

# Pacific Journal of Mathematics

**ON SUB-ALGEBRAS OF A  $C^*$ -ALGEBRA**

JOHN R. RINGROSE

## ON SUB-ALGEBRAS OF A C\*-ALGEBRA

J. R. RINGROSE

The following noncommutative extension of the Stone-Weierstrass approximation theorem has been obtained by Glimm.

**Theorem.** Let  $\mathcal{A}$  be a C\*-algebra with identity I, and let  $\mathcal{B}$  be a C\*-sub-algebra containing I. Suppose that  $\mathcal{B}$  separates the pure state space of  $\mathcal{A}$ . Then  $\mathcal{B} = \mathcal{A}$ .

In the present paper, we apply Glimm's theorem to obtain the following noncommutative generalisation of another result of Stone.

Let  $\mathcal{A}$  be a C\*-algebra with identity I and pure state space  $\mathcal{P}$ . Let  $\mathcal{B}$  be a C\*-sub-algebra of  $\mathcal{A}$ , and define

$$\begin{aligned}\mathcal{N} &= \{f: f \text{ is a pure state of } \mathcal{A} \text{ and } f(B) = 0 \ (B \in \mathcal{B})\}, \\ \mathcal{E} &= \{(g, h): g, h \in \mathcal{P} \text{ and } g(B) = h(B) \ (B \in \mathcal{B})\}, \\ \mathcal{H}_{\mathcal{B}} &= \{A: A \in \mathcal{A}, f(A) = 0 \ (f \in \mathcal{N}) \text{ and } g(A) = h(A) \\ &\quad ((g, h) \in \mathcal{E})\}.\end{aligned}$$

Then  $\mathcal{B} = \mathcal{H}_{\mathcal{B}}$ .

We will refer to this as Theorem 2 in the sequel. Glimm's theorem is to be found in [1]; Stone's, in [3].

Once it is known that  $\mathcal{H}_{\mathcal{B}}$  is a C\*-sub-algebra of  $\mathcal{A}$ , Theorem 2 is an almost immediate consequence of Glimm's theorem (see § 4). It is clear that  $\mathcal{H}_{\mathcal{B}}$  is a closed self-adjoint linear subspace of  $\mathcal{A}$ ; accordingly, most of this paper is devoted to proving that  $\mathcal{H}_{\mathcal{B}}$  is closed under multiplication (see § 3).

We remark that, if  $\mathcal{A}$  is commutative, then  $\mathcal{P}$  consists exactly of all homomorphism from  $\mathcal{A}$  on to the complex plane  $\mathbb{C}$ ; so in this case, it is immediate from its definition that  $\mathcal{H}_{\mathcal{B}}$  is a C\*-sub-algebra. However, this seems not to be obvious in the general case. Indeed, for a *general* set  $\mathcal{N}$  of pure states of  $\mathcal{A}$  and a *general* subset  $\mathcal{E}$  of  $\mathcal{P} \times \mathcal{P}$ , the class

$$\{A: A \in \mathcal{A}, f(A) = 0 \ (f \in \mathcal{N}) \text{ and } g(A) = h(A) \ ((g, h) \in \mathcal{E})\}$$

need not be a sub-algebra of  $\mathcal{A}$ ; for example, let  $\mathcal{A}$  consist of all bounded linear operators on a Hilbert space  $H$ , let  $\mathcal{N}$  be void, and let  $\mathcal{E}$  consist of a single pair of vector states arising from orthogonal unit vectors.

2. Notation. Throughout,  $\mathcal{A}$  is a C\*-algebra-by which we shall mean a uniformly closed self-adjoint algebra of operators acting on a (complex) Hilbert space  $H$ . We shall always assume that  $\mathcal{A}$  contains

the identity operator  $I$  on  $H$ . A *state* of  $\mathcal{A}$  is a linear functional  $f$  on  $\mathcal{A}$  such that  $f(A^*A) \geq 0$  ( $A \in \mathcal{A}$ ) and  $f(I) = 1$ . The set of all states is convex and weak \* compact; the Krein-Milman theorem ensures the existence of extreme points, and these are called *pure states*. The *pure state space* of  $\mathcal{A}$ , denoted by  $\mathcal{P}$  (or  $\mathcal{P}(\mathcal{A})$  if  $\mathcal{A}$  has to be specified), is the weak \* closure of the set of all pure states.

