THE 2-LENGTH OF A FINITE SOLVABLE GROUP

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One measure of the structure of a finite solvable group $G$ is its $p$-length $l_p(G)$. A problem connected with this measure is to obtain an upper bound for $l_p(G)$ in terms of $e_p(G)$, which is a numerical invariant of the Sylow $p$-subgroups of $G$. This problem has been solved but the best-possible result is not known for $p = 2$. The main result of this paper is that $l_2(G) \leq 2e_2(G) - 1$, which is an improvement on earlier results. A secondary objective of this paper is to investigate finite solvable groups in which the Sylow 2-group is of exponent 4. In particular it is proved that if $G$ is a finite group of exponent 12, then the 2-length is at most 2.

Introduction and discussion of results. The object of this paper is to obtain bounds for the 2-length of a finite solvable group. Following Hall and Higman [4], we call a finite group $G$ $p$-solvable if it possesses a normal series such that each factor group is either a $p$-group or a $p'$-group. The $p$-length, $l_p(G)$, of such a group is the smallest number of $p$-groups which can occur as factor groups in such a normal series. $e_p(G)$ is defined to be the smallest $n$ such that $x^{pn} = 1$ for all $x$ belonging to a Sylow $p$-subgroup of $G$.

For an odd prime $p$, it is proved in [4] that $l_p(G) \leq e_p(G)$ if $p$ is not a Fermat prime and $l_p(G) \leq 2e_p(G)$ if $p$ is a Fermat prime. Furthermore these results are best-possible. A. H. M. Hoare [6] then proved that in a 2-solvable group $G$, $l_2(G) \leq 3e_2(G) - 2$ provided that $l_2(G) \geq 1$. The primary purpose of this paper is to prove the following improvement:

**THEOREM A.** If $G$ is a finite solvable group and $l_2(G) \geq 1$, then $l_2(G) \leq 2e_2(G) - 1$.

Feit and Thompson [1] have proved that solvability and 2-solvability are equivalent notions for finite groups. Thus no loss of generality is involved in requiring $G$ to be solvable in the theorem.

Theorem A will be shown to be an easy consequence of the following theorem about linear groups:

**THEOREM B.** Let $G$ be a finite solvable linear group over a field...
F of characteristic 2 and assume G has no nontrivial normal 2-
subgroup. Then if N is the largest normal 2'-subgroup of G and
if g is an exceptional element of order $2^m$ in G, it follows that $g^{2^m-1}$
is in the largest normal 2-subgroup of $G/N$.

Here, following [4], an element $x$ of order $p^n$ in a linear group over
a field of characteristic $p$ is said to be exceptional if $(x - 1)^{p^n-1} = 0$.

Whether or not Theorem A represents a best-possible result is not
known, but it seems likely that further improvements can be made. Indeed, the author knows of no group whose 2-length exceeds its
2-exponent. In the special case of finite solvable groups satisfying
$e_2(G) = 2$, i.e., solvable groups whose Sylow 2-subgroups are of exponent
4, I think it likely that $l_2(G) \leq 2$ instead of the bound $l_2(G) \leq 3$
furnished by Theorem A.

In § 4 of this paper, groups satisfying $e_2(G) = 2$ are studied in
more detail. A sufficient condition for $l_2(G) \leq 2$ in this special case is
established, and, as an application, we prove that $l_2(G) \leq e_2(G)$ if $G$
is a finite group of exponent 12.

2. Proof of Theorem A from Theorem B. For the rest of
this paper we adopt the convention that all groups referred to are
assumed finite, and, if $G$ is such a group, then $|G|$ denotes its order.
If $H$ is a normal subgroup of $G$, we write $H \triangleleft G$.

We now recall the definition of the upper 2-series of the solvable
group $G$:

$$1 = P_0 \leq N_0 < P_1 < N_1 < \cdots < P_i \leq N_i = G.$$ 

Here $N_k/P_k$ is defined to be the greatest normal 2'-subgroup of $G/P_k$
and $P_{k+1}/N_k$ the greatest normal 2-subgroup of $G/N_k$. The least inte-
erg $l$ such that $N_l = G$ is the 2-length $l_2(G)$. (If there is no danger
of confusion we write simply $l_2$.)

It is proved in [4] that the automorphisms of $P_i/F$, where $F/N_0$ is
the Frattini subgroup of $P_i/N_0$, induced by $G$ represent $G/P_i$ faithfully.
Thus $G/P_i$ is faithfully represented as a linear group operating on $P_i/F$
($P_i/F$ is an elementary abelian 2-group and so is considered as a vector
space over the field with 2 elements).

Now if $l_2(G) = 1$, the conclusion of A is trivial. Also the $p$-length
group is at most equal to the class of a Sylow $p$-subgroup [4, Theorem
1.2.6]. An immediate consequence of this is that if $G$ is solvable and
$e_2(G) = 1$, then $l_2(G) = 1$. Thus $l_2 = 2$ implies that $e_2 \geq 2$ so the result
again follows. Now if $l_2 > 2$, then $l_2(G/P_2) = l_2(G) - 2 \geq 1$ so that
Theorem A would follow by induction on $l_2$ if we could prove that
Now suppose $g$ is an element of maximal order $2^m$ in a Sylow $2'$-Sylow subgroup of $G/P_i$. If $g$ is not exceptional, then [4, Lemma 3.1.2] we have $e_i(G) \geq m + 1$. If $g$ is exceptional, then, since $G/P_i$ satisfies the hypothesis of Theorem B, $g^{i(m-1)}$ is in $P_i/P_i$ if Theorem B is true. Thus, assuming the validity of B, we obtain in all cases $e_i(G/P_i) \leq e_i(G) - 1$ and Theorem A follows.

3. Proof of Theorem B. Neither the hypothesis nor the conclusion of the theorem is affected by an extension of the field $F$. Hence, without loss of generality, we assume that $F$ is algebraically closed. Since an element of order 2 cannot be exceptional, $m$ must be greater than 1. Let $h = g^{2^{m-2}}$ and so $h^2 = g^{2^{m-1}}$.

In proving B we will define subgroups $H$ and $H_1$ such that $H < G$, $H_1 < H$, $h^2 \in H_1$, and $g$ normalizes $H_1$. It then will be shown that if $x$ is any element in the largest normal $2'$-subgroup of $H_1/H_1 \cap N$ then $(h^2, x) = (h, x)^2$. From this it will follow that $h^2$ is in the largest normal $2'$-subgroup of $H_1/H_1 \cap N$, and, finally, from this the theorem will follow.

