THE 2-LENGTH OF A FINITE SOLVABLE GROUP

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One measure of the structure of a finite solvable group $G$ is its $p$-length $l_p(G)$. A problem connected with this measure is to obtain an upper bound for $l_p(G)$ in terms of $e_p(G)$, which is a numerical invariant of the Sylow $p$-subgroups of $G$. This problem has been solved but the best-possible result is not known for $p = 2$. The main result of this paper is that $l_2(G) \leq 2e_2(G) - 1$, which is an improvement on earlier results. A secondary objective of this paper is to investigate finite solvable groups in which the Sylow 2-group is of exponent 4. In particular it is proved that if $G$ is a finite group of exponent 12, then the 2-length is at most 2.

Introduction and discussion of results. The object of this paper is to obtain bounds for the 2-length of a finite solvable group. Following Hall and Higman [4], we call a finite group $G$ $p$-solvable if it possesses a normal series such that each factor group is either a $p$-group or a $p'$-group. The $p$-length, $l_p(G)$, of such a group is the smallest number of $p$-groups which can occur as factor groups in such a normal series. $e_p(G)$ is defined to be the smallest $n$ such that $x^{2^n} = 1$ for all $x$ belonging to a Sylow $p$-subgroup of $G$.

For an odd prime $p$, it is proved in [4] that $l_p(G) \leq e_p(G)$ if $p$ is not a Fermat prime and $l_p(G) \leq 2e_p(G)$ if $p$ is a Fermat prime. Furthermore these results are best-possible. A. H. M. Hoare [6] then proved that in a 2-solvable group $G$, $l_2(G) \leq 3e_2(G) - 2$ provided that $l_2(G) \geq 1$. The primary purpose of this paper is to prove the following improvement:

**Theorem A.** If $G$ is a finite solvable group and $l_2(G) \geq 1$, then $l_2(G) \leq 2e_2(G) - 1$.

Feit and Thompson [1] have proved that solvability and 2-solvability are equivalent notions for finite groups. Thus no loss of generality is involved in requiring $G$ to be solvable in the theorem.

Theorem A will be shown to be an easy consequence of the following theorem about linear groups:

**Theorem B.** Let $G$ be a finite solvable linear group over a field
F of characteristic 2 and assume G has no nontrivial normal 2-subgroup. Then if N is the largest normal 2'-subgroup of G and if g is an exceptional element of order 2^m in G, it follows that g^m-1 is in the largest normal 2-subgroup of G/N.

Here, following [4], an element x of order p^n in a linear group over a field of characteristic p is said to be exceptional if (x - 1)^p^n - 1 = 0.

Whether or not Theorem A represents a best-possible result is not known, but it seems likely that further improvements can be made. Indeed, the author knows of no group whose 2-length exceeds its 2-exponent. In the special case of finite solvable groups satisfying e_2(G) = 2, i.e., solvable groups whose Sylow 2-subgroups are of exponent 4, I think it likely that l_2(G) ≤ 2 instead of the bound l_2(G) ≤ 3 furnished by Theorem A.

In § 4 of this paper, groups satisfying e_2(G) = 2 are studied in more detail. A sufficient condition for l_2(G) ≤ 2 in this special case is established, and, as an application, we prove that l_2(G) ≤ e_2(G) if G is a finite group of exponent 12.

2. Proof of Theorem A from Theorem B. For the rest of this paper we adopt the convention that all groups referred to are assumed finite, and, if G is such a group, then |G| denotes its order. If H is a normal subgroup of G, we write H ≤ G.

We now recall the definition of the upper 2-series of the solvable group G:

\[ 1 = P_0 \leq N_0 < P_1 < N_1 < \cdots < P_i \leq N_i = G. \]

Here N_k/P_k is defined to be the greatest normal 2'-subgroup of G/P_k and P_{k+1}/N_k the greatest normal 2-subgroup of G/N_k. The least integer l such that N_l = G is the 2-length l_2(G). (If there is no danger of confusion we write simply l_n.)

It is proved in [4] that the automorphisms of P/F, where F/N_0 is the Frattini subgroup of P_0/N_0, induced by G represent G/P_i faithfully. Thus G/P_i is faithfully represented as a linear group operating on P_i/F (P_i/F is an elementary abelian 2-group and so is considered as a vector space over the field with 2 elements).

Now if l_2(G) = 1, the conclusion of A is trivial. Also the p-length group is at most equal to the class of a Sylow p-subgroup [4, Theorem 1.2.6]. An immediate consequence of this is that if G is solvable and e_2(G) = 1, then l_2(G) = 1. Thus l_2 = 2 implies that e_2 ≥ 2 so the result again follows. Now if l_2 ≥ 2, then l_2(G/P_2) = l_2(G) - 2 ≥ 1 so that Theorem A would follow by induction on l_2 if we could prove that
Now suppose $g$ is an element of maximal order $2^m$ in a Sylow $2'$-subgroup of $G/P_1$. If $g$ is not exceptional, then [4, Lemma 3.1.2] we have $e_2(G) \geq m + 1$. If $g$ is exceptional, then, since $G/P_1$ satisfies the hypothesis of Theorem B, $g^{m-1}$ is in $P_2/P_1$ if Theorem B is true. Thus, assuming the validity of B, we obtain in all cases $e_2(G/P_2) \leq e_2(G) - 1$ and Theorem A follows.

3. Proof of Theorem B. Neither the hypothesis nor the conclusion of the theorem is affected by an extension of the field $F$. Hence, without loss of generality, we assume that $F$ is algebraically closed. Since an element of order 2 cannot be exceptional, $m$ must be greater than 1. Let $h = g^{m-1}$ and so $h^2 = g^{m-2}$.

In proving B we will define subgroups $H$ and $H_1$ such that $H \triangleleft G, H_1 \triangleleft H, h^2 \in H_1$, and $g$ normalizes $H_1$. It then will be shown that if $x$ is any element in the largest normal 2-subgroup of $H_1/H_\cap N$ then $(h^2, x) = (h, x)^2$. From this it will follow that $h^2$ is in the largest normal 2-subgroup of $H_1/H_\cap N$, and, finally, from this the theorem will follow.

First we need two lemmas which are of use later and which motivate the definition of $H$. Here, and elsewhere, we denote the space on which $G$ operates by $V$.