Given a state  $f$  of  $\mathcal{A}$ , there is a \*-representation  $\phi_f$  of  $\mathcal{A}$  on a Hilbert space  $H_f$ , and a unit vector  $x_f$  in  $H_f$ , such that  $\phi_f(\mathcal{A})x_f$  is dense in  $H_f$ , and

$$f(A) = \langle \phi_f(A)x_f, x_f \rangle \quad (A \in \mathcal{A}).$$

To within unitary equivalence,  $\phi_f$  is unique. Furthermore,  $\phi_f$  is irreducible if and only if  $f$  is a pure state (see, for example, [2] 245, 265, 266). We shall always use the symbols  $\phi_f, H_f, x_f$  in the sense just described.

3. **Some lemmas.** Throughout this section we shall assume that  $\mathcal{B}$  is a  $C^*$ -sub-algebra of  $\mathcal{A}$ , and that  $I \in \mathcal{B}$ . We use the notations introduced in the statement of Theorem 2; note that, since  $I \in \mathcal{B}$ ,  $\mathcal{N}$  is empty and

$$\mathcal{H}_\omega = \{A : A \in \mathcal{A} \text{ and } g(A) = h(A) \ ((g, h) \in \mathcal{E})\}.$$

For completeness, we give a proof of the following simple result.

LEMMA 1. (i) *Let  $f \in \mathcal{P}$ ,  $S \in \mathcal{A}$  and suppose that  $f(S^*S) = 1$ . Define  $g(A) = f(S^*AS)$  ( $A \in \mathcal{A}$ ). Then  $g \in \mathcal{P}$ .*

(ii) *Let  $f \in \mathcal{P}$ ,  $x \in H_f$ ,  $\|x\| = 1$ , and define  $g(A) = \langle \phi_f(A)x, x \rangle$  ( $A \in \mathcal{A}$ ). Then  $g \in \mathcal{P}$ .*

*Proof.* (i) Clearly  $g$  is a state. Suppose first that  $f$  is a pure state, and let  $x = \phi_f(S)x_f$ . Then for each  $A \in \mathcal{A}$ ,

$$(1) \quad \langle \phi_f(A)x, x \rangle = \langle \phi_f(S^*AS)x_f, x_f \rangle = f(S^*AS) = g(A).$$

With  $A = I$  we obtain  $\|x\| = 1$ ; and since  $f$  is a pure state,  $\phi_f$  is irreducible, so  $\phi_f(\mathcal{A})x$  is dense in  $H_f$ . This, with (1), implies that  $\phi_f$  and  $\phi_g$  are unitarily equivalent. Thus  $\phi_g$  is irreducible, so  $g$  is pure.

Now suppose only that  $f \in \mathcal{P}$ . There is a net  $(f_i)$  of pure states which converges to  $f$  in the weak \* topology. Since  $f_i(S^*S) \rightarrow f(S^*S) = 1$ , we may suppose that  $f_i(S^*S) > 0$  for each  $i$ . Let  $k_i = [f_i(S^*S)]^{-1/2}$ ,  $S_i = k_i S$ , and define  $g_i(A) = f_i(S_i^*AS_i)$  ( $A \in \mathcal{A}$ ). Then  $f_i(S_i^*S_i) = 1$ , and the argument of the preceding paragraph shows that  $g_i$  is a pure state. For each  $A \in \mathcal{A}$ ,

$$g_i(A) = \frac{f_i(S^*AS)}{f_i(S^*S)} \rightarrow f(S^*AS) = g(A).$$

Hence  $(g_i)$  is a net of pure states which converges to  $g$  in the weak \* topology, so  $g \in \mathcal{P}$ .