First we need two lemmas which are of use later and which motivate the definition of $H$. Here, and elsewhere, we denote the space on which $G$ operates by $V$.

**Lemma 3.1.** If $Q$ is any $2'$-subgroup of $G$ which is normalized by $g$, then $h^2$ fixes every minimal characteristic $F - Q$ submodule of $V$.

**Proof.** A minimal characteristic $F - Q$ submodule is simply the join of all those $F - Q$ submodules operator isomorphic to a given irreducible $F - Q$ submodule. Now if $Q$ is a $2'$-group, $V$ can be written as the direct sum of the minimal characteristic $F - Q$ submodules. $g$ normalizes $Q$ so $g$ must permute the minimal characteristic $F - Q$ submodules. If the lemma were not true, then $g$, as a permutation of these submodules, would have a cycle of length $2^m$ which would contradict the assumption that $g$ is exceptional.

**Lemma 3.2.** If $Q$ is any abelian $2'$-subgroup of $G$ and $x$ is any element of $G$ normalizing $Q$ and fixing every minimal characteristic $F - Q$ submodule of $V$, then $x$ centralizes $Q$.

**Proof.** Let $V_i$ be any minimal characteristic $F - Q$ submodule of $V$. Since $Q$ is abelian and $F$ is algebraically closed, $Q$ operates on $V_i$ as a scalar multiplication, i.e., if $y \in Q$ and $v \in V_i$ then $yv = \chi_i(y)v$ where $\chi_i(y)$ is a scalar. We now obtain
Thus \((y, x)\) is the identity on \(V_i\) for all \(y \in Q\) and the lemma follows.

Now let \(H\) be the normal subgroup of \(G\) consisting of all elements which fix every minimal characteristic \(F - Q\) submodule for every normal \(2'\)-subgroup \(Q\). Since the largest normal \(2\)-subgroup and the largest normal \(2'\)-subgroup of \(H\) are normal in \(G\), we see that \(H\) has no normal \(2\)-subgroup greater than the identity and the largest normal \(2'\)-subgroup of \(H\) is \(H \cap N\). By Lemma 3.1 \(h^2\) must belong to \(H\).

Let \(M\) be the largest normal nilpotent subgroup of \(H\). Clearly \(M\) is a \(2'\)-group and \(M \trianglelefteq G\). Furthermore, since \(H\) is solvable, \(M\) contains its own centralizer in \(H\) [2].

**Lemma 3.3.** \(M\) is of class 2.

*Proof.* Since \(h^2 \in H\), \(h^2\) does not centralize \(M\). Thus by Lemmas 3.1 and 3.2, \(M\) is not abelian. Now let \(c\) be the class of \(M\) and suppose \(c \geq 3\). Then if \(\Gamma_i(M)\) is the \(i\)th term in the lower central series of \(M\) \((\Gamma_0(M) = M\) and \(\Gamma_{i+1}(M) = (\Gamma_i(M), M)\) and if \(d\) is the first integer \(\geq (c+1)/2\), we have [3, Chap. 10]

\[
\Gamma_d(M), M = \Gamma_{d+1}(M) \neq 1 \quad \text{since } d \leq c - 1,
\]

and

\[
\Gamma_d(M), \Gamma_d(M) \trianglelefteq \Gamma_{\varphi d}(M) = 1.
\]

Thus \(\Gamma_d(M)\) is abelian and, of course, normal in \(G\) but is not centralized by \(M\). From Lemma 3.2 and the definition of \(H\) we see that this is impossible, and so \(c = 2\).

\(M = M_1 \times M_2 \times \cdots\) where \(M_i\) is the Sylow \(q_i\)-subgroup of \(M\) and \(q_i\) is an odd prime. Each \(M_i\) is of class at most 2 and so \(M_i\) is a regular \(q_i\)-group [3, p. 183]. Then the elements of order at most \(q_i\) form a characteristic subgroup \(K_i\) of \(M_i\). Let \(K = K_1 \times K_2 \times \cdots\) An automorphism of \(M_i\) of order prime to \(q_i\) centralizes \(K_i\) only if it is the identity automorphism [7, Hilfssatz 1.5]. Therefore no 2-element of \(H\), except for the identity, centralizes \(K\). Hence \(K\) cannot be abelian (since \(h^2\) is a nonidentity 2-element of \(H\)) and so \(K\) must be of class 2.

We now are prepared to define the subgroup \(H_i\). For this purpose decompose \(V\) for each \(K_i\) into the sum

\[V = V_{i1} \oplus V_{i2} \oplus \cdots\]

where the \(V_{ij}\) are the minimal characteristic \(F - K_i\) submodules. Let
\[ C_{ij} = \{ x \mid x \in H \text{ and } (K_i, x) = 1 \text{ on } V_{ij} \}. \] \( C_{ij} \) is a normal subgroup of \( H \) although not necessarily normal in \( G \).

Take \( H_i \) to be the intersection of all the \( C_{ij} \) which contain \( h^2 \). If \( h^2 \) is not in any \( C_{ij} \) then set \( H_i \) equal to \( H \). In any event \( H_i \triangleleft H \) and \( H_i \) is normalized by \( g \). As was the case with \( H, H_i \) has no normal 2-subgroup greater than the identity and the greatest normal 2'-subgroup is \( H_i \cap N \).

Now let \( P \) be a 2-subgroup of \( H_i \) such that \( P \) and \( g \) belong to the same Sylow 2-subgroup of \( G \) and \( P(H_i \cap N)/(H_i \cap N) \) is the largest normal 2-subgroup of \( H_i/(H_i \cap N) \). Since, modulo \( N, P \) is normalized by \( g \), it follows that \( g \) normalizes \( P \).

**Lemma 3.4.** If \( x \in P \), then \((h^2, x) = (h, x)^2\).

**Proof.** First we show that this lemma finishes the proof of Theorem B: \( h \) normalizes \( P \) so that \((h, x)^2 \in \Phi(P) \) where \( \Phi(P) \) is the Frattini subgroup of \( P \). Thus the lemma implies that \( h^2 \) centralizes \( P/\Phi(P) \). Therefore from [4] we conclude that \( h^2 \in P \). Since \( h^2 \) is in the greatest normal 2-subgroup of \( H_i/(H_i \cap N) \), it follows that \( h^2 \) is in the greatest normal 2-subgroup of \( H/(H \cap N) \) from which the conclusion of Theorem B follows.