**Lemma 3.1.** If $Q$ is any $2'$-subgroup of $G$ which is normalized by $g$, then $h^2$ fixes every minimal characteristic $F - Q$ submodule of $V$.

**Proof.** A minimal characteristic $F - Q$ submodule is simply the join of all those $F - Q$ submodules operator isomorphic to a given irreducible $F - Q$ submodule. Now if $Q$ is a $2'$-group, $V$ can be written as the direct sum of the minimal characteristic $F - Q$ submodules. $g$ normalizes $Q$ so $g$ must permute the minimal characteristic $F - Q$ submodules. If the lemma were not true, then $g$, as a permutation of these submodules, would have a cycle of length $2^m$ which would contradict the assumption that $g$ is exceptional.

**Lemma 3.2.** If $Q$ is any abelian $2'$-subgroup of $G$ and $x$ is any element of $G$ normalizing $Q$ and fixing every minimal characteristic $F - Q$ submodule of $V$, then $x$ centralizes $Q$.

**Proof.** Let $V_i$ be any minimal characteristic $F - Q$ submodule of $V$. Since $Q$ is abelian and $F$ is algebraically closed, $Q$ operates on $V_i$ as a scalar multiplication, i.e., if $y \in Q$ and $v \in V_i$ then $yv = \chi_i(y)v$ where $\chi_i(y)$ is a scalar. We now obtain
Thus \((y, x)\) is the identity on \(V_i\) for all \(y \in Q\) and the lemma follows.

Now let \(H\) be the normal subgroup of \(G\) consisting of all elements which fix every minimal characteristic \(F - Q\) submodule for every normal \(2'\)-subgroup \(Q\). Since the largest normal \(2\)-subgroup and the largest normal \(2'\)-subgroup of \(H\) are normal in \(G\), we see that \(H\) has no normal \(2\)-subgroup greater than the identity and the largest normal \(2'\)-subgroup of \(H\) is \(H \cap N\). By Lemma 3.1 \(h^2\) must belong to \(H\).

Let \(M\) be the largest normal nilpotent subgroup of \(H\). Clearly \(M\) is a \(2'\)-group and \(M \triangleleft G\). Furthermore, since \(H\) is solvable, \(M\) contains its own centralizer in \(H\) [2].

**Lemma 3.3.** \(M\) is of class 2.

*Proof.* Since \(h^2 \in H\), \(h^2\) does not centralize \(M\). Thus by Lemmas 3.1 and 3.2, \(M\) is not abelian. Now let \(c\) be the class of \(M\) and suppose \(c \geq 3\). Then if \(\Gamma_i(M)\) is the \(i\)th term in the lower central series of \(M\) (\(\Gamma_1(M) = M\) and \(\Gamma_{i+1}(M) = (\Gamma_i(M), M)\)) and if \(d\) is the first integer \(\geq (c + 1)/2\), we have [3, Chap. 10] \[(\Gamma_d(M), M) = \Gamma_{d+1}(M) \neq 1 \text{ (since } d \leq c - 1)\, ,
\]
and
\[(\Gamma_d(M), \Gamma_{d}(M)) \leq \Gamma_{c_0}(M) = 1\, .
\]
Thus \(\Gamma_d(M)\) is abelian and, of course, normal in \(G\) but is not centralized by \(M\). From Lemma 3.2 and the definition of \(H\) we see that this is impossible, and so \(c = 2\).

\(M = M_1 \times M_2 \times \cdots\) where \(M_i\) is the Sylow \(q_i\)-subgroup of \(M\) and \(q_i\) is an odd prime. Each \(M_i\) is of class at most 2 and so \(M_i\) is a regular \(q_i\)-group [3, p. 183]. Then the elements of order at most \(q_i\) form a characteristic subgroup \(K_i\) of \(M_i\). Let \(K = K_1 \times K_2 \times \cdots\) An automorphism of \(M_i\) of order prime to \(q_i\) centralizes \(K_i\) only if it is the identity automorphism [7, Hilfssatz 1.5]. Therefore no 2-element of \(H\), except for the identity, centralizes \(K\). Hence \(K\) cannot be abelian (since \(h^2\) is a nonidentity 2-element of \(H\)) and so \(K\) must be of class 2.

We now are prepared to define the subgroup \(H_i\). For this purpose decompose \(V\) for each \(K_i\) into the sum
\[V = V_{i_1} \oplus V_{i_2} \oplus \cdots\]
where the \(V_{i_j}\) are the minimal characteristic \(F - K_i\) submodules. Let
\( C_{ij} = \{ x \mid x \in H \text{ and } (K_i, x) = 1 \text{ on } V_{ij} \} \). \( C_{ij} \) is a normal subgroup of \( H \) although not necessarily normal in \( G \).

Take \( H_i \) to be the intersection of all the \( C_{ij} \) which contain \( h^2 \). If \( h^2 \) is not in any \( C_{ij} \) then set \( H_i \) equal to \( H \). In any event \( H_i \lhd H \) and \( H_i \) is normalized by \( g \). As was the case with \( H, H_i \) has no normal 2-subgroup greater than the identity and the greatest normal 2'-subgroup is \( H_i \cap N \).

Now let \( P \) be a 2-subgroup of \( H_i \) such that \( P \) and \( g \) belong to the same Sylow 2-subgroup of \( G \) and \( P(H_i \cap N)/(H_i \cap N) \) is the largest normal 2-subgroup of \( H_i/(H_i \cap N) \). Since, modulo \( N, P \) is normalized by \( g \), it follows that \( g \) normalizes \( P \).

**Lemma 3.4.** If \( x \in P \), then 
\[
(h^2, x) = (h, x)^2.
\]

**Proof.** First we show that this lemma finishes the proof of Theorem B: \( h \) normalizes \( P \) so that \( (h, x)^2 \in \Phi(P) \) where \( \Phi(P) \) is the Frattini subgroup of \( P \). Thus the lemma implies that \( h^2 \) centralizes \( P/\Phi(P) \). Therefore from [4] we conclude that \( h^2 \in P \). Since \( h^2 \) is in the greatest normal 2-subgroup of \( H_i/(H_i \cap N) \), it follows that \( h^2 \) is in the greatest normal 2-subgroup of \( H/(H \cap N) \) from which the conclusion of Theorem B follows.