(ii) Since  $\phi_f(\mathcal{A})x_f$  is dense in  $H_f$ , we may choose  $S_n \in \mathcal{A}$  ( $n = 1, 2, \dots$ ) such that

$$\|\phi_f(S_n)x_f\| = 1, \quad \|\phi_f(S_n)x_f - x\| \rightarrow 0.$$

Thus  $f(S_n^*S_n) = 1$ , and by part (i) of this lemma, we may define  $g_n$  in  $\mathcal{P}$  by  $g_n(A) = f(S_n^*AS_n)$  ( $A \in \mathcal{A}$ ). Then for each  $A \in \mathcal{A}$ ,

$$g_n(A) = \langle \phi_f(A)\phi_f(S_n)x_f, \phi_f(S_n)x_f \rangle \rightarrow \langle \phi_f(A)x, x \rangle = g(A).$$

Thus  $g \in \mathcal{P}$ .

LEMMA 2. Let  $T \in \mathcal{H}_{\mathcal{B}}$ ,  $S \in \mathcal{B}$ . Then  $S^*TS \in \mathcal{H}_{\mathcal{B}}$ .

*Proof.* Let  $(f_1, f_2) \in \mathcal{E}$ . We have to show that  $f_1(S^*TS) = f_2(S^*TS)$ . Since  $S^*S \in \mathcal{B}$ , we have  $f_1(S^*S) = f_2(S^*S)$ ; and after multiplying  $S$  by a suitable scalar, we may clearly suppose that  $f_1(S^*S)$  is either 0 or 1.

If  $f_i(S^*S) = 0$ , then  $S$  is in the left kernel of  $f_i$  ( $i = 1, 2$ ), and  $f_i(S^*TS) = f_2(S^*TS) = 0$ .

If  $f_i(S^*S) = 1$ , define  $g_i(A) = f_i(S^*AS)$  ( $A \in \mathcal{A}$ ). By Lemma 1 (i),  $g_i \in \mathcal{P}$ . If  $B \in \mathcal{B}$ , then  $S^*BS \in \mathcal{B}$ , so  $f_1(S^*BS) = f_2(S^*BS)$ ; that is,  $g_1(B) = g_2(B)$ . Hence  $(g_1, g_2) \in \mathcal{E}$ , and since  $T \in \mathcal{H}_{\mathcal{B}}$ , it follows that  $g_1(T) = g_2(T)$ ; that is,  $f_1(S^*TS) = f_2(S^*TS)$ . This completes the proof.

LEMMA 3. Let  $T \in \mathcal{H}_{\mathcal{B}}$  and  $R, S \in \mathcal{B}$ . Then  $R^*TS \in \mathcal{H}_{\mathcal{B}}$ .

*Proof.* This follows from Lemma 2 since

$$\begin{aligned} 4R^*TS &= (R + S)^*T(R + S) - (R - S)^*T(R - S) \\ &\quad - i(R + iS)^*T(R + iS) + (R - iS)^*T(R - iS). \end{aligned}$$

LEMMA 4. Let  $f \in \mathcal{P}$  and let  $M$  be a closed subspace of  $H_f$  which is invariant under  $\phi_f(\mathcal{B})$ . Then  $M$  is a invariant under  $\phi_f(\mathcal{H}_{\mathcal{B}})$ .

*Proof.* Suppose that the lemma is false. Then we may choose  $T \in \mathcal{H}_{\mathcal{B}}$  and  $x \in M$  such that  $\phi_f(T)x \notin M$ . Let  $y = (I - E)\phi_f(T)x$ , where  $E$  is the projection from  $H_f$  on to  $M$ . Given  $t$  in  $[0, 2\pi)$ , define  $y_t = x + \exp(it)y$ ,  $z_t = ky_t$ , where

$$k = [\|x\|^2 + \|y\|^2]^{-1/2} = \|y_t\|^{-1}.$$

Thus  $z_t \in H_f$ ,  $\|z_t\| = 1$ , and by Lemma 1 (ii) we may define  $g_t \in \mathcal{P}$  by  $g_t(A) = \langle \phi_f(A)z_t, z_t \rangle$  ( $A \in \mathcal{A}$ ). Since  $\phi_f(\mathcal{B})$  leaves both  $M$  and  $H_f \ominus M$  invariant, it follows that for each  $B \in \mathcal{B}$ ,

$$\begin{aligned} g_t(B) &= k^2 \langle \phi_f(B)(x + e^{it}y), x + e^{it}y \rangle \\ &= k^2 [\langle \phi_f(B)x, x \rangle + \langle \phi_f(B)y, y \rangle], \end{aligned}$$

