Φ-BOUNDED HARMONIC FUNCTIONS AND CLASSIFICATION OF RIEMANN SURFACES

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Let \(\phi(t)\) be a nonnegative real valued function defined for \(t\) in \([0, \infty)\) such that \(\phi(t)\) is unbounded in \([0, \infty)\) and bounded in a neighborhood of a point in \([0, \infty)\). A harmonic function \(u\) on a Riemann surface \(R\) is said to be \(\phi\)-bounded if the composite function \(\phi(|u|)\) has a harmonic majorant on \(R\). Denote by \(O_{H\phi}\) the class of all Riemann surfaces on which every \(\phi\)-bounded harmonic function reduces to a constant. The main result in this paper is the following: \(O_{H\phi} = O_{H_P}\) (resp. \(O_{HB}\)) if and only if \(d(\phi) < \infty\) (resp. \(d(\phi) = \infty\)), where \(d(\phi) = \limsup_{t \to \infty} \phi(t)/t\). This is the best possible improvement of a result of M. Parreau.

We also prove a similar theorem for the classification of subsurfaces of Riemann surfaces using \(\phi\)-bounded harmonic functions vanishing on the relative boundaries of subsurfaces.

The chief tool of our proof is the theory of Wiener compactifications of Riemann surfaces.

Consider a nonnegative real valued function \(\phi(t)\) defined for all real numbers \(t\) in \([0, \infty)\). A harmonic function \(u\) on a Riemann surface \(R\) is said to be \(\phi\)-bounded if the composite function \(\phi(|u|)\) has a harmonic majorant on \(R\). The totality of \(\phi\)-bounded harmonic functions on \(R\) is denoted by \(H\phi(R)\), or simply \(H\phi\). We denote by \(O_{H\phi}\) the class of all Riemann surfaces \(R\) on which every \(\phi\)-bounded harmonic function reduces to a constant. Our problem is to determine \(O_{H\phi}\) for every \(\phi\).

First assume that \(\phi(t)\) is bounded on \([0, \infty)\). Then every harmonic function is \(\phi\)-bounded. Hence \(R\) belongs to \(O_{H\phi}\) if and only if there exists no nonconstant harmonic function on \(R\). Thus the class \(O_{H\phi}\) consists of all closed Riemann surfaces if \(\phi\) is bounded. Soon we see that the converse is also valid. Hence, hereafter, we always assume that

1. \(\phi(t)\) is unbounded on \([0, \infty)\).

We say that \(\phi(t)\) is bounded at a point \(t_0\) in \([0, \infty)\) if there exists a neighborhood of \(t_0\) relative to \([0, \infty)\) in which \(\phi(t)\) is bounded. Now assume that \(\phi(t)\) is not bounded at any point of \([0, \infty)\). Let \(u\) be a nonconstant harmonic function on \(R\). Then \(\phi(|u|)\) is not bounded at any neighborhood of any point of \(R\) and so \(u\) is not \(\phi\)-bounded. Thus the class \(O_{H\phi}\) consists of all Riemann surfaces if \(\phi(t)\) is not bounded at

Received September 5, 1964.
any point of \([0, \infty)\). Soon we see that the converse is also true. Hence, hereafter, we always assume that

\((2) \ \Phi(t) \text{ is bounded at least at one point in } [0, \infty).\)

Now our problem which is left is to determine \(O_{H\Phi}\) for functions \(\Phi\) satisfying the two conditions \((1)\) and \((2)\). For the aim, we put

\[d(\Phi) = \limsup_{t \to \infty} \frac{\Phi(t)}{t}.\]

Clearly \(0 \leq d(\Phi) \leq \infty\). Our result is stated as follows:

**Theorem 1.** Assume that \(\Phi\) satisfies \((1)\) and \((2)\). If \(d(\Phi)\) is finite (resp. infinite), then \(O_{H\Phi} = O_{H\Phi}\) (resp. \(O_{H\Phi}\)).

Since the restrictions on \(\Phi\) are exclusive each other, we also see that \(O_{H\Phi} = O_{H\Phi}\) (resp. \(O_{H\Phi}\)) implies that \(\Phi\) satisfies \((1)\) and \((2)\) and \(d(\Phi)\) is finite (resp. infinite). This theorem is proved by Parreau [3] for the special \(\Phi\) which is increasing and convex (and so continuous) (see also Ahlfors-Sario’s book [1], pp. 216–219). Parreau’s proof keenly uses the increasingness and convexity of \(\Phi\) and one might suspect that these assumptions are inevitable. We are interested in the fact that for the validity of Parreau’s result, no assumption is needed for \(\Phi\) except the inevitable conditions \((1)\) and \((2)\). Thus our Theorem 1 is the best possible generalization of Parreau’s result at least in the above formulation.