To prove the lemma, let \( k = (h^2, x)(h, x)^{-2} \) and suppose \( k \neq 1 \). Since \( k \) cannot centralize \( K \), \( (K, k) \) is not the identity on some \( V_{ij} \). Since \( k \in H_i \), we must have \( (K_i, h^2) \) also not the identity on \( V_{ij} \). (This last statement is the motivation for our choice of \( H_i \).)

In what follows let \( V' = V_{ij}, q = q_i, \) and \( Q, x_i, k_i \) the restrictions of \( K_i, x, k \), respectively, to \( V' \). Let \( g^{2m-n} \) be the first power of \( g \) fixing \( V' \) and let \( g/2 \) be the restriction of \( g^{2m-n} \) to \( V' \). Now \( h^2 \) is not the identity on \( V' \) and [4, p. 13] \( g_i \) must be exceptional

\[
(\text{i.e., } (g_i - 1)^{2n-1} = 0),
\]

and thus \( n \) must be at least 2. Let \( h_i = g_i^{2m-2}. k_i = (h_i, x_i)(h_i, x_i)^{-2} \) and both \( (Q, h_i^2) \) and \( (Q, k_i) \) are not the identity.

Since \( g_i \) is exceptional and \( (Q, h_i^2) \neq 1 \), \( Q \) cannot be abelian. Thus \( Q \) must be of class 2. \( V' \) is the sum of absolutely irreducible \( F - Q \) submodules all of which are operator isomorphic to each other. Hence \( Z(Q) \), the center of \( Q \), is cyclic and is generated by a scalar matrix. Since \( Q \) is of exponent \( q \) and \( Q' \neq 1 \), we see that

\[
Z(Q) = Q' = \Phi(Q)
\]

and so \( Q \) is an extra-special \( q \)-group [4, p. 15]. We note also that if \( S \) is the 2-group generated by \( x_i \) and \( g_i \), then \( (Z(Q), S) = 1 \) since \( Z(Q) \) is generated by a scalar matrix.
Now let $V''$ be an irreducible $F - QS$ submodule of $V$. $V''$ is an irreducible $F - Q$ module \cite[Lemma 2.2.3]{4}, and $V'$ is the sum of $F - Q$ modules operator isomorphic to $V''$. Thus $(Q, h_i^2) \neq 1$ on $V''$ and $g_i$ is exceptional on $V''$. From \cite[Theorem 2.5.4]{4} we have the following:

(1) $2^n - 1$ is a power of $q$, and

(2) if $g_i$ is faithfully and irreducibly represented on $Q/Q'$ (such a $Q_i$ can always be found since $h^2$ is not the identity on $Q/Q'$), then $Q$ can be written as the central product of $Q_i$ and a group $Q_2$ and $g_i$ transforms $Q_2$ trivially. It now follows \cite{6} that $2^n - 1 = q$ and $|Q_i/Q'| = q^2$.

The representation of $Q$ on $V''$ is isomorphic to the representation of $Q$ on $V'$ so that $(g_i, Q_i) = 1$ on $V''$ implies that $(g_i, Q_i) = 1$. Thus the centralizer of $g_i$ in the space $Q/Q'$ has co-dimension 2 over $GF(q)$. The minimal equation of $h_i$ on $Q_i/Q'$ must be $t^2 + 1 = 0$ so that $h_i^2$ must have the representation

$\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}$

on $Q_i/Q'$. We now can conclude that for every power of $g_i$ (except for the identity, of course), the co-dimension of its centralizer in $Q/Q'$ is 2. Also, since $q = 3 \pmod{4}$, $GF(q)$ contains no primitive 4th root of unity. Thus if $n \neq 2$ then in the completely reduced representation of $g_i^2$ on $Q/Q'$ there is only one nontrivial block. If $n = 2$, there are two nontrivial blocks.

Now if $c$ is a generator of $Q'$, define $\rho(a, b)$ for $a, b \in Q$ by the equation

$$(a, b) = \rho(a, b) = _{\rho}^{\omega_{a,b}} .$$

$\rho(a, b)$ is bilinear and skew symmetric and gives $Q/Q'$ the structure of a symplectic space over $GF(q)$ \cite{4}.

$\rho$ is of maximum rank since $Q' = Z(Q)$ so $Q/Q'$ must have dimension 2$r$. Since $(S, Q') = 1$, $S$ preserves the symplectic structure of $Q/Q'$. Thus the representation of $S$ on $Q/Q'$ may be considered as a subgroup of a Sylow 2-subgroup of the symplectic group on $Q/Q'$.

$Q/Q'$ is of dimension 2$r$ over $GF(q)$ so that $Q/Q'$ can be provided with the structure of a vector space $U$ of dimension $r$ over $GF(q^2)$. If $u_1, \ldots, u_r$ is a basis for $U$, the expression \cite{4}

$$\rho(\Sigma \alpha_i u_i, \Sigma \beta_i u_i) = \Sigma (\alpha_i \beta_i^* - \alpha_i^* \beta_i) \gamma ,$$

where $\alpha' = \alpha^*$ and $\gamma$ is a primitive 4th root of unity, is a skew symmetric bilinear form on $U$ of rank 2$r$ with values in $GF(q)$.

Let $\theta$ be a primitive $2^{r+1}$-th root of unity in $GF(q^2)$ and let $T$ be
the group of transformations of $GF(q^2)$ generated by the two transformations $\alpha \rightarrow \theta \alpha$ and $\alpha \rightarrow \theta \alpha'$. All transformations $y$ of $U$ of the form

$$y(\Sigma \alpha_i u_i) = \Sigma (T_i \alpha_i) u_{\sigma(1)} ,$$

where the $T_i$ are taken from $T$ and $\sigma$ is a permutation taken from a Sylow 2-subgroup of the symmetric group on the numbers $1, 2, \cdots, r$, form a Sylow 2-subgroup of the Symplectic group on $Q/Q'$ [4].

Thus we may assume that $x, g, h, i$, the representations of $x_i, g_i, h_i$, respectively, on $Q/Q'$, are of this form. Since $(Q, h_i^2) \neq 1$ and $(Q, k_i) \neq 1$, we have $h_i^2 \neq 1$ and $(h_i^2, x_i) \neq (h_i, x_i)^2$. We now need more information on $g_i$.