which is independent of  $t$ . Hence, for each  $s, t$  in  $[0, 2\pi)$ , we have  $(g_s, g_t) \in \mathcal{E}$ . Since  $T \in \mathcal{H}_{\mathcal{B}}$ , it follows that  $g_s(T) = g_t(T)$ ; so  $g_t(T)$  is independent of  $t \in [0, 2\pi)$ . However,

$$\begin{aligned} g_t(T) &= k^2 \langle \phi_f(T)(x + e^{it}y), x + e^{it}y \rangle \\ &= p + qe^{it} + re^{-it}, \end{aligned}$$

where  $p, q, r$  are independent of  $t$  and

$$r = k^2 \langle \phi_f(T)x, y \rangle = k^2 \|y\|^2 \neq 0.$$

Thus  $g_t(T)$  is not independent of  $t \in [0, 2\pi)$ , and we have obtained a contradiction. This proves the lemma.

LEMMA 5.  $\mathcal{H}_{\mathcal{B}}$  is a  $C^*$ -sub-algebra of  $\mathcal{A}$ .

Proof. Suppose that  $(g, h) \in \mathcal{E}$ . Let  $M_g$  be the closed subspace of  $H_g$  which is generated by  $\phi_g(\mathcal{B})x_g$ . It follows from Lemma 4 that  $M_g$  is invariant under  $\phi_g(\mathcal{H}_{\mathcal{B}})$ . When  $T \in \mathcal{H}_{\mathcal{B}}$ , we shall write  $\phi_g(T) | M_g$  for the operator (from  $M_g$  into  $M_g$ ) obtained by restricting  $\phi_g(T)$  to  $M_g$ . Similar notations will be used with  $h$  in place of  $g$ .

Given  $T \in \mathcal{H}_{\mathcal{B}}$  and  $R, S \in \mathcal{B}$ , we have (Lemma 3)  $R^*TS \in \mathcal{H}_{\mathcal{B}}$ . Since  $(g, h) \in \mathcal{E}$ , it follows that  $g(R^*TS) = h(R^*TS)$ , or equivalently that

$$(2) \quad \langle \phi_g(T)\phi_g(S)x_g, \phi_g(R)x_g \rangle = \langle \phi_h(T)\phi_h(S)x_h, \phi_h(R)x_h \rangle.$$

By taking  $T = I$ , we deduce the existence of a unitary operator  $U$  from  $M_g$  on to  $M_h$  such that

$$(3) \quad U\phi_g(S)x_g = \phi_h(S)x_h \quad (S \in \mathcal{B}).$$

Equation (2) then implies that

$$\langle \phi_g(T)v, w \rangle = \langle \phi_h(T)Uv, Uw \rangle \quad (T \in \mathcal{H}_{\mathcal{B}})$$

for all  $v, w \in \phi_g(\mathcal{B})x_g$ , hence for all  $v, w \in M_g$ . The last equation is equivalent to

$$(4) \quad \phi_g(T) | M_g = U^*[\phi_h(T) | M_h]U \quad (T \in \mathcal{H}_{\mathcal{B}}).$$

Now suppose that  $T_1, T_2 \in \mathcal{H}_{\mathcal{B}}$ . Given  $(g, h) \in \mathcal{E}$ , construct  $U$  as

above. Since  $\phi_g(T_i)$  leaves  $M_g$  invariant ( $i = 1, 2$ ), so does  $\phi_g(T_1T_2)$ , and

$$\phi_g(T_1T_2) | M_g = [\phi_g(T_1) | M_g][\phi_g(T_2) | M_g] ;$$

similar considerations apply with  $h$  in place of  $g$ . From (4), with  $T = T_1, T_2$ , we deduce that

$$\phi_g(T_1T_2) | M_g = U^*[\phi_h(T_1T_2) M_h]U .$$

Since  $x_g \in M_g$  and  $Ux_g = x_h$ , the last equation implies that

$$\langle \phi_g(T_1T_2)x_g, x_g \rangle = \langle \phi_h(T_1T_2)x_h, x_h \rangle ;$$

that is,  $g(T_1T_2) = h(T_1T_2)$ . This holds whenever  $(g, h) \in \mathcal{E}$ , so  $T_1T_2 \in \mathcal{H}_{\mathcal{B}}$ .