To prove the lemma, let \( k = (h^2, x)(h, x)^{-2} \) and suppose \( k \neq 1 \). Since \( k \) cannot centralize \( K_i, (K_i, k) \) is not the identity on some \( V_{ij} \). Since \( k \in H_i \), we must have \((K_i, h^2) \) also not the identity on \( V_{ij} \). (This last statement is the motivation for our choice of \( H_i \)).

In what follows let \( V' = V_{ij}, q = q_i, \) and \( Q, x_i, k_i \) the restrictions of \( K_i, x, k \), respectively, to \( V' \). Let \( g^{2m-n} \) be the first power of \( g \) fixing \( V' \) and let \( g_i \) be the restriction of \( g^{2m-n} \) to \( V' \). Now \( h^2 \) is not the identity on \( V' \) and [4, p. 13] \( g_i \) must be exceptional

\[ (i.e., \ (g_i - 1)^{2^n-1} = 0), \]

and thus \( n \) must be at least 2. Let \( h_i = g_i^{2^{m-2}}, k_i = (h_i, x_i)(h_i, x_i)^{-2} \) and both \((Q, h_i^2)\) and \((Q, k_i)\) are not the identity.

Since \( g_i \) is exceptional and \((Q, h_i^2) \neq 1, Q \) cannot be abelian. Thus \( Q \) must be of class 2. \( V' \) is the sum of absolutely irreducible \( F - Q \) submodules all of which are operator isomorphic to each other. Hence \( Z(Q), \) the center of \( Q \), is cyclic and is generated by a scalar matrix. Since \( Q \) is of exponent \( q \) and \( Q' \neq 1 \), we see that

\[ Z(Q) = Q' = \Phi(Q) \]

and so \( Q \) is an extra-special \( q \)-group [4, p. 15]. We note also that if \( S \) is the 2-group generated by \( x_i \) and \( g_i \), then \((Z(Q), S) = 1 \) since \( Z(Q) \) is generated by a scalar matrix.
Now let $V''$ be an irreducible $F - QS$ submodule of $V'$. $V''$ is an irreducible $F - Q$ module [4, Lemma 2.2.3], and $V'$ is the sum of $F - Q$ modules operator isomorphic to $V''$. Thus $(Q, h_i^2) \neq 1$ on $V''$ and $g_i$ is exceptional on $V''$. From [4, Theorem 2.5.4] we have the following:

1. $2^s - 1$ is a power of $q$, and
2. if $g_i$ is faithfully and irreducibly represented on $Q_i/Q'$ (such a $Q_i$ can always be found since $h^2$ is not the identity on $Q/Q'$), then $Q$ can be written as the central product of $Q_i$ and a group $Q_2$ and $g_i$ transforms $Q_2$ trivially. It now follows [6] that $2^n - 1 = q$ and $|Q_i/Q'| = q^a$.

The representation of $Q$ on $V''$ is isomorphic to the representation of $Q$ on $V'$ so that $(g_i, Q_2) = 1$ on $V''$ implies that $(g_i, Q_2) = 1$. Thus the centralizer of $g_i$ in the space $Q/Q'$ has co-dimension 2 over $GF(q)$. The minimal equation of $h_i$ on $Q_i/Q'$ must be $t^2 + 1 = 0$ so that $h_i^2$ must have the representation

$$
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
$$

on $Q_i/Q'$. We now can conclude that for every power of $g_i$ (except for the identity, of course), the co-dimension of its centralizer in $Q/Q'$ is 2. Also, since $q = 3 \mod 4$, $GF(q)$ contains no primitive 4th root of unity. Thus if $n \neq 2$ then in the completely reduced representation of $g_i^2$ on $Q/Q'$ there is only one nontrivial block. If $n = 2$, there are two nontrivial blocks.

Now if $c$ is a generator of $Q'$, define $\rho(a, b)$ for $a, b \in Q$ by the equation

$$
(a, b) = e^{\rho(a, b)}.
$$

$\rho(a, b)$ is bilinear and skew symmetric and gives $Q/Q'$ the structure of a symplectic space over $GF(q)$ [4].

$\rho$ is of maximum rank since $Q' = Z(Q)$ so $Q/Q'$ must have dimension $2r$. Since $(S, Q') = 1$, $S$ preserves the symplectic structure of $Q/Q'$. Thus the representation of $S$ on $Q/Q'$ may be considered as a subgroup of a Sylow 2-subgroup of the symplectic group on $Q/Q'$.

$Q/Q'$ is of dimension $2r$ over $GF(q)$ so that $Q/Q'$ can be provided with the structure of a vector space $U$ of dimension $r$ over $GF(q^2)$. If $u_1, \ldots, u_r$ is a basis for $U$, the expression [4]

$$
\rho(\Sigma \alpha_i u_i, \Sigma \beta_i u_i) = \Sigma (\alpha_i \beta_i' - \alpha_i' \beta_i)/\gamma ,
$$

where $\gamma^r = \alpha'$ and $\gamma$ is a primitive 4th root of unity, is a skew symmetric bilinear form on $U$ of rank $2r$ with values in $GF(q)$.

Let $\theta$ be a primitive $2^{n+1}$-th root of unity in $GF(q^2)$ and let $T$ be
the group of transformations of \( GF(q^2) \) generated by the two transformations \( \alpha \rightarrow \theta^2 \alpha \) and \( \alpha \rightarrow \theta \alpha' \). All transformations \( y \) of \( U \) of the form

\[
y(\Sigma \alpha_i u_i) = \Sigma (T_i \alpha_i) u_{\sigma_1(i)} ,
\]

where the \( T_i \) are taken from \( T \) and \( \sigma \) is a permutation taken from a Sylow 2-subgroup of the symmetric group on the numbers \( 1, 2, \ldots, r \), form a Sylow 2-subgroup of the Symplectic group on \( Q/\overline{Q}' \) [4].

Thus we may assume that \( x_1, g_1, h_1 \), the representations of \( x_1, g_1, h_1 \), respectively, on \( Q/\overline{Q}' \), are of this form. Since \((Q, h_1) \neq 1\) and \((Q, k_1) \neq 1\), we have \( h_1^2 \neq 1 \) and \((h_1^2, x_1) \neq (h_1, x_1)^2\). We now need more information on \( g_1 \).