2. Before entering the proof of Theorem 1, for convenience, we explain an outline of the *Wiener compactification* of a Riemann surface and its some properties which we use in the proof of Theorem 1. For details, consult Constantinescu-Cornea’s book [2], §6, 8 and 9.

Let \(F\) be a Riemann surface not belonging to \(O_\theta\) and \(f\) be a real valued function on \(F\). Let \(\overline{W}_f^\rho\) (resp. \(\overline{W}_f^\rho\)) be the totality of superharmonic (resp. subharmonic) functions \(s\) on \(F\) such that there exists a compact subset \(K_s\) of \(F\) with the property that \(f \leq s\) (resp. \(f \geq s\)) on \(F - K_s\). If \(\overline{W}_f^\rho\) and \(\overline{W}_f^\rho\) are nonvoid, then \(\overline{W}_f^\rho\) and \(\overline{W}_f^\rho\) are Perron’s families and so

\[\overline{h}_f^\rho(p) = \inf (s(p); s \in \overline{W}_f^\rho)\]

and

\[\overline{h}_f^\rho(p) = \sup (s(p); s \in \overline{W}_f^\rho)\]

are harmonic and \(\overline{h}_f^\rho \geq \overline{h}_f^\rho\). If \(\overline{h}_f^\rho = \overline{h}_f^\rho\) on \(F\), then we write \(h_f^\rho = \overline{h}_f^\rho\) and \(h_f^\rho\) and we call \(f\) to be harmonizable on \(F\).

Let \(R\) be an arbitrary Riemann surface. A real-valued function \(f\) on \(R\) is said to be a continuous Wiener function if (a) for any sub-surface \(F\) of \(R\) as a Riemann surface, the restriction of \(f\) on \(F\) is harmonizable on \(F\) and the restriction of \(|f|\) on \(F\) has a super-harmonic majorant on \(F\); and if (b) \(f\) is finitely continuous on \(R\). We denote by \(WC = WC(R)\) the totality of continuous Wiener functions
on $R$. We also denote by $WB = WB(R)$ the totality of bounded members in $WC$. Observe that $WC$ (resp. $WB$) is a vector space and closed under max and min operations. Any continuous superharmonic function on $R$ which has a harmonic majorant clearly belongs to $WC$. Hence $HP \subset WC$ and $HB \subset WB$.

There exists a unique compact Hausdorff space $R^*$ containing $R$ as its open and dense subset such that $C(R^*)|R = WB(R)$, where $C(R^*)$ is the totality of finitely continuous functions on $R^*$ and $C(R^*)|R$ is the totality of restrictions of functions in $C(R^*)$ to $R$. We call $R^*$ the Wiener compactification of $R$. By the obvious identification, we may simply write as $C(R^*) = WB(R)$. It is clear that any function in $WC(R)$ is (not necessarily finitely) continuous on $R^*$, or more accurately, is continuously extended to $R^*$. Hereafter, we use topological notions relative to $R^*$ only. For example, $\bar{A}$ for $A \subset R$ means the closure of $A$ in $R^*$. But the notation $\partial A$ for $A \subset R^*$ is the only exceptional. $\partial A$ means the boundary of $A \cap R$ relative to $R$.

Let $W_0C(R) = \{f \in WC; h^f_R = 0\}$ if $R \in O_o$ and $W_0C(R) = WC$ if $R \in O_o$. We set $\Delta = (p \in R^*; f(p) = 0$ for any $f \in W_0C)$. This is a compact subset of $I^* = R^* - R$ and called the (Wiener) harmonic boundary of $R$. It is seen that $W_0C = (f \in WC; f = 0$ on $\Delta)$. From the definition, it is obvious that $R \in O_o$ if and only if $\Delta = \emptyset$. Moreover,

**Lemma 1.** $R \in O_{HB} - O_o$ if and only if $\Delta$ consists of only one point.

Let $F$ be an open subset of $R$ each boundary point of which is regular for Dirichlet problem and $\partial F \neq \emptyset$. Such an $F$ is called a regular open subset of $R$. We say that $F \in SO_{HB}$ if any connected component of $F$ does not carry any nonconstant bounded harmonic functions vanishing continuously at $\partial F$. The most important is the following

**Lemma 2.** $F \in SO_{HB}$ if and only if $\bar{F} - \partial \bar{F}$ contains a point of $\Delta$.

As an corollary of this, we can easily see the following useful

**Lemma 3.** Let $F$ be a regular open subset of $R$ and $s$ be a superharmonic function on $F$ bounded from below. If

$$\liminf_{F \ni q \to p} s(p) \geq 0$$

for any $q$ in $\partial F \cup (\bar{F} \cap \Delta)$, then $s \geq 0$ on $F$.