We have now shown that  $\mathcal{H}_{\mathcal{B}}$  admits multiplication; since  $\mathcal{H}_{\mathcal{B}}$  is clearly a closed self-adjoint linear subspace of  $\mathcal{A}$ , the lemma is proved.

**4. Proof of Theorem 2.** We shall use the notations introduced in the statement of Theorem 2. It is immediate from the definition of  $\mathcal{H}_{\mathcal{B}}$  that  $\mathcal{B} \subseteq \mathcal{H}_{\mathcal{B}}$ .

We first consider the case in which  $I \in \mathcal{B}$ , so that the theory developed in § 3 applies to show that  $\mathcal{H}_{\mathcal{B}}$  is a C\*-algebra. We remark that each element  $f$  of the pure state space  $\mathcal{P}(\mathcal{H}_{\mathcal{B}})$  can be extended to an element  $\bar{f}$  of  $\mathcal{P}(\mathcal{A})$ . For there is a net  $(f_i)$  of pure states of  $\mathcal{H}_{\mathcal{B}}$ , converging to  $f$  in the weak \* topology. Each  $f_i$  can be extended to a pure state  $\bar{f}_i$  of  $\mathcal{A}$  (see, for example, [2] 304). Since  $\mathcal{P}(\mathcal{A})$  is compact, the net  $(\bar{f}_i)$  has at least one weak \* limit point  $\bar{f} \in \mathcal{P}(\mathcal{A})$ , and  $\bar{f}$  is an extension of  $f$ .

Suppose that  $\mathcal{B} \neq \mathcal{H}_{\mathcal{B}}$ . Then by Glimm's theorem there exist distinct  $g, h \in \mathcal{P}(\mathcal{H}_{\mathcal{B}})$  such that  $g(B) = h(B)$  ( $B \in \mathcal{B}$ ). We may extend  $g, h$  to elements,  $\bar{g}, \bar{h}$  respectively of  $\mathcal{P}(\mathcal{A})$ . Clearly  $(\bar{g}, \bar{h}) \in \mathcal{E}$ . Thus, by the definition of  $\mathcal{H}_{\mathcal{B}}$ ,  $\bar{g}(T) = \bar{h}(T)$  whenever  $T \in \mathcal{H}_{\mathcal{B}}$ ; that is,  $g = h$ , contrary to hypothesis. This proves Theorem 2 for the case in which  $I \in \mathcal{B}$ .

If  $I \notin \mathcal{B}$ , let  $\mathcal{B}_1 = \mathcal{B} + CI$  be the C\*-algebra generated by  $I, \mathcal{B}$  ( $C$  denotes the complex field). With an obvious modification of the notation introduced in Theorem 2, it is clear that  $\mathcal{N}(\mathcal{B}_1)$  is empty and that  $\mathcal{E}(\mathcal{B}_1) = \mathcal{E}(\mathcal{B})$ . Thus  $\mathcal{H}_{\mathcal{B}} \subseteq \mathcal{H}_{\mathcal{B}_1}$ ; since  $I \in \mathcal{B}_1$ , the first part of this proof shows that  $\mathcal{B}_1 = \mathcal{H}_{\mathcal{B}_1}$ , so  $\mathcal{H}_{\mathcal{B}} \subseteq \mathcal{B}_1$ .

Now let  $f$  be the pure state of  $\mathcal{B}_1$  defined by  $f(\lambda I + B) = \lambda$  ( $\lambda \in C, B \in \mathcal{B}$ ), and let  $g$  be any extension of  $f$  to a pure state of  $\mathcal{A}$ . Clearly  $g \in \mathcal{N}(\mathcal{B})$ . Hence  $g(\mathcal{H}_{\mathcal{B}}) = (0)$ , and

$$\mathcal{H}_{\mathcal{B}} \subseteq \mathcal{B}_1 \cap g^{-1}(0) = f^{-1}(0) ;$$

that is,  $\mathcal{H}_a \subseteq \mathcal{B}$ . The reverse inclusion has already been noted, so  $\mathcal{B} = \mathcal{H}_a$ .

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