**Lemma 3.5.** The permutation \( \sigma \) associated with \( g_1 \) is the identity permutation.

**Proof.** \( \sigma \) is of order less than the order of \( g_1 \) from [4, p. 23]. First suppose \( \sigma \) is of order \( > 2 \). Then \( n > 2 \) and so the representation of \( g_1^2 \) on \( Q/\overline{Q}' \) has only one nontrivial irreducible block. But the permutation associated with \( g_1^2 \) is \( \sigma^2 \) which has at least 2 disjoint nontrivial cycles. Clearly this is a contradiction. Thus \( \sigma^2 = 1 \).

Now suppose \( \sigma \neq 1 \). Assume, say, \( \sigma(1) = 2, \sigma(2) = 1 \). The representation of \( g_1 \) on \( Q/\overline{Q}' \) has only one nontrivial irreducible block so \( g_1 \) must be the identity on

\[
\sum_{i \neq 1, 2} \alpha_i u_i .
\]

Now \( g_1^2(\alpha_i u_1 + \alpha_2 u_2) = T_2 T_1 \alpha_i u_1 + T_1 T_2 \alpha_2 u_2 \) and so one of \( T_2 T_1 \) or \( T_1 T_2 \) must not be the identity of \( T \). But then neither one can be the identity. Therefore the representation of \( Q/\overline{Q}' \) would have 2 nontrivial irreducible blocks. This can happen only if \( n = 2 \). This implies that \( T_2 T_1 \) and \( T_1 T_2 \) are of order 2 and thus must equal the transformation \( \alpha \rightarrow -\alpha \). (This is the only element of order 2 in \( T \).) Thus the centralizer of \( g_1^2 \) in \( Q/\overline{Q}' \) has co-dimension 4 over \( GF(q) \) whereas it should have co-dimension 2. This proves that \( \sigma = 1 \).

Hence \( g_1 \) fixes each \( u_i \) and must act trivially on \( \alpha_i \) for all but one value of \( i \), \( i = 1 \), say. Therefore

\[
g_1(\Sigma \alpha_i u_i) = A \alpha_i u_1 + \sum_{i \neq 1} \alpha_i u_i
\]

where \( A \) is an element of order \( 2^n \) in \( T \). Then

\[
h_1(\Sigma \alpha_i u_i) = A^{2n-2} \alpha_i u_1 + \sum_{i \neq 1} \alpha_i u_i ,
\]

and
\[ h_i(\Sigma \alpha_i u_i) = -\alpha_i u_i + \sum_{i \neq 1} \alpha_i u_i. \]

We may assume that
\[ x_i(\Sigma \alpha_i u_i) = \Sigma T_i \alpha_i u_{\pi(i)}. \]

**Case 1.** \( \pi(1) \neq 1. \) Assume, say, that \( \pi^{-1}(1) = 2. \) Straightforward calculation yields
\[ (h_i, x_i)(\Sigma \alpha_i u_i) = A^{-2^{n-2}} \alpha_i u_i + T_2^{-1} A^{2^{n-2}} T_2 \alpha_i u_2 + \sum_{i \neq 1,2} \alpha_i u_i. \]

But \( A^{2^{n-1}} \) is the unique element of order 2 in \( T. \) Thus
\[ (h_i, x_i)^2(\Sigma \alpha_i u_i) = -\alpha_i u_i - \alpha_i u_2 + \sum_{i \neq 1,2} \alpha_i u_i \]
and it is easily verified that this is the same result as \( (h_i^2, x_i). \)

**Case 2:** \( \pi(1) = 1. \) In this case we easily find that \( (h_i^2, x_i) \) is the identity while
\[ (h_i, x_i)^2(\Sigma \alpha_i u_i) = (A^{2^{n-2}}, T_1)^2 \alpha_i u_i + \sum_{i \neq 1} \alpha_i u_i. \]

Now the group \( T \) easily is seen to be a generalized quaternion group of order \( 2^{n+1} \) so that the only conjugates of \( A \) in \( T \) are \( A \) and \( A^{-1} \). Thus
\[ (A^{2^{n-1}}, T_1)^2 = A^{2^{n-1}} T_1^{-1} (A^{2^{n-1}}) T_1 = 1. \]

Thus \( (h_i, x_i) \) is also the identity.

Therefore it has been shown that
\[ (h_i, x_i)^2 = (h_i^2, x_i) \]
in all cases. This completes the proof of lemma 3.4, and, by a previous argument, Theorem B now is proved.

4. **Groups with** \( e_2 = 2. \) If \( G \) is a solvable group whose Sylow 2-groups are of exponent 4, then we know from Theorem A that \( l_2(G) \leq 3. \) We now investigate conditions for \( l_2(G) \leq 2 \) to hold. The argument is similar to that used in proving Theorem B, but a more restrictive hypothesis is needed. That no loss of generality is involved in assuming the stronger hypothesis is insured by the following reduction theorem, which is stated in a slightly more general form than needed.

A proposition \( R \) will be said to be of type 4.1 if it is of the following form:

If \( G \) is a finite \( p \)-solvable group satisfying condition \( C, \) then
the 2-length of a finite solvable group $l_p(G) \leq f(e_p(G))$, where $f$ is a monotonically increasing function defined for nonnegative integral arguments, $f(0) = 0$, and condition C either is vacuous or states that $e_p(G) \leq a_i$ for some set, possibly infinite, of primes $p_i$ and nonnegative integers $a_i$.

Note that the proposition that $l_p(G) \leq e_p(G)$ if $G$ is a finite solvable group satisfying $e_p(G) \leq 2$ is of type 4.1. One of the results of this section is that $l_p(G) \leq e_p(G)$ if $G$ is a finite group of exponent 12. This statement is also of type 4.1 since the condition that $G$ be of exponent 12 is equivalent to stating that $e_p(G) \leq 2$, $e_{p'}(G) \leq 1$, and $e_p(G) \leq 0$ for all other primes.

**Theorem 4.1.** To prove a proposition $R$ of type 4.1 it is sufficient to prove the proposition for the following special case:

1. $G$ is the normal product of $V$ by $G_1$, where $V$ is a vector space over $F$, a finite field of characteristic $p$, and $G_1$ is a $p$-solvable linear group on $V$ having no normal $p$-subgroup other than the identity.

2. Any irreducible representation of any $p'$-subgroup of $G_1$ over $F$ is in fact absolutely irreducible.

3. All groups of order at most $|G_1|$ satisfy $R$.

4. $V$ is an irreducible $F - G_1$ module.

**Proof.** In proving this theorem we assume $R$ is valid for the special case and then prove it is valid for the general case.