**Lemma 3.5.** The permutation $\sigma$ associated with $g_i$ is the identity permutation.

**Proof.** $\sigma$ is of order less than the order of $g_i$ from [4, p. 23]. First suppose $\sigma$ is of order $> 2$. Then $n > 2$ and so the representation of $g_i^2$ on $Q/Q'$ has only one nontrivial irreducible block. But the permutation associated with $g_i^2$ is $\sigma^2$ which has at least 2 disjoint nontrivial cycles. Clearly this is a contradiction. Thus $\sigma^2 = 1$.

Now suppose $\sigma \neq 1$. Assume, say, $\sigma(1) = 2, \sigma(2) = 1$. The representation of $g_i$ on $Q/Q'$ has only one nontrivial irreducible block so $g_i$ must be the identity on

$$\sum_{i=1,2} \alpha_i u_i .$$

Now $g_i(\alpha_i u_i + \alpha_i u_i) = T_i T_i \alpha_i u_i + T_i T_i \alpha_i u_i$ and so one of $T_i T_i$ or $T_i T_i$ must not be the identity of $T$. But then neither one can be the identity. Therefore the representation of $Q/Q'$ would have 2 nontrivial irreducible blocks. This can happen only if $n = 2$. This implies that $T_i T_i$ and $T_i T_i$ are of order 2 and thus must equal the transformation $\alpha \rightarrow -\alpha$. (This is the only element of order 2 in $T$.) Thus the centralizer of $g_i^2$ in $Q/Q'$ has co-dimension 4 over $GF(q)$ whereas it should have co-dimension 2. This proves that $\sigma = 1$.

Hence $g_i$ fixes each $u_i$ and must act trivially on $\alpha_i$ for all but one value of $i, i = 1$, say. Therefore

$$g_i(\Sigma \alpha_i u_i) = A \alpha_i u_i + \sum_{i \neq 1} \alpha_i u_i$$

where $A$ is an element of order $2^s$ in $T$. Then

$$h_i(\Sigma \alpha_i u_i) = A^{2^s-2} \alpha_i u_i + \sum_{i \neq 1} \alpha_i u_i ,$$

and
We may assume that

\[ x_i(\Sigma \alpha_i u_i) = \Sigma T_i \alpha_i u_{x(i)} . \]

**Case 1.** \( \pi(1) \neq 1 \). Assume, say, that \( \pi^{-1}(1) = 2 \). Straight forward calculation yields

\[ (h_i, x_i)(\Sigma \alpha_i u_i) = A^{2^{n-2}} \alpha_i u_1 + T_2 A^{2^{n-2}} T_2 \alpha_i u_2 + \sum_{i \neq 1, 2} \alpha_i u_i . \]

But \( A^{2^{n-1}} \) is the unique element of order 2 in \( T \). Thus

\[ (h_i, x_i)^2(\Sigma \alpha_i u_i) = -\alpha_i u_1 - \alpha_i u_2 + \sum_{i \neq 1, 2} \alpha_i u_i \]

and it is easily verified that this is the same result as \( (h_i^2, x_i) \).

**Case 2:** \( \pi(1) = 1 \). In this case we easily find that \( (h_i^2, x_i) \) is the identity while

\[ (h_i, x_i)^2(\Sigma \alpha_i u_i) = (A^{2^{n-2}}, T_i)^2 \alpha_i u_1 + \sum_{i \neq 1} \alpha_i u_i . \]

Now the group \( T \) easily is seen to be a generalized quaternion group of order \( 2^{n+1} \) so that the only conjugates of \( A \) in \( T \) are \( A \) and \( A^{-1} \). Thus

\[ (A^{2^{n-1}}, T_i)^2 = A^{2^{n-1}} T_i^{-1}(A^{2^{n-1}}) T_i = 1 . \]

Thus \( (h_i, x_i) \) is also the identity.

Therefore it has been shown that

\[ (h_i, x_i)^2 = (h_i^2, x_i) \]

in all cases. This completes the proof of lemma 3.4, and, by a previous argument, Theorem B now is proved.

**4. Groups with \( e_2 = 2 \).** If \( G \) is a solvable group whose Sylow 2-groups are of exponent 4, then we know from Theorem A that \( l_2(G) \leq 3 \). We now investigate conditions for \( l_2(G) \leq 2 \) to hold. The argument is similar to that used in proving Theorem B, but a more restrictive hypothesis is needed. That no loss of generality is involved in assuming the stronger hypothesis is insured by the following reduction theorem, which is stated in a slightly more general form than needed.

A proposition \( R \) will be said to be of type 4.1 if it is of the following form:

If \( G \) is a finite \( p \)-solvable group satisfying condition \( C \), then
$l_p(G) \leq f(e_p(G))$, where $f$ is a monotonically increasing function defined for nonnegative integral arguments, $f(0) = 0$, and condition C either is vacuous or states that $e_{p_i}(G) \leq a_i$ for some set, possibly infinite, of primes $p_i$ and nonnegative integers $a_i$.

Note that the proposition that $l_p(G) \leq e_p(G)$ if $G$ is a finite solvable group satisfying $e_p(G) \leq 2$ is of type 4.1. One of the results of this section is that $l_p(G) \leq e_p(G)$ if $G$ is a finite group of exponent 12. This statement is also of type 4.1 since the condition that $G$ be of exponent 12 is equivalent to stating that $e_p(G) \leq 2$, $e_p(G) \leq 1$, and $e_p(G) \leq 0$ for all other primes.

**Theorem 4.1.** To prove a proposition $R$ of type 4.1 it is sufficient to prove the proposition for the following special case:

1. $G$ is the normal product of $V$ by $G_1$ where $V$ is a vector space over $F$, a finite field of characteristic $p$, and $G_1$ is a $p$-solvable linear group on $V$ having no normal $p$-subgroup other than the identity.

2. Any irreducible representation of any $p'$-subgroup of $G_1$ over $F$ is in fact absolutely irreducible.

3. All groups of order at most $|G_1|$ satisfy $R$.

4. $V$ is an irreducible $F - G_1$ module.

**Proof.** In proving this theorem we assume $R$ is valid for the special case and then prove it is valid for the general case.

Now suppose $G$ is the group of smallest order which satisfies the hypothesis but not the conclusion of $R$, and let

$$1 = P_0 \leq N_0 < P_1 < \cdots < P_i \leq N_i = G$$

be the upper $p$-series of $G$. Since $f(0) = 0$ we must have $l_p(G) > 0$. If $F_1/N_0$ is the Frattini subgroup of $P_i/N_0$, then, as is shown in [4], $l_p(G/F_1) = l_p(G)$ so that if $F_1 \neq 1$ we would have a proper factor group of $G$ satisfying the hypothesis but not the conclusion of $R$.