3. **Proof of Theorem 1 for $d(\Phi) < \infty$.** Since $d(\Phi) < \infty$, we can find a positive number $c$ and a point $t_0$ in $[0, \infty)$ such that $\Phi(t) \leq ct$ for any $t \geq t_0$. Assume that there exists a nonconstant $HP$-function
$u_i$ on $R$. Then $u = u_2 + t_0$ is also a nonconstant harmonic function on $R$ with $u \geq t_0 \geq 0$ on $R$. Thus $\Phi(|u|) \leq c |u| = cu$ and $cu$ is an $HP$-function on $R$. Hence $O_{H\Phi} \subset O_{HP}$.

Conversely, assume that there exists a nonconstant $H\Phi$-function $u$ on $R$. We have to prove the existence of a nonconstant $HP$-function on $R$. By the definition, there exists an $HP$-function $v$ on $R$ with $\Phi(|u|) \leq v$ on $R$. If $v$ is not a constant or $u$ is bounded, then nothing is left to prove and so we assume that $v$ is a constant and $u$ is not bounded. Then the connected open set $D = \{(u(p)); \; p \in R\}$ in $[0, \infty)$ does not contain $0$. Contrary to the assertion, assume that $D \ni 0$. Then $D = [0, \infty)$ and so $(\Phi(|u(p)|); \; p \in R) = (\Phi(t); \; t \in [0, \infty))$ is unbounded in $[0, \infty)$ by the assumption (1) for $\Phi$. But this is impossible, since $\Phi(|v|) = \Phi(v) \leq c$ on $R$. Thus $0 \notin D$. This shows that $u$ does not change sign on $R$. Hence $u$ or $-u$ is a nonconstant $HP$-function on $R$. Therefore, $O_{H\Phi} \supset O_{HP}$. Thus $O_{H\Phi} = O_{HP}$ for $\Phi$ with $d(\Phi) < \infty$.

4. Proof of Theorem 1 for $d(\Phi) = \infty$. First assume that there exists a nonconstant $HB$-function $u$ on $R$. By the assumption (2) for $\Phi$, there exists an interval $(a, b) \subset [0, \infty)$ in which $\Phi(t) \leq c$ (constant). By choosing a suitable constants $A$ and $B$, the range of $v = Au + B$ is contained in $(a, b)$. Then $\Phi(|v|) = \Phi(v) \leq c$ on $R$. Thus $v$ is a nonconstant $H\Phi$-function on $R$. Hence $O_{HB} \supset O_{H\Phi}$.

Next we prove the converse inclusion $O_{HB} \subset O_{H\Phi}$, or equivalently, $R \in O_{H\Phi}$ implies $R \notin O_{HB}$. Assume that there exists a nonconstant $H\Phi$-function $u$ on $R$. We have to prove that $R \notin O_{HB}$. Contrary to the assumption, assume that $R \in O_{HB}$. By the definition, there exists an $HP$-function $v$ such that $\Phi(|u|) \leq v$ on $R$. From this, we see that $R \notin O_{HP}$. For, if $R \in O_{HB}$, then $\Phi(|u|) \leq v$ (constant) and since $d(\Phi) = \infty$, $|u|$ is bounded. This contradicts $R \in O_{HB}$. Hence $R \notin O_{HP}$ and a fortiori $R \in O_\Phi$. Thus $R \in O_{HB} - O_\Phi$ and so by Lemma 1, the harmonic boundary $\Delta$ of $R$ consists of only one point $\delta$, i.e. $\Delta = \{\delta\}$. By $d(\Phi) = \infty$, we can find a strictly increasing sequence $(r_n)_{n=1}^\infty$ of positive numbers such that

$$\lim_{n \to \infty} \Phi(r_n)/r_n = \infty \text{ and } \lim_{n \to \infty} r_n = \infty.$$ 