Now suppose $G$ is the group of smallest order which satisfies the hypothesis but not the conclusion of $R$, and let

$$1 = P_0 \leq N_0 < P_1 < \cdots < P_i \leq N_i = G$$

be the upper $p$-series of $G$. Since $f(0) = 0$ we must have $l_p(G) > 0$. If $F_i/N_0$ is the Frattini subgroup of $P_i/N_0$, then, as is shown in [4], $l_p(G/F_i) = l_p(G)$ so that if $F_i \neq 1$ we would have a proper factor group of $G$ satisfying the hypothesis but not the conclusion of $R$.

Hence assume $F_i = 1$. Thus $P_1$ is an elementary abelian $p$-group which we identify with a vector space $V_1$ over $GF(p)$. $G/P_1$ is faithfully represented as a linear group $G_1$ on $V_1$ and $G_1$ has no normal $p$-group greater than the identity.

From [4, p. 4] we may assume that $G$ has only one minimal normal subgroup. This subgroup must be contained in $V_1$ and we denote it with $M$. If $M \neq V_1$ and $G_1$ is faithfully represented on $V_1/M$ then we have $l_p(G/M) = l_p(G)$ so that we would have a contradiction to the minimality of $G$.

Now suppose $M \neq V_1$ and $G_1$ is not faithfully represented on $V_1/M$. 


Then the elements of \( G_1 \) centralizing \( V/M \) form a normal subgroup of \( G_1 \) greater than the identity. If \( Q \) is a minimal normal subgroup of \( G_1 \) centralizing \( V/M \), then \( Q \) must be a \( p' \)-group so that \( V \) as a \( Q \)-module is completely reducible. Thus there exists a \( Q \)-module \( M_1 \) such that \( V_1 = M \oplus M_1 \). \( Q \) is the identity on \( M \) but not on \( M_1 \) since \( Q \) is faithfully represented on \( V_1 \). Now if \( M_2 \) is the centralizer of \( Q \) in \( V_1 \) then \( M_2 \) is normal in \( G \), \( M_0 \) is not the identity, and \( M_0 \) does not contain \( M \). This contradicts the minimality of \( M \).

Thus we see that \( M = V_1 \) which implies that \( G_1 \) is irreducibly represented on \( V_1 \). A consequence of this is that if \( H \) is any normal subgroup greater than the identity in \( G \) then \( H \) can have no nonzero fixed vector in \( V_1 \). Otherwise all the vectors fixed by \( H \) would form a nontrivial submodule of \( V_1 \).

Now pick \( F \) to be a large enough finite extension of \( GF(q) \) such that any irreducible representation of any \( p' \)-subgroup of \( G \) over \( F \) is absolutely irreducible. Let \( 1 = \theta_0, \theta_1, \cdots, \theta_r \) be a basis for \( F \) over \( GF(p) \) and let \( v_1, v_2, \cdots, v_s \) be a basis for \( V_1 \) over \( GF(p) \). Finally let \( V \) be the vector space over \( F \) with basis \( v_1, \cdots, v_s \), i.e., the vectors of \( V \) are the formal sums

\[
\sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij} \theta_i v_j
\]

where \( c_{ij} \in GF(p) \). \( G_1 \) acts on \( V \) in the obvious way.

Consider the group \( G^* = G_1 V \), i.e., the normal product of \( V \) by \( G_1 \). If \( g^* \) is of order \( p^m \) in \( G^* \) then either the image \( g \) of \( g^* \) in \( G_1 \) is of order \( p^m \) or \( g \) is of order \( p^{m-1} \) and \( g \) is not exceptional on \( V \). In the latter case \( (g - 1)^{p^{m-1}} v_i \neq 0 \) for some \( v_i \) from which it follows that \( g \) is not exceptional on \( V_1 \). Thus \( e_p(G) \geq (m - 1) + 1 = m \).

Therefore in any event \( e_p(G) \geq e_p(G^*) \). Since \( e_q(G^*) = e_q(G) \) for \( q \neq p \), \( G^* \) satisfies condition C. Furthermore \( l_p(G) = l_p(G^*) \) so that if \( G^* \) satisfies \( R \) so does \( G \).

Now suppose \( H \) is any normal \( p' \)-subgroup other than the identity in \( G_1 \) and suppose

\[
v = \sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij} \theta_i v_j
\]

is a nonzero vector fixed by \( H \). Since \( v \neq 0 \) the coefficient of \( v_j \) is not zero for some \( j \), \( j = 1 \) say. Then there exists \( \alpha \in F \) such that \( \alpha(\sum_{i=0}^{r} c_{ij} \theta_i) = 1 \). \( H \) must fix \( \alpha v \) which can be written in the form \( \alpha v = v' + v'' \) where

\[
v' = v_1 + \sum_{j=2}^{s} c'_{0j} v_j, \quad v'' = \sum_{j=2}^{s} \sum_{i=1}^{r} c'_{ij} \theta_i v_j .
\]

For \( H \) to fix \( \alpha v \) it must also fix \( v' \) which contradicts the fact that
$H$ has no nonzero fixed vector in $V$. Thus $H$ has no nonzero fixed vector in $V$.

If $V$ is an irreducible $F - G_1$ module then we have arrived at the special case of the theorem. Therefore assume $U$ is a proper submodule.

If $G_1$ is not faithfully represented on $V/U$, then let $Q$ be a minimal normal subgroup of $G_1$ centralizing $V/U$. $Q$ must be a $p'$-group so that $V$ is completely reducible as an $F - Q$ module. Thus there exists a nontrivial $F - Q$ submodule on which $Q$ is the identity. This is impossible since $Q$ can have no nonzero fixed vector.

Hence $G_1$ is faithfully represented on $V/U$. Thus $l_p(G^*) = l_p(G^*/U)$ and, of course, $e_p(G^*) \geq e_p(G^*/U)$ so that if $G^*/U$ satisfies $R$ so does $G^*$ and then so does $G$.

We still have that any normal nonidentity $p'$-subgroup $H$ of $G_1$ has no nonzero fixed vector in $V/U$ since $V$ is completely reducible as an $F - H$ module. Therefore if $G_1$ is not irreducibly represented on $V/U$ then the same argument as before yields that $G_1$ is faithfully represented on a nontrivial factor module of $V/U$. Continuing in this way we ultimately arrive at the special case where $G_1$ is faithfully and irreducibly represented on some vector space over the field $F$. This finishes the proof of Theorem 4.1.