Hence assume $F_1 = 1$. Thus $P_i$ is an elementary abelian $p$-group which we identify with a vector space $V_i$ over $GF(p)$. $G/P_i$ is faithfully represented as a linear group $G_1$ on $V_i$ and $G_1$ has no normal $p$-group greater than the identity.

From [4, p. 4] we may assume that $G$ has only one minimal normal subgroup. This subgroup must be contained in $V_1$ and we denote it with $M$. If $M \neq V_1$ and $G_1$ is faithfully represented on $V_1/M$ then we have $l_p(G/M) = l_p(G)$ so that we would have a contradiction to the minimality of $G$.

Now suppose $M \neq V_1$ and $G_1$ is not faithfully represented on $V_1/M$. 


Then the elements of $G_1$ centralizing $V/M$ form a normal subgroup of $G$, greater than the identity. If $Q$ is a minimal normal subgroup of $G_1$ centralizing $V/M$, then $Q$ must be a $p'$-group so that $V$ as a $Q$-module is completely reducible. Thus there exists a $Q$-module $M_i$ such that $V_i = M_i \oplus M_i$. $Q$ is the identity on $M_i$, but not on $M$ since $Q$ is faithfully represented on $V$. Now if $M_2$ is the centralizer of $Q$ in $V_1$ then $M_2$ is normal in $G$, $M_2$ is not the identity, and $M_2$ does not contain $M$. This contradicts the minimality of $M$.

Thus we see that $M = V_1$ which implies that $G_1$ is irreducibly represented on $V_1$. A consequence of this is that if $H$ is any normal subgroup greater than the identity in $G_1$ then $H$ can have no nonzero fixed vector in $V_1$. Otherwise all the vectors fixed by $H$ would form a nontrivial submodule of $V_1$.

Now pick $F$ to be a large enough finite extension of $GF(q)$ such that any irreducible representation of any $p'$-subgroup of $G_1$ over $F$ is absolutely irreducible. Let $1 = \theta_0, \theta_1, \ldots, \theta_r$ be a basis for $F$ over $GF(p)$ and let $v_1, v_2, \ldots, v_s$ be a basis for $V_1$ over $GF(p)$. Finally let $V$ be the vector space over $F$ with basis $v_1, \ldots, v_s$, i.e., the vectors of $V$ are the formal sums

$$\sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij} \theta_i v_j$$

where $c_{ij} \in GF(p)$. $G_1$ acts on $V$ in the obvious way.

Consider the group $G^* = G_1 V$, i.e., the normal product of $V$ by $G_1$. If $g^*$ is of order $p^m$ in $G^*$ then either the image $g$ of $g^*$ in $G_1$ is of order $p^m$ or $g$ is of order $p^{m-1}$ and $g$ is not exceptional on $V$. In the latter case $(g - 1)^{p^m-1} v_i \neq 0$ for some $v_i$ from which it follows that $g$ is not exceptional on $V_1$. Thus $e_p(G) \geq (m - 1) + 1 = m$.

Therefore in any event $e_p(G) \geq e_p(G^*)$. Since $e_q(G^*) = e_q(G)$ for $q \neq p$, $G^*$ satisfies condition C. Furthermore $l_p(G) = l_p(G^*)$ so that if $G^*$ satisfies $R$ so does $G$.

Now suppose $H$ is any normal $p'$-subgroup other than the identity in $G_1$ and suppose

$$v = \sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij} \theta_i v_j$$

is a nonzero vector fixed by $H$. Since $v \neq 0$ the coefficient of $v_j$ is not zero for some $v_j, j = 1$ say. Then there exists $\alpha \in F$ such that $\alpha(\sum_{i=0}^{s} c_{ij} \theta_i) = 1$. $H$ must fix $\alpha v$ which can be written in the form $\alpha v = v' + v''$ where

$$v' = v_1 + \sum_{j=2}^{s} c'_{0j} v_j, \quad v'' = \sum_{j=2}^{s} \sum_{i=1}^{r} c'_{ij} \theta_i v_j.$$

For $H$ to fix $\alpha v$ it must also fix $v'$ which contradicts the fact that
\[ H \] has no nonzero fixed vector in \( V_i \). Thus \( H \) has no nonzero fixed vector in \( V \).

If \( V \) is an irreducible \( F - G_i \) module then we have arrived at the special case of the theorem. Therefore assume \( U \) is a proper submodule.

If \( G_i \) is not faithfully represented on \( V/U \), then let \( Q \) be a minimal normal subgroup of \( G_i \) centralizing \( V/U \). \( Q \) must be a \( p' \)-group so that \( V \) is completely reducible as an \( F - Q \) module. Thus there exists a nontrivial \( F - Q \) submodule on which \( Q \) is the identity. This is impossible since \( Q \) can have no nonzero fixed vector.

Hence \( G_i \) is faithfully represented on \( V/U \). Thus \( l_p(G^*) = l_p(G^*/U) \) and, of course, \( e_p(G^*) \geq e_p(G^*/U) \) so that if \( G^*/U \) satisfies \( R \) so does \( G^* \) and then so does \( G \).

We still have that any normal nonidentity \( p' \)-subgroup \( H \) of \( G_i \) has no nonzero fixed vector in \( V/U \) since \( V \) is completely reducible as an \( F - H \) module. Therefore if \( G_i \) is not irreducibly represented on \( V/U \) then the same argument as before yields that \( G_i \) is faithfully represented on a nontrivial factor module of \( V/U \). Continuing in this way we ultimately arrive at the special case where \( G_i \) is faithfully and irreducibly represented on some vector space over the field \( F \). This finishes the proof of Theorem 4.1.

Among the results we now shall prove is that if \( G \) is of exponent 12 then \( e_2(G) \geq e_3(G) \). Before doing this it might be well to justify this work. For in a group of order \( 2^a3^b \) the 2-length and the 3-length can vary at most by one. Thus if it were true that the 3-length of a group of exponent 12 was one, then it would be trivial to state that the 2-length was at most two. However in \([5, p. 5]\) is found an example of a group of exponent 12 but with 3-length two.

For the rest of this paper we make the following standing assumptions.