Let $G_n = \{p \in R; \; |u(p)| < r_n\}$. Since $u$ is not a constant and $u$ is unbounded by $R \in O_{HB}$, $G_n$ is a regular open subset of $R$ with $\partial G_n \neq \emptyset$ and $G_n \cap R$. We see that $G_n \in SO_{HB}$ for some $n$. For, if this is not the case, then $G_n \in SO_{HB}$ for all $n = 1, 2, \cdots$. Let $a_n = r_n/\Phi(r_n)$. Then $a_n \rightarrow 0(n \to \infty)$. Consider the function $a_nv - |u|$, which is superharmonic and bounded from below on $G_n$ and continuous in $G_n \cup \partial G_n$. If $q \in \partial G_n$, then
Thus $a_n v - |u| \geq 0$ on $\partial G_n$. Hence $a_n v - |u| \geq 0$ in $G_n$. For, if $a_n v(p_0) - |u(p_0)| < d < 0$ for some $p_0$ in $G_n$, then $G'_n = (p \in G_n; a_n v(p) - |u(p)| < d)$ is a nonempty regular open subset with $G'_n \cup \partial G'_n \subset G_n$. The function $d - (a_n v - |u|)$ is a positive and bounded (with bound $d + r_n$) subharmonic function in $G'_n$ vanishing continuously at $\partial G'_n$. So $G'_n \notin SO_{\Omega B}$. But this is a contradiction, since $G_n \supset G'_n \cup \partial G'_n$ and $G_n \in SO_{\Omega B}$. Hence $a_n v - |u| \geq 0$ in $G_n$. Now let $p$ be an arbitrary point in $\Omega$. There exists an $n_0$ such that $p \in G_n$ for all $n \geq n_0$. Then $|u(p)| \leq a_n v(p)$ for all $n \geq n_0$. Thus by making $n \nearrow \infty$, $|u(p)| = 0$, i.e. $u = 0$ on $\Omega$, which is a contradiction. Hence $G_{n_1} \notin SO_{\Omega B}$ for some $n_1$ and so $G_n \in SO_{\Omega B}$ for all $n \geq n_1$ and so without loss of generality, we may assume that $G_n \in SO_{\Omega B}$ for all $n = 1, 2, \ldots$. In particular, $G_1 \notin SO_{\Omega B}$ implies that $\Gamma_1 - \partial \Gamma_1$ contains $\delta$ by Lemma 2 (recall that $A = (\delta)$), i.e. $\Gamma_1$ is a neighborhood of $\delta$ in the Wiener compactification $\Omega^*$ of $\Omega$. Hence in the topology of $\Omega^*$,

\[
(*) \quad \lim_{\Omega \ni p \to \delta} \sup_{\Omega \ni p \to \delta} |u(p)| = \lim_{\Omega \ni p \to \delta} |u(p)| \leq r_1.
\]

Now consider the function $f_n = a_n v + r_n - |u|$, which is superharmonic and bounded from below on $G_n$ and continuous in $G_n \cup \partial G_n$. If $q \in \partial G_n$, then as before,

\[
|u(q)| = r_n = (r_n/\Phi(r_n)) \Phi(r_n) = a_n \Phi(|u(q)|) \leq a_n v(q) \leq a_n v(q) + r_n,
\]

and so $f_n(q) \geq 0$ on $\partial G_n$. This with $(*)$ gives that

\[
\lim_{\Omega \ni p \to \delta} \inf_{\Omega \ni p \to \delta} f_n(p) \geq 0
\]

for any $q$ in $\partial G_n \cup (\delta) = \partial G_n \cup (\bar{G}_n \cap \Delta)$. Hence by Lemma 3, $f_n \geq 0$ in $G_n$, or

\[
|u| \leq a_n v + r_1
\]

in $G_n$. Let $p$ be an arbitrary point in $\Omega$. There exists an $n_0$ such that $p \in G_n$ for all $n \geq n_0$. Thus $|u(p)| \leq a_n v(p) + r_1$ for all $n \geq n_0$. Hence by making $n \nearrow \infty$, $|u(p)| \leq r_1$, i.e. $|u| \leq r_1$ on $\Omega$. Hence $\Omega \notin O_{\Omega B}$. This is a contradiction, since we assumed that $\Omega \in O_{\Omega B}$. Thus $\Omega \in O_{\Omega B}$.

5. Finally we make a few remark to the classification of Riemann surfaces with regular boundaries. Let $\Phi(t)$ be a non-negative real-valued function defined in $[0, \infty)$. Let $\Omega$ be a Riemann surface and $F$ be a regular open subset of $\Omega$. We denote by $H^H_\Omega \Phi = H^H_\Omega \Phi(\Omega, F)$ the totality of harmonic functions $u$ in $F$ vanishing continuously at $\partial F$ such that $\Phi(|u|)$ admits a harmonic majorant in $F$. We say that
We want to determine $S_0 H \Phi$ for every $\Phi$. As before, unless $\Phi$ satisfies (1), then $F \in SO_{H \Phi}$ if and only if $F$ does not carry any nonzero harmonic function in $F$ vanishing continuously at $\partial F$. Thus $SO_{H \Phi}$ consists of all relatively compact regular open subsets of Riemann surfaces if $\Phi(t)$ is bounded in $[0, \infty)$. Similarly as before, $SO_{H \Phi}$ consists of all regular open subsets of Riemann surfaces if $\Phi(t)$ is not bounded at $t = 0$. Hence we have only to consider the problem of determining $SO_{H \Phi}$ under the condition

(3) $\Phi(t)$ is bounded at $t = 0$ and unbounded in $[0, \infty)$.