Among the results we now shall prove is that if $G$ is of exponent 12 then $l_3(G) \leq e_3(G)$. Before doing this it might be well to justify this work. For in a group of order $2^a 3^b$ the 2-length and the 3-length can vary at most by one. Thus if it were true that the 3-length of a group of exponent 12 was one, then it would be trivial to state that the 2-length was at most two. However in [5, p. 5] is found an example of a group of exponent 12 but with 3-length two.

For the rest of this paper we make the following standing assumptions.

(1) $G = G_1 V$, the normal product of $V$ by $G_1$, where $V$ is a vector space over a finite field $F$ of characteristic 2 and $G_1$ is a finite, solvable linear group having no normal 2-subgroup other than the identity.

(2) $V$ is an irreducible $F - G_1$ module.

(3) Any representation over $F$ of any $p'$-subgroup of $G_1$ is absolutely irreducible.

(4) $e_3(G) \leq 2$.

We are interested in seeing under what conditions can $l_3(G)$ exceed $e_3(G)$. But if $e_3(G_1) = 0$ then both $e_3(G)$ and $l_3(G)$ are 1, and if $e_3(G_1) = 1$ then $l_3(G_1) = 1$ so that $l_3(G) = e_3(G) = 2$. Thus we may as well assume
Later we shall add to these assumptions the further one that $G$ is of exponent 12. Actually, until we restrict ourselves to groups of exponent 12, we will make no use of the fact that $G_1$ is irreducibly represented on $V$.

Now let $N$ be the largest normal $2'$-subgroup of $G$. We shall show that a certain $2$-subgroup, to be described later, must be contained in the greatest normal $2$-subgroup of $G_1/N$. In particular if $\ell(G) > 2$ (which is the same as $\ell(G_1) > 1$), we shall see that there must exist an element of order 4 of a special type in $G_1$.

First let $H$ be the following normal subgroup of $G$: $x \in H$ if, and only if, for every normal nilpotent subgroup $Q$ of class at most 2 in $G_1$, $x$ fixes every minimal characteristic $F - Q$ submodule of $V$. A normal nilpotent subgroup of $G_1$ must be a $2'$-group so that $V$ splits into the sum of minimal characteristic $F - Q$ modules.

From (5) there are elements of order 4 in $G_1$, and from (4) all such elements must be exceptional. Thus if $g$ is of order 4 in $G_1$ then $g^2$ must be in $H$ by lemma 3.1. Hence $H$ is greater than the identity. $H$ has no normal 2-subgroup except for the identity and the largest normal $2'$-subgroup is $H \cap N$.

Let $D$ be the greatest normal nilpotent subgroup of $H$. $D = D_1 \times D_2 \times \cdots$ where $D_i$ is a Sylow $q_i$-subgroup of $D$ for an odd prime $q_i$. $H$ centralizes any normal abelian subgroup of $G_1$ so that, by the proof of Lemma 3.3, we obtain $c(D) = 2$. Now, as before, let $K_i$ be the subgroup of $D_i$ consisting of all elements of order at most $q_i$ and let $K = K_1 \times K_2 \times \cdots$ We again have that no non-identity 2-element of $H$ centralizes $K$.

Now take $H_1$ to be the subgroup of $G_1$ consisting of all elements which fix every minimal characteristic $F - K_i$ module for all $i$. $H_1 \leq G_1$, and, since $c(K_i) \leq 2$, $H_1 \leq H_i$. $H_1$ has no normal 2-subgroup except for the identity and its greatest normal $2'$-subgroup is $H_1 \cap N$.

Let $P$ be a Sylow 2-subgroup of $H_i$. $P \neq 1$ since if $g$ is any element of order 4 in $G_1$ then $g^2 \in H$. Now the square of any element of $P$ must be in $H$. Thus $P/(P \cap H)$ is of exponent 2 and thus abelian.

Therefore $P' < H$. We now prove two lemmas which enable us to show directly that $PN/N$ is normal in $G/N$.

**Lemma 4.2.** Suppose that $g$ and $h$ are two elements of $P$ and $V'$ is a minimal characteristic $F - K_i$ submodule of $V$. Let $Q$, $g_i$, and $h_i$ be the restrictions of $K_i$, $g$, and $h$, respectively, to $V'$. Then if $(Q, h_i) = 1$ it follows that $(Q, (g_i, h_i)) = 1$.

**Proof.** Assume $(Q, (g_i, h_i)) \neq 1$. Therefore neither $g_i$ nor $h_i$ central-
izes Q. If \((Q, g_i^2) = 1\), then straight forward calculation yields
\[
(Q, (g_i h_i)^2) = (Q, (g_i, h_i)) \neq 1,
\]
\[
(Q, (g_i h_i, h_i)) = (Q, (g_i h_i)^{-1}) \neq 1.
\]
Thus, replacing \(g_i\) by \(g_i h_i\) if \((Q, g_i^2) = 1\), we may assume that 
\((Q, g_i^2) \neq 1\) along with \((Q, h_i^2) = 1\) and \((Q, (g_i, h_i)) \neq 1\).

Now exactly as in the proof of Lemma 3.4 we obtain that \(Q\) is an extra special \(q\)-group (actually \(q = 3\) since \(g_i\) is of order 4 and thus exceptional so that \(4 - 1\) must be a power of \(q\), \(Q/Q'\) is a symplectic space, \(g_i\) and \(h_i\) preserve the symplectic structure of \(Q/Q'\), and we may assume that \(g_i\) and \(h_i\) operate on \(Q/Q'\) as follows:

\[
g_i(\sum \alpha_i u_i) = A\alpha_i u_i + \sum_{\sigma \neq 1} \alpha_i u_i,
\]
\[
h_i(\sum \alpha_i u_i) = \Sigma T_i \alpha_i u_{\sigma(i)},
\]
where \(\sigma\) is a permutation of order \(\leq 2\) (since \((Q, h_i^2) = 1\)), and \(A\) and the \(T_i\) are chosen from a group isomorphic to the quaternion group of order 8 (since \(q = 3\)). In addition \(A\) must be of order 4 since \((Q, g_i^2) \neq 1\).

If \(\sigma\) does not fix 1 then \((g_i, h_i)\) would be of order 4 but its centralizer in \(Q/Q'\) would have co-dimension 4 over \(GF(3)\). Thus \((g_i, h_i)\) would be of order 4 but not exceptional which is impossible.

Hence \(\sigma\) fixes 1 and, since \((Q, h_i^2) = 1\), we must have
\[
h_i(\sum \alpha_i u_i) = \pm \alpha_i n_i + \sum_{\sigma \neq 1} T_i \alpha_i u_{\sigma(i)}.
\]
It is now an easy matter to verify that \((g_i, h_i) = 1\) and the lemma is proved.