(1) \( G = G_1V \), the normal product of \( V \) by \( G_1 \), where \( V \) is a vector space over a finite field \( F \) of characteristic 2 and \( G_1 \) is a finite, solvable linear group having no normal 2-subgroup other than the identity.

(2) \( V \) is an irreducible \( F - G_i \) module.

(3) Any representation over \( F \) of any \( p' \)-subgroup of \( G_i \) is absolutely irreducible.

(4) \( e_2(G) \leq 2 \).

We are interested in seeing under what conditions can \( l_2(G) \) exceed \( e_2(G) \). But if \( e_2(G_i) = 0 \) then both \( e_2(G) \) and \( l_2(G) \) are 1, and if \( e_2(G_i) = 1 \) then \( l_2(G_i) = 1 \) so that \( l_2(G) = e_2(G) = 2 \). Thus we may as well assume
Later we shall add to these assumptions the further one that $G$ is of exponent 12. Actually, until we restrict ourselves to groups of exponent 12, we will make no use of the fact that $G_i$ is irreducibly represented on $V$.

Now let $N$ be the largest normal 2'-subgroup of $G_i$. We shall show that a certain 2-subgroup, to be described later, must be contained in the greatest normal 2-subgroup of $G_i/N$. In particular if $l_2(G_i) > 2$ (which is the same as $l_2(G_i^s) > 1$), we shall see that there must exist an element of order 4 of a special type in $G_i$.

First let $H$ be the following normal subgroup of $G_i$: $x \in H$ if, and only if, for every normal nilpotent subgroup $Q$ of class at most 2 in $G_i$, $x$ fixes every minimal characteristic $F - Q$ submodule of $V$. A normal nilpotent subgroup of $G_i$ must be a 2'-group so that $V$ splits into the sum of minimal characteristic $F - Q$ modules.

From (5) there are elements of order 4 in $G_i$, and from (4) all such elements must be exceptional. Thus if $g$ is of order 4 in $G_i$ then $g^2$ must be in $H$ by lemma 3.1. Hence $H$ is greater than the identity. $H$ has no normal 2-subgroup except for the identity and the largest normal 2'-subgroup is $H \cap N$.

Let $D$ be the greatest normal nilpotent subgroup of $H$. $D = D_1 \times D_2 \times \cdots$ where $D_i$ is a Sylow $q_i$-subgroup of $D$ for an odd prime $q_i$. $H$ centralizes any normal abelian subgroup of $G_i$ so that, by the proof of Lemma 3.3, we obtain $c(D) = 2$. Now, as before, let $K_i$ be the subgroup of $D_i$ consisting of all elements of order at most $q_i$ and let $K = K_1 \times K_2 \times \cdots$. We again have that no non-identity 2-element of $H$ centralizes $K$.

Now take $H_i$ to be the subgroup of $G_i$ consisting of all elements which fix every minimal characteristic $F - K_i$ module for all $i$. $H_i \leq G_i$, and, since $c(K_i) \leq 2$, $H_i$ has no normal 2-subgroup except for the identity and its greatest normal 2'-subgroup is $H_i \cap N$.

Let $P$ be a Sylow 2-subgroup of $H_i$. $P \neq 1$ since if $g$ is any element of order 4 in $G_i$ then $g^2 \in H$. Now the square of any element of $P$ must be in $H$. Thus $P/(P \cap H)$ is of exponent 2 and thus abelian. Therefore $P' < H$. We now prove two lemmas which enable us to show directly that $PN/N$ is normal in $G_i/N$.

**Lemma 4.2.** Suppose that $g$ and $h$ are two elements of $P$ and $V'$ is a minimal characteristic $F - K_i$ submodule of $V$. Let $Q$, $g_i$, and $h_i$ be the restrictions of $K_i$, $g$, and $h$, respectively, to $V'$. Then if $(Q, h_i) = 1$ it follows that $(Q, (g_i, h_i)) = 1$.

*Proof.* Assume $(Q, (g_i, h_i)) \neq 1$. Therefore neither $g_i$ nor $h_i$ central-
izes $Q$. If $(Q, g_i^2) = 1$, then straightforward calculation yields

$$(Q, (g_i h_i)^2) = (Q, (g_i, h_i)) \neq 1,$$

$$(Q, (g_i h_i, h_i)) = (Q, (g_i h_i)^{-1}) \neq 1.$$  

Thus, replacing $g_i$ by $g_i h_i$ if $(Q, g_i^2) = 1$, we may assume that $(Q, g_i^2) \neq 1$ along with $(Q, h_i^2) = 1$ and $(Q, (g_i, h_i)) \neq 1$.

Now exactly as in the proof of Lemma 3.4 we obtain that $Q$ is an extra-special $q$-group (actually $q = 3$ since $g_i$ is of order 4 and thus exceptional so that $4 - 1$ must be a power of $q$), $Q/Q'$ is a symplectic space, $g_i$ and $h_i$ preserve the symplectic structure of $Q/Q'$, and we may assume that $g_i$ and $h_i$ operate on $Q/Q'$ as follows:

$$g_i(\Sigma \alpha_i u_i) = A \alpha_i u_i + \sum_{i \neq i} \alpha_i u_i,$$

$$h_i(\Sigma \alpha_i u_i) = \Sigma T_i \alpha_i u_{\sigma(i)},$$

where $\sigma$ is a permutation of order $\leq 2$ (since $(Q, h_i^2) = 1$), and $A$ and the $T_i$ are chosen from a group isomorphic to the quaternion group of order 8 (since $q = 3$). In addition $A$ must be of order 4 since $(Q, g_i^2) \neq 1$.

If $\sigma$ does not fix 1 then $(g_i, h_i)$ would be of order 4 but its centralizer in $Q/Q'$ would have co-dimension 4 over $GF(3)$. Thus $(g_i, h_i)$ would be of order 4 but not exceptional which is impossible.

Hence $\sigma$ fixes 1 and, since $(Q, h_i^2) = 1$, we must have

$$h_i(\Sigma \alpha_i u_i) = \pm \alpha_i n_i + \sum_{i \neq i} T_i \alpha_i u_{\sigma(i)}.$$  

It is now an easy matter to verify that $(g_i, h_i) = 1$ and the lemma is proved.

**Corollary.** If $g, h \in P$ and $h^2 = 1$, then $(g, h) = 1$.

**Proof.** $(g, h)$ is in $P'$ and thus in $H$. So if $(g, h) \neq 1$ then $(K_i, (g, h)) \neq 1$ for some $K_i$. Then lemma states that this cannot happen.