As before $d(\Phi) = \limsup_{t \to \infty} \Phi(t)/t$. By (3), $SO_{H \Phi} \subset SO_{H \Phi}$ is always valid. Without assuming (3), we can show $SO_{H \Phi} \supset SO_{H \Phi}$ if $d(\Phi) = \infty$ (see the proof of Theorem 2 below). If $d(\Phi) < \infty$, then we cannot get any definite conclusion in general. So we prove only the following

**Theorem 2.** Assume that $\Phi$ satisfies (3) and $d(\Phi) = \infty$. Then $SO_{H \Phi} = SO_{H \Phi}$.

*Proof.* Assume that there exists a nonconstant $H_\Phi$-function $u$ in $F$. Then $\Phi(|u|) \leq v$ in $F$ for some harmonic function $v$ in $F$. We want to show that $F \notin SO_{H \Phi}$. Contrary to the assertion, assume that $F \in SO_{H \Phi}$. By $d(\Phi) = \infty$, there exists an increasing sequence $(r_n)_{n=1}^\infty$ of positive numbers such that $a_n = r_n/\Phi(r_n) \downarrow 0$ and $r_n \nearrow \infty$ as $n \nearrow \infty$. Let $F_n = \{ p \in F : |u(p)| < r_n \}$. Clearly $F_n \nearrow F$ and $F_n \in SO_{H \Phi}$. As in the proof of Theorem 1 for $d(\Phi) = \infty$, $a_n v - |u| \geq 0$ on $\partial F_n$ and $a_n v - |u|$ is lower bounded superharmonic function in $F_n$ and so $F_n \in SO_{H \Phi}$ implies that $a_n v \geq |u|$ in $F_n$ and finally $u = 0$ in $F$. This is a contradiction and so $F \notin SO_{H \Phi}$, or $SO_{H \Phi} \supset SO_{H \Phi}$.

Now we change the definition of $H_\Phi = H_\Phi(R, F)$ as follows: $H_\Phi$ is the totality of harmonic functions $u$ in $F$ vanishing continuously at $\partial F$ such that $\Phi(|u|)$ admits a harmonic majorant in $R$, where we define $u = 0$ in $R - F$. Under this new definition, Theorem 2 is again valid. In fact, $SO_{H \Phi} \subset SO_{H \Phi}$ is clear by (3) and the above proof for $SO_{H \Phi} \supset SO_{H \Phi}$ for $d(\Phi) = \infty$ can be applied with an obvious modification to the present case. Moreover, we can show the following

**Theorem 3.** Assume that $\Phi$ satisfies (3). If $F$ is a regular open subset of $R$ with the compact complement in $R$, then $F \in SO_{H \Phi}$ if and only if $F \in SO_{H \Phi}$, or equivalently, $R \in O_\Phi$.

*Proof.* Clearly $F \in SO_{H \Phi}$ implies $F \in SO_{H \Phi}$ by the condition (3). Hence we have to show that $F \in SO_{H \Phi}$ implies $F \in SO_{H \Phi}$. Evidently, $F \in SO_{H \Phi}$ is equivalent to $R \in O_\Phi$. Let $u$ be a nonconstant $H_\Phi$-function in $F$. Then there exists an HP-function $v$ in $R$ such that $\Phi(|u|) \leq v$ on $R$, where we define $u = 0$ in $R - F$. Contrary to the
assertion, assume that \( F \in SO_{HB} \), or equivalently \( R \in O_\sigma \). Then the inclusion \( O_\sigma \subset O_{HP} \) implies that \( v \) is a constant, i.e. \( \Phi(|u|) \) is a bounded function on \( R \). Let \( D = (|u(p)|; p \in R) \). Since \( D \) is connected and \( |u| \) is not bounded, \( D = [0, \infty) \). Thus \( (\Phi(|u(p)|); p \in R) = (\Phi(t); t \in [0, \infty)) \). From this, the boundedness of \( \Phi(|u|) \) implies the boundedness of \( \Phi(t) \), which contradicts the assumption (3).

REFERENCES


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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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* Basil Gordon, Acting Managing Editor until February 1, 1966.