

Pacific Journal of Mathematics

ON SUB-ALGEBRAS OF A C^* -ALGEBRA

JOHN R. RINGROSE

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The following noncommutative extension of the Stone-Weierstrass approximation theorem has been obtained by Glimm.

Theorem. Let \mathcal{A} be a C^* -algebra with identity I , and let \mathcal{B} be a C^* -sub-algebra containing I . Suppose that \mathcal{B} separates the pure state space of \mathcal{A} . Then $\mathcal{B} = \mathcal{A}$.

In the present paper, we apply Glimm's theorem to obtain the following noncommutative generalisation of another result of Stone.

Let \mathcal{A} be a C^* -algebra with identity I and pure state space \mathcal{P} . Let \mathcal{B} be a C^* -sub-algebra of \mathcal{A} , and define

$$\begin{aligned} \mathcal{N} &= \{f: f \text{ is a pure state of } \mathcal{A} \text{ and } f(B) = 0 \ (B \in \mathcal{B})\}, \\ \mathcal{E} &= \{(g, h): g, h \in \mathcal{P} \text{ and } g(B) = h(B) \ (B \in \mathcal{B})\}, \\ \mathcal{H}_{\mathcal{B}} &= \{A: A \in \mathcal{A}, f(A) = 0 \ (f \in \mathcal{N}) \text{ and } g(A) = h(A) \\ &\quad ((g, h) \in \mathcal{E})\}. \end{aligned}$$

Then $\mathcal{B} = \mathcal{H}_{\mathcal{B}}$.

We will refer to this as Theorem 2 in the sequel. Glimm's theorem is to be found in [1]; Stone's, in [3].

Once it is known that $\mathcal{H}_{\mathcal{B}}$ is a C^* -sub-algebra of \mathcal{A} , Theorem 2 is an almost immediate consequence of Glimm's theorem (see § 4). It is clear that $\mathcal{H}_{\mathcal{B}}$ is a closed self-adjoint linear subspace of \mathcal{A} ; accordingly, most of this paper is devoted to proving that $\mathcal{H}_{\mathcal{B}}$ is closed under multiplication (see § 3).

We remark that, if \mathcal{A} is commutative, then \mathcal{P} consists exactly of all homomorphism from \mathcal{A} on to the complex plane \mathbb{C} ; so in this case, it is immediate from its definition that $\mathcal{H}_{\mathcal{B}}$ is a C^* -sub-algebra. However, this seems not to be obvious in the general case. Indeed, for a *general* set \mathcal{N} of pure states of \mathcal{A} and a *general* subset \mathcal{E} of $\mathcal{P} \times \mathcal{P}$, the class

$$\{A: A \in \mathcal{A}, f(A) = 0 \ (f \in \mathcal{N}) \text{ and } g(A) = h(A) \ ((g, h) \in \mathcal{E})\}$$

need not be a sub-algebra of \mathcal{A} ; for example, let \mathcal{A} consist of all bounded linear operators on a Hilbert space H , let \mathcal{N} be void, and let \mathcal{E} consist of a single pair of vector states arising from orthogonal unit vectors.

2. Notation. Throughout, \mathcal{A} is a C^* -algebra-by which we shall mean a uniformly closed self-adjoint algebra of operators acting on a (complex) Hilbert space H . We shall always assume that \mathcal{A} contains

the identity operator I on H . A *state* of \mathcal{A} is a linear functional f on \mathcal{A} such that $f(A^*A) \geq 0$ ($A \in \mathcal{A}$) and $f(I) = 1$. The set of all states is convex and weak * compact; the Krein-Milman theorem ensures the existence of extreme points, and these are called *pure states*. The *pure state space* of \mathcal{A} , denoted by \mathcal{P} (or $\mathcal{P}(\mathcal{A})$ if \mathcal{A} has to be specified), is the weak * closure of the set of all pure states.

Given a state f of \mathcal{A} , there is a *-representation ϕ_f of \mathcal{A} on a Hilbert space H_f , and a unit vector x_f in H_f , such that $\phi_f(\mathcal{A})x_f$ is dense in H_f , and

$$f(A) = \langle \phi_f(A)x_f, x_f \rangle \quad (A \in \mathcal{A}).$$

To within unitary equivalence, ϕ_f is unique. Furthermore, ϕ_f is irreducible if and only if f is a pure state (see, for example, [2] 245, 265, 266). We shall always use the symbols ϕ_f, H_f, x_f in the sense just described.

3. **Some lemmas.** Throughout this section we shall assume that \mathcal{B} is a C^* -sub-algebra of \mathcal{A} , and that $I \in \mathcal{B}$. We use the notations introduced in the statement of Theorem 2; note that, since $I \in \mathcal{B}$, \mathcal{N} is empty and

$$\mathcal{H}_{\mathcal{B}} = \{A : A \in \mathcal{A} \text{ and } g(A) = h(A) \ ((g, h) \in \mathcal{E})\}.$$

For completeness, we give a proof of the following simple result.

LEMMA 1. (i) *Let $f \in \mathcal{P}$, $S \in \mathcal{A}$ and suppose that $f(S^*S) = 1$. Define $g(A) = f(S^*AS)$ ($A \in \mathcal{A}$). Then $g \in \mathcal{P}$.*

(ii) *Let $f \in \mathcal{P}$, $x \in H_f$, $\|x\| = 1$, and define $g(A) = \langle \phi_f(A)x, x \rangle$ ($A \in \mathcal{A}$). Then $g \in \mathcal{P}$.*

Proof. (i) Clearly g is a state. Suppose first that f is a pure state, and let $x = \phi_f(S)x_f$. Then for each $A \in \mathcal{A}$,

$$(1) \quad \langle \phi_f(A)x, x \rangle = \langle \phi_f(S^*AS)x_f, x_f \rangle = f(S^*AS) = g(A).$$

With $A = I$ we obtain $\|x\| = 1$; and since f is a pure state, ϕ_f is irreducible, so $\phi_f(\mathcal{A})x$ is dense in H_f . This, with (1), implies that ϕ_f and ϕ_g are unitarily equivalent. Thus ϕ_g is irreducible, so g is pure.

Now suppose only that $f \in \mathcal{P}$. There is a net (f_i) of pure states which converges to f in the weak * topology. Since $f_i(S^*S) \rightarrow f(S^*S) = 1$, we may suppose that $f_i(S^*S) > 0$ for each i . Let $k_i = [f_i(S^*S)]^{-1/2}$, $S_i = k_i S$, and define $g_i(A) = f_i(S_i^*AS_i)$ ($A \in \mathcal{A}$). Then $f_i(S_i^*S_i) = 1$, and the argument of the preceding paragraph shows that g_i is a pure state. For each $A \in \mathcal{A}$,

$$g_i(A) = \frac{f_i(S^*AS)}{f_i(S^*S)} \rightarrow f(S^*AS) = g(A) .$$

Hence (g_i) is a net of pure states which converges to g in the weak * topology, so $g \in \mathcal{P}$.

(ii) Since $\phi_f(\mathcal{A})x_f$ is dense in H_f , we may choose $S_n \in \mathcal{A}$ ($n = 1, 2, \dots$) such that

$$\|\phi_f(S_n)x_f\| = 1, \quad \|\phi_f(S_n)x_f - x\| \rightarrow 0 .$$

Thus $f(S_n^*S_n) = 1$, and by part (i) of this lemma, we may define g_n in \mathcal{P} by $g_n(A) = f(S_n^*AS_n)$ ($A \in \mathcal{A}$). Then for each $A \in \mathcal{A}$,

$$g_n(A) = \langle \phi_f(A)\phi_f(S_n)x_f, \phi_f(S_n)x_f \rangle \rightarrow \langle \phi_f(A)x, x \rangle = g(A) .$$

Thus $g \in \mathcal{P}$.

LEMMA 2. Let $T \in \mathcal{H}_{\mathcal{B}}$, $S \in \mathcal{B}$. Then $S^*TS \in \mathcal{H}_{\mathcal{B}}$.

Proof. Let $(f_1, f_2) \in \mathcal{E}$. We have to show that $f_1(S^*TS) = f_2(S^*TS)$. Since $S^*S \in \mathcal{B}$, we have $f_1(S^*S) = f_2(S^*S)$; and after multiplying S by a suitable scalar, we may clearly suppose that $f_1(S^*S)$ is either 0 or 1.

If $f_i(S^*S) = 0$, then S is in the left kernel of f_i ($i = 1, 2$), and $f_1(S^*TS) = f_2(S^*TS) = 0$.

If $f_i(S^*S) = 1$, define $g_i(A) = f_i(S^*AS)$ ($A \in \mathcal{A}$). By Lemma 1 (i), $g_i \in \mathcal{P}$. If $B \in \mathcal{B}$, then $S^*BS \in \mathcal{B}$, so $f_1(S^*BS) = f_2(S^*BS)$; that is, $g_1(B) = g_2(B)$. Hence $(g_1, g_2) \in \mathcal{E}$, and since $T \in \mathcal{H}_{\mathcal{B}}$, it follows that $g_1(T) = g_2(T)$; that is, $f_1(S^*TS) = f_2(S^*TS)$. This completes the proof.

LEMMA 3. Let $T \in \mathcal{H}_{\mathcal{B}}$ and $R, S \in \mathcal{B}$. Then $R^*TS \in \mathcal{H}_{\mathcal{B}}$.

Proof. This follows from Lemma 2 since

$$\begin{aligned} 4R^*TS &= (R + S)^*T(R + S) - (R - S)^*T(R - S) \\ &\quad - i(R + iS)^*T(R + iS) + (R - iS)^*T(R - iS) . \end{aligned}$$

LEMMA 4. Let $f \in \mathcal{P}$ and let M be a closed subspace of H_f which is invariant under $\phi_f(\mathcal{B})$. Then M is a invariant under $\phi_f(\mathcal{H}_{\mathcal{B}})$.

Proof. Suppose that the lemma is false. Then we may choose $T \in \mathcal{H}_{\mathcal{B}}$ and $x \in M$ such that $\phi_f(T)x \notin M$. Let $y = (I - E)\phi_f(T)x$, where E is the projection from H_f on to M . Given t in $[0, 2\pi)$, define $y_t = x + \exp(it)y$, $z_t = ky_t$, where

$$k = [\|x\|^2 + \|y\|^2]^{-1/2} = \|y_t\|^{-1} .$$

Thus $z_t \in H_f$, $\|z_t\| = 1$, and by Lemma 1 (ii) we may define $g_t \in \mathcal{P}$ by $g_t(A) = \langle \phi_f(A)z_t, z_t \rangle$ ($A \in \mathcal{A}$). Since $\phi_f(\mathcal{B})$ leaves both M and $H_f \ominus M$ invariant, it follows that for each $B \in \mathcal{B}$,

$$\begin{aligned} g_t(B) &= k^2 \langle \phi_f(B)(x + e^{it}y), x + e^{it}y \rangle \\ &= k^2 [\langle \phi_f(B)x, x \rangle + \langle \phi_f(B)y, y \rangle], \end{aligned}$$

which is independent of t . Hence, for each s, t in $[0, 2\pi)$, we have $(g_s, g_t) \in \mathcal{E}$. Since $T \in \mathcal{H}_{\mathcal{B}}$, it follows that $g_s(T) = g_t(T)$; so $g_t(T)$ is independent of $t \in [0, 2\pi)$. However,

$$\begin{aligned} g_t(T) &= k^2 \langle \phi_f(T)(x + e^{it}y), x + e^{it}y \rangle \\ &= p + qe^{it} + re^{-it}, \end{aligned}$$

where p, q, r are independent of t and

$$r = k^2 \langle \phi_f(T)x, y \rangle = k^2 \|y\|^2 \neq 0.$$

Thus $g_t(T)$ is not independent of $t \in [0, 2\pi)$, and we have obtained a contradiction. This proves the lemma.

LEMMA 5. $\mathcal{H}_{\mathcal{B}}$ is a C^* -sub-algebra of \mathcal{A} .

Proof. Suppose that $(g, h) \in \mathcal{E}$. Let M_g be the closed subspace of H_g which is generated by $\phi_g(\mathcal{B})x_g$. It follows from Lemma 4 that M_g is invariant under $\phi_g(\mathcal{H}_{\mathcal{B}})$. When $T \in \mathcal{H}_{\mathcal{B}}$, we shall write $\phi_g(T) | M_g$ for the operator (from M_g into M_g) obtained by restricting $\phi_g(T)$ to M_g . Similar notations will be used with h in place of g .

Given $T \in \mathcal{H}_{\mathcal{B}}$ and $R, S \in \mathcal{B}$, we have (Lemma 3) $R^*TS \in \mathcal{H}_{\mathcal{B}}$. Since $(g, h) \in \mathcal{E}$, it follows that $g(R^*TS) = h(R^*TS)$, or equivalently that

$$(2) \quad \langle \phi_g(T)\phi_g(S)x_g, \phi_g(R)x_g \rangle = \langle \phi_h(T)\phi_h(S)x_h, \phi_h(R)x_h \rangle.$$

By taking $T = I$, we deduce the existence of a unitary operator U from M_g on to M_h such that

$$(3) \quad U\phi_g(S)x_g = \phi_h(S)x_h \quad (S \in \mathcal{B}).$$

Equation (2) then implies that

$$\langle \phi_g(T)v, w \rangle = \langle \phi_h(T)Uv, Uw \rangle \quad (T \in \mathcal{H}_{\mathcal{B}})$$

for all $v, w \in \phi_g(\mathcal{B})x_g$, hence for all $v, w \in M_g$. The last equation is equivalent to

$$(4) \quad \phi_g(T) | M_g = U^*[\phi_h(T) | M_h]U \quad (T \in \mathcal{H}_{\mathcal{B}}).$$

Now suppose that $T_1, T_2 \in \mathcal{H}_{\mathcal{B}}$. Given $(g, h) \in \mathcal{E}$, construct U as

above. Since $\phi_g(T_i)$ leaves M_g invariant ($i = 1, 2$), so does $\phi_g(T_1T_2)$, and

$$\phi_g(T_1T_2) | M_g = [\phi_g(T_1) | M_g][\phi_g(T_2) | M_g] ;$$

similar considerations apply with h in place of g . From (4), with $T = T_1, T_2$, we deduce that

$$\phi_g(T_1T_2) | M_g = U^*[\phi_h(T_1T_2) M_h]U .$$

Since $x_g \in M_g$ and $Ux_g = x_h$, the last equation implies that

$$\langle \phi_g(T_1T_2)x_g, x_g \rangle = \langle \phi_h(T_1T_2)x_h, x_h \rangle ;$$

that is, $g(T_1T_2) = h(T_1T_2)$. This holds whenever $(g, h) \in \mathcal{E}$, so $T_1T_2 \in \mathcal{H}_g$.

We have now shown that \mathcal{H}_g admits multiplication; since \mathcal{H}_g is clearly a closed self-adjoint linear subspace of \mathcal{A} , the lemma is proved.

4. Proof of Theorem 2. We shall use the notations introduced in the statement of Theorem 2. It is immediate from the definition of \mathcal{H}_g that $\mathcal{B