**Lemma 4.3.** If $g, h \in P$, then $(g, h)^2 = 1$.

**Proof.** Suppose that $(g, h)^2 \neq 1$. Then for some $K_i$, $(K_i, (g, h)^2) \neq 1$. Choose $V'$ to be a minimal characteristic $F - K_i$ submodule of $V$ such that $(K_i, (g, h)^2)$ is not the identity on $V'$. If $Q, g_i, h_i$ are defined as in the previous lemma, then, if either $(Q, g_i^2)$ or $(Q, h_i^2)$ is the identity, $(g_i, h_i) = 1$. Therefore assume neither $g_i^2$ nor $h_i^2$ centralize $Q$. Thus $g_i$ and $h_i$ are both exceptional of order 4. $Q$ is an extra-special
3-group and we may assume $g_i$ and $h_i$ operate on $Q/Q'$ as follows:

$$g_i(\sum \alpha_i u_i) = A\alpha_i u_i + \sum_{i \neq i} \alpha_i u_i,$$

$$h_i(\sum \alpha_i u_i) = B\alpha_i u_i + \sum_{j \neq j} \alpha_i u_i.$$ 

Now if $j \neq 1$ then $(g_i, h_i) = 1$ and if $j = 1$ then

$$(g_i, h_i)^{(\sum \alpha_i u_i)} = (A, B)^{(\sum \alpha_i u_i)}.$$ 

But $A$ and $B$ are elements of a quaternion group so that $(A, B)^2$ is the identity and the lemma is proved.

**Theorem 4.4.** $P \cap N \triangleleft G_i/N$.

**Proof.** We shall prove that $P(H_i \cap N)/(H_i \cap N) \triangleleft H_i/(H_i \cap N)$ which is equivalent to the theorem since $H_i \triangleleft G_i$.

Let $P_1$ be the subgroup of $P$ such that $P_i(H_i \cap N)/(H_i \cap N)$ is the largest normal 2-subgroup of $H_i/(H_i \cap N)$. $P \triangleleft P_1$ and $P_1$ contains the center of $P$ [4, Lemma 1.2.3]. Thus by the corollary to Lemma 4.2, $P_1$ contains all elements of order 2 in $P$. The elements of order 2 in $P$ form an elementary abelian group $P_2$ which is normal, modulo $H_i \cap N$, in $H_i$. The elements of $H_i/(H_i \cap N)$ which centralize both $P_1$ and $P/P_2$ form a normal subgroup of $H_i/(H_i \cap N)$. But if any 2'-element centralized both $P_2$ and $P_i/P_2$, then, as easily may be seen, this element would centralize $P_1$ contrary to the fact [4, Lemma 1.2.3] that $P_1$ contains its centralizer in $H_i/(H_i \cap N)$. Thus the elements centralizing both $P_2$ and $P_i/P_2$ form a normal 2-subgroup of $H_i/(H_i \cap N)$, and from the corollary to Lemma 4.2 and from Lemma 4.3, $P$ must be contained in this normal 2-subgroup. But $P$ is a Sylow 2-subgroup of $H_i$ and thus it follows that, modulo $H_i \cap N$, $P$ is normal in $H_i$.

**Corollary.** $l_p(H_i) = 1$.

Now let $S$ be a Sylow 2-subgroup of $G_i$ which contains $P$. From the theorem it follows that $P$ is normal in $S$.

**Lemma 4.5.** If $P$ contains all elements of order 4 in $S$, then $l_p(G_i) = 1$.

**Proof.** If $S = P$ we are done. Therefore assume $S \neq P$. Then if $x \in S - P$ we must have $x^2 = 1$. Also $x \in S - P$, $y \in P$ imply that $xy \in S - P$ so that $(xy)^2 = 1$ which implies that $x^{-1}yx = y^{-1}$. Thus $x$ induces the automorphism $y \rightarrow y^{-1}$ of $P$. This can be an automorphism only if $P$ is abelian. Now if both $x_1$ and $x_2$ are in $S - P$ then $x_1x_2$. 


centralizes $P$. But $e_2(G_j) = 2$ so that $P$ does contain elements of order 4. Hence $x, x_i$ cannot be in $S - P$.

Therefore $|S/P| = 2$ and $P$ is abelian. Now if $x \in S - P$, $y \in P$, then $(x, y) = x^{-i}y^{-i}xy = y^2 \in \Phi(P)$ and thus $x$ centralizes $P/\Phi(P)$. Hence [4, Lemma 1.2.5] $PN/N$ cannot be the largest normal 2-subgroup of $G_i/N$. But $P$ is maximal in $S$ so that $SN/N$ must be the largest normal 2-subgroup of $G_i/N$. This implies that $l_2(G_i) = 1$.

To our assumptions (1)~(5) we now add

(6) $G$ is of exponent 12.

This implies that $K$ must be a group of exponent 3 and class at most 2. We prove that $l_2(G_i) = 1$ in this case by showing that the hypothesis of Lemma 4.5 are satisfied.

For this purpose assume that $g$ is an element of order 4 in $S - P$. $g^2$ is in $H$ so $(K, g^2) \neq 1$. Let $V = V_1 \oplus V_2 \oplus \cdots$ be the decomposition of $V$ into minimal characteristic $F - K$ modules. Since $g \in S - P$, $g$ does not fix some $V_i$. $g^2$ does fix each $V_i$ and if $g^2$ is not the identity on a $V_i$ then $g$ must fix that $V_i$ for otherwise $g$ could not be exceptional [4, p. 13]. We now need the following result:

**Lemma 4.6.** There exist $x$ and $y$ in $K$ such that $((x, g^2), (y, g^2)) \neq 1$.

**Proof.** Let $C = \{x \mid x \in K, (x, g^2) \in Z(K)\}$. Clearly $C \supseteq Z(K)$ but $C \neq K$ since then $g^2$ would centralize $Z(K)$ and $K/Z(K)$ which would imply that $(K, g^2) = 1$. ($g^2$ centralizes $Z(K)$ by Lemma 3.1 and 3.2.) $K/Z(K)$ is an elementary abelian 3-group so that there must be a $GF(3) - g$ module of $K/Z(K)$ complementary to $C/Z(K)$. Thus $K/Z(K) = L/Z(K) \oplus C/Z(K)$ and $g$ normalizes $L$. For all $x \in L - Z(K)$, $(x, g^2)$ is not in $Z(K)$.