**COROLLARY.** If \(g, h \in P\) and \(h^2 = 1\), then \((g, h) = 1\).

**Proof.** \((g, h)\) is in \(P'\) and thus in \(H\). So if \((g, h) \neq 1\) then 
\((K_i, (g, h)) \neq 1\) for some \(K_i\). Then lemma states that this cannot happen.

**LEMMA 4.3.** If \(g, h \in P\), then \((g, h)^2 = 1\).

**Proof.** Suppose that \((g, h)^2 \neq 1\). Then for some \(K_i\), \((K_i, (g, h)^2) \neq 1\). Choose \(V'\) to be a minimal characteristic \(F - K_i\) submodule of \(V\) such that \((K_i, (g, h)^2)\) is not the identity on \(V'\). If \(Q, g_i\), and \(h_i\) are defined as in the previous lemma, then, if either \((Q, g_i^2)\) or \((Q, h_i^2)\) is the identity, \((g_i, h_i) = 1\). Therefore assume neither \(g_i^2\) nor \(h_i^2\) centralize \(Q\). Thus \(g_i\) and \(h_i\) are both exceptional of order 4. \(Q\) is an extra-special
3-group and we may assume $g_i$ and $h_i$ operate on $Q/Q'$ as follows:

$$g_i(\sum \alpha_i u_i) = A\alpha_i u_i + \sum_{i \neq j} \alpha_i u_i,$$

$$h_i(\sum \alpha_i u_i) = B\alpha_i u_i + \sum_{i \neq j} \alpha_i u_i.$$

Now if $j \neq 1$ then $(g_i, h_i) = 1$ and if $j = 1$ then

$$(g_i, h_i)^2(\sum \alpha_i u_i) = (A, B)^2\alpha_i u_i + \sum_{i \neq j} \alpha_i u_i.$$

But $A$ and $B$ are elements of a quaternion group so that $(A, B)^2$ is the identity and the lemma is proved.

**Theorem 4.4.** $PN/N \triangleleft G_i/N$.

**Proof.** We shall prove that $P(H_1 \cap N)/(H_1 \cap N) \triangleleft H_i/(H_1 \cap N)$ which is equivalent to the theorem since $H_i \triangleleft G_i$.

Let $P_1$ be the subgroup of $P$ such that $P_1(H_1 \cap N)/(H_1 \cap N)$ is the largest normal 2-subgroup of $H_i/(H_1 \cap N)$. $P_1 \triangleleft P$ and $P_1$ contains the center of $P$ [4, Lemma 1.2.3]. Thus by the corollary to Lemma 4.2, $P_1$ contains all elements of order 2 in $P$. The elements of order 2 in $P$ form an elementary abelian group $P_2$ which is normal, modulo $H_i \cap N$, in $H_i$. The elements of $H_i/(H_1 \cap N)$ which centralize both $P_2$ and $P_1/P_2$ form a normal subgroup of $H_i/(H_1 \cap N)$. But if any 2'-element centralized both $P_2$ and $P_1/P_2$, then, as easily may be seen, this element would centralize $P_1$ contrary to the fact [4, Lemma 1.2.3] that $P_1$ contains its centralizer in $H_i/(H_1 \cap N)$. Thus the elements centralizing both $P_2$ and $P_1/P_2$ form a normal 2-subgroup of $H_i/(H_1 \cap N)$, and from the corollary to Lemma 4.2 and from Lemma 4.3, $P$ must be contained in this normal 2-subgroup. But $P$ is a Sylow 2-subgroup of $H_i$ and thus it follows that, modulo $H_1 \cap N$, $P$ is normal in $H_i$.

**Corollary.** $l_2(H_i) = 1$.

Now let $S$ be a Sylow 2-subgroup of $G_i$ which contains $P$. From the theorem it follows that $P$ is normal in $S$.

**Lemma 4.5.** If $P$ contains all elements of order 4 in $S$, then $l_2(G_i) = 1$.

**Proof.** If $S = P$ we are done. Therefore assume $S \neq P$. Then if $x \in S - P$ we must have $x^2 = 1$. Also $x \in S - P$, $y \in P$ imply that $xy \in S - P$ so that $(xy)^2 = 1$ which implies that $x^{-1}yx = y^{-1}$. Thus $x$ induces the automorphism $y \to y^{-1}$ of $P$. This can be an automorphism only if $P$ is abelian. Now if both $x_1$ and $x_2$ are in $S - P$ then $x_1x_2$.
centralizes \( P \). But \( e_2(G) = 2 \) so that \( P \) does contain elements of order 4. Hence \( x, x_2 \) cannot be in \( S - P \).

Therefore \( |S/P| = 2 \) and \( P \) is abelian. Now if \( x \in S - P, y \in P \), then \((x, y) = x^{-1}y^{-1}xy = y^2 \in \Phi(P) \) and thus \( x \) centralizes \( P/\Phi(P) \). Hence [4, Lemma 1.2.5] \( PN/N \) cannot be the largest normal 2-subgroup of \( GJN \). But \( P \) is maximal in \( S \) so that \( SN/N \) must be the largest normal 2-subgroup of \( GJN \). This implies that \( l_2(G) = 1 \).

To our assumptions (1)~(5) we now add

\((6)\) \( G \) is of exponent 12.

This implies that \( K \) must be a group of exponent 3 and class at most 2. We prove that \( l_2(G) = 1 \) in this case by showing that the hypothesis of Lemma 4.5 are satisfied.

For this purpose assume that \( g \) is an element of order 4 in \( S - P \). \( g^2 \) is in \( H \) so \((K, g^2) \neq 1 \). Let \( V = V_1 \oplus V_2 \oplus \cdots \) be the decomposition of \( V \) into minimal characteristic \( F - K \) modules. Since \( g \in S - P \), \( g \) does not fix some \( V_i \). \( g^2 \) does fix each \( V_i \) and if \( g^2 \) is not the identity on a \( V_i \), then \( g \) must fix that \( V_i \) for otherwise \( g \) could not be exceptional [4, p. 13]. We now need the following result:

**Lemma 4.6.** There exist \( x \) and \( y \) in \( K \) such that \((x, g^2), (y, g^2) \neq 1 \).

**Proof.** Let \( C = \{x \mid x \in K, (x, g^2) \in Z(K)\} \). Clearly \( C \supseteq Z(K) \) but \( C \neq K \) since then \( g^2 \) would centralize \( Z(K) \) and \( K/Z(K) \) which would imply that \((K, g^2) = 1 \). \((g^2 \) centralizes \( Z(K) \) by Lemma 3.1 and 3.2.) \( K/Z(K) \) is an elementary abelian 3-group so that there must be a \( GF(3) - g \) module of \( K/Z(K) \) complementary to \( C/Z(K) \). Thus \( K/Z(K) = L/Z(K) \oplus C/Z(K) \) and \( g \) normalizes \( L \). For all \( x \in L - Z(K), (x, g^2) \) is not in \( Z(K) \).

Now suppose \( x, y \in L - Z(K) \) and \((x, g^2)(y, g^2)^{-1} \in Z(K) \). Since \( K/Z(K) \) is abelian, straight forward calculation yields

\[
(xy^{-1}, g^2) \equiv (x, g^2)(y^{-1}, g^2) \pmod{Z(K)},
\]
\[
1 = (yy^{-1}, g^2) \equiv (y, g^2)(y^{-1}, g^2) \pmod{Z(K)}.
\]

Thus \((xy^{-1}, g^2) \equiv (x, g^2)(y, g^2)^{-1} \equiv 1 \pmod{Z(K)} \). This implies that \( xy^{-1} \in Z(K) \). Therefore we have shown that \((x, g^2) \equiv (y, g^2)(\mod Z(K)) \) if, and only if, \( x \equiv y \pmod{Z(K)} \) for \( x, y \in L \).

It immediately follows from this that for any \( x \in L \), there exists a \( y \) such that \( x \equiv (y, g^2)(\mod Z(K)) \). Now \( L \) cannot be abelian since \( g \) normalizes \( L \) and \( g^2 \) does not centralize it. From all this we see that there exist \( x, y \in L \) such that \(((x, g^2), (y, g^2)) \neq 1 \).

Now taking \( x \) and \( y \) to satisfy the lemma, we may assume without
loss of generality that \(((x, g^2), (y, g^2))\) is not the identity on \(V_i\). This implies that \(g^2\) is not the identity on \(V_i\) so \(g\) must fix \(V_i\).

Since \(g\) does not fix every \(V_i\), assume \(g\) does not fix \(V_2\). Therefore \(g^2\) is the identity on \(V_2\) which then also must be the case for \((x, g^2)\) and \((y, g^2)\).

\(V\) is an irreducible \(F - G_i\) module so that there must be an element taking \(V_i\) into \(V_i\). Such an element must be of the form \(zh\) where \(h \in S\) and \(z\) is from a Sylow 3-subgroup of \(G_i\) which necessarily must contain \(K\). We shall derive a contradiction by showing that \(z\) and \(K\) generate elements of order 9 which is impossible in a group of exponent 12.

If \(hV_1 = V_m\) then \(zV_m = V_2\). Set \(g_1 = hg^{-1}\). Then

\[\left(\left((x^{h^{-1}}, g_i), (y^{h^{-1}}, g_i)\right)\right)\]

is not the identity on \(V_m\). Now suppose \(g_1V_2 = V_2\). Then \(gh^{-1}V_2 = h^{-1}V_2\), and, since \(gV_2 \neq V_2\), this implies that \(h^{-1}V_2 = V_j, j \neq 2\). Then we would have \(gV_j = V_j\). But \(gh^{-1} \in S\) so that \((gh^{-1})^2 \in H\). Thus \((gh^{-1})^2\) fixes \(V_2\) and, therefore, \(gh^{-1}V_j = V_2\). \((h^{-1})^2\) also must fix \(V_2\) so we have \(h^{-1}V_j = V_2\). From this we conclude that \(V_m \neq V_2\) for \(V_m = V_2\) would imply that \(h^{-1}V_2 = V_1\) which would imply that \(g_V_2 = hV_1 = V_2\). Since \(V_m \neq V_2\) it follows that \(z\) is not the identity and so is of order 3.

If we replace \(V_i, g, x,\) and \(y\) by \(V_m, g_1, x^{h^{-1}},\) and \(y^{h^{-1}},\) respectively, we may assume that \(zV_1 = V_2, gV_2 \neq V_2,\) and \(((x, g^2), (y, g^2))\) is not the identity on \(V_i\). Let \(x_i = (x, g^2)\) and \(y_i = (y, g^2)\). \(x_i\) and \(y_i\) must be the identity on \(V_2\) since \(g_2\) is. Since \(z\) is of order 3, we have \(zV_1 = V_1, zV_2 = V_3(n \neq 1, 2),\) and \(zV_n = V_1\).

Let \(V' = V_1 \oplus V_2 \oplus V_n\). \(V'\) is fixed by \(z\) and the restrictions of \(x_i, y_i,\) and \(z\) to \(V'\) are

\[z = \begin{pmatrix} 0 & 0 & A \\ B & 0 & 0 \\ 0 & C & 0 \end{pmatrix}, \]

\[x_i = \begin{pmatrix} 0 & 0 & 0 \\ M & 0 & 0 \\ 0 & I & 0 \end{pmatrix}, \]

\[y_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & N_i \end{pmatrix}, \]

where \(I\) is the identity and 0 the zero matrix. Now \((x_i, y_i)\) is not the identity on \(V_1\) but \((x_i, y_i) \in Z(K)\) and \(Z(K)\) is represented on \(V_1\) as a cyclic group generated by a scalar matrix. Thus \((M, N) = \omega I\) where \(\omega\) is a primitive third root of unity. From \(z^3 = 1\) we obtain \(C = A^{-1}B^{-1}\).

Now \(z, x_i,\) and \(y_i\) all belong to the same Sylow 3-subgroup of \(G_i\). Thus \((zx_i)^3 = (zy_i)^3 = 1\). From this direct computation yields that \(M_i = A^{-1}M^{-1}A, N_i = A^{-1}N^{-1}A\). Thus \((M_i, N_i) = A^{-1}(M^{-1}, N^{-1})A\). But \(M\) and \(N\) generate a group of exponent 3 and class 2. It follows easily that \((M^{-1}, N^{-1}) = (M, N) = \omega I\). Thus
It is now a simple matter to verify that $(z(x_1, y_1))^p \neq 1$. Hence $z(x_1, y_1)$ is a 3-element of order greater than 3 which is impossible in a group of exponent 12. This contradiction proves that the hypothesis of Lemma 4.5 is satisfied and thus:

**Theorem 4.7.** If $G$ is a finite group of exponent 12, then $l_2(G) \leq e_2(G)$.

**References**


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