelian lattice-groups .................... 1347
Frank Albert Raymond, Some remarks on the coefficients used in the theory of homology manifolds ..................................................................................................................... 1365
John R. Ringrose, On sub-algebras of a C*-algebra ....................................................................... 1377
Jack Max Robertson, Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres ........................................................................... 1383
Zalman Rubinstein, Some results in the location of zeros of polynomials .................................... 1391
Arthur Argyle Sagle, On simple algebras obtained from homogeneous general Lie triple systems ................................................................................................................................. 1397
Hans Samelson, On small maps of manifolds ............................................................................... 1401
Annette Sinclair, |φ(z)|-closeness of approximation .................................................................... 1405
Edsel Ford Stiel, Isometric immersions of manifolds of nonnegative constant sectional curvature ................................................................................................................................. 1415
Earl J. Taft, Invariant splitting in Jordan and alternative algebras ............................................... 1421
L. E. Ward, On a conjecture of R. J. Koch .................................................................................. 1429
Neil Marchand Wigley, Development of the mapping function at a corner ................................. 1435
Horace C. Wiser, Embedding a circle of trees in the plane ............................................................ 1463
Adil Mohamed Yaqub, Ring-logics and residue class rings ............................................................ 1465
John W. Lamperti and Patrick Colonel Suppes, Correction to: Chains of infinite order and their application to learning theory ................................................................. 1471
Charles Vernon Coffman, Correction to: Non-linear differential equations on cones in Banach spaces ................................................................................................................................. 1472
P. H. Doyle, III, Correction to: A sufficient condition that an arc in S^n be cellular .................... 1474
P. P. Saworotnow, Correction to: On continuity of multiplication in a complemented algebra ................................................................................................................................. 1474
Basil Gordon, Correction to: A generalization of the coset decomposition of a finite group .... 1474