Now suppose $x, y \in L - Z(K)$ and $(x, g^2)(y, g^2)^{-1} \in Z(K)$. Since $K/Z(K)$ is abelian, straight forward calculation yields

$$(xy^{-1}, g^2) \equiv (x, g^2)(y^{-1}, g^2) \quad (\text{mod } Z(K)),
1 = (yy^{-1}, g^2) \equiv (y, g^2)(y^{-1}, g^2) \quad (\text{mod } Z(K)).$$

Thus $(xy^{-1}, g^2) \equiv (x, g^2)(y, g^2)^{-1} \equiv 1$ (mod $Z(K)$). This implies that $xy^{-1} \in Z(K)$. Therefore we have shown that $(x, g^2) \equiv (y, g^2)(\text{mod } Z(K))$ if, and only if, $x \equiv y$ (mod $Z(K)$) for $x, y \in L$.

It immediately follows from this that for any $x \in L$, there exists a $y$ such that $x \equiv (y, g^2)(\text{mod } Z(K))$. Now $L$ cannot be abelian since $g$ normalizes $L$ and $g^2$ does not centralize it. From all this we see that there exist $x, y \in L$ such that $((x, g^2), (y, g^2)) \neq 1$.

Now taking $x$ and $y$ to satisfy the lemma, we may assume without
loss of generality that \(((x, g^2), (y, g^2))\) is not the identity on \(V_1\). This implies that \(g^2\) is not the identity on \(V_1\) so \(g\) must fix \(V_1\).

Since \(g\) does not fix every \(V_\iota\), assume \(g\) does not fix \(V_2\). Therefore \(g^2\) is the identity on \(V_2\) which then also must be the case for \((x, g^2)\) and \((y, g^2)\).

\(V\) is an irreducible \(F - G\) module so that there must be an element taking \(V_1\) into \(V_2\). Such an element must be of the form \(zh\) where \(h \in S\) and \(z\) is from a Sylow 3-subgroup of \(G\) which necessarily must contain \(K\). We shall derive a contradiction by showing that \(z\) and \(K\) generate elements of order 9 which is impossible in a group of exponent 12.

If \(hV = V_m\) then \(zV = V_2\). Set \(g_1 = hgh^{-1}\). Then

\[ ((x^{h^{-1}}, g^2), (y^{h^{-1}}, g^2)) \]

is not the identity on \(V_m\). Now suppose \(g_1V_2 = V_2\). Then \(gh^{-1}V_2 = h^{-1}V_2\), and, since \(gV_2 \neq V_2\), this implies that \(h^{-1}V_2 = V_j, j \neq 2\). Then we would have \(gV_j = V_j\). But \(gh^{-1} \in S\) so that \((gh^{-1})^2 \in H\). Thus \((gh^{-1})^2\) fixes \(V_3\) and, therefore, \(gh^{-1}V_j = V_2\). \((h^{-1})^3\) also must fix \(V_3\) so we have \(h^{-1}V_j = V_2\). From this we conclude that \(V_2 = gh^{-1}V_j = gV_2\) which is a contradiction. Hence \(g_1V_2 \neq V_2\). A consequence of this is that \(V_m \neq V_2\) for \(V_m = V_2\) would imply that \(h^{-1}V_j = V_1\) which would imply that \(g_1V_2 = hgV_1 = V_2\). Since \(V_m \neq V_2\) it follows that \(z\) is not the identity and so is of order 3.

If we replace \(V_1, g, x,\) and \(y\) by \(V_m, g_1, x^{h^{-1}},\) and \(y^{h^{-1}}\), respectively, we may assume that \(zV_1 = V_3, gV_2 \neq V_2,\) and \(((x, g^i), (y, g^i))\) is not the identity on \(V_1\). Let \(x_i = (x, g^i)\) and \(y_i = (y, g^i)\). \(x_i\) and \(y_i\) must be the identity on \(V_3\) since \(g_i\) is. Since \(z\) is of order 3, we have \(zV_i = V_3, zV_2 = V_4, zV_m = V_1\).

Let \(V' = V_1 \oplus V_2 \oplus V_3\). \(V'\) is fixed by \(z\) and the restrictions of \(x_i, y_i,\) and \(z\) to \(V'\) are

\[
z = \begin{pmatrix}
0 & 0 & A \\
B & 0 & 0 \\
0 & C & 0
\end{pmatrix},
\]

\[
x_i = \begin{pmatrix}
M & 0 & 0 \\
0 & I & 0 \\
0 & 0 & M_i
\end{pmatrix},
\]

\[
y_i = \begin{pmatrix}
N & 0 & 0 \\
0 & I & 0 \\
0 & 0 & N_i
\end{pmatrix},
\]

where \(I\) is the identity and 0 the zero matrix. Now \((x_i, y_i)\) is not the identity on \(V_3\) but \((x_i, y_i) \in Z(K)\) and \(Z(K)\) is represented on \(V_3\) as a cyclic group generated by a scalar matrix. Thus \((M, N) = \omega I\) where \(\omega\) is a primitive third root of unity. From \(z^3 = 1\) we obtain \(C = A^{-1}B^{-1}\).

Now \(z, x_i,\) and \(y_i\) all belong to the same Sylow 3-subgroup of \(G\). Thus \((xz_i)^3 = (yz_i)^3 = 1\). From this direct computation yields that \(M_1 = A^{-1}M^{-1}A, N_1 = A^{-1}N^{-1}A\). Thus \((M_1, N_1) = A^{-1}(M^{-1}, N^{-1})A\). But \(M\) and \(N\) generate a group of exponent 3 and class 2. It follows easily that \((M^{-1}, N^{-1}) = (M, N) = \omega I\). Thus
It is now a simple matter to verify that \((x, y) = (\omega I, 0, 0)\).

\[
(x, y) = \begin{pmatrix}
\omega I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & \omega I
\end{pmatrix}.
\]

It is now a simple matter to verify that \((z(x, y))^3 \neq 1\). Hence \(z(x, y)\) is a 3-element of order greater than 3 which is impossible in a group of exponent 12. This contradiction proves that the hypothesis of Lemma 4.5 is satisfied and thus:

**Theorem 4.7.** If \(G\) is a finite group of exponent 12, then \(L_2(G) \leq e_3(G)\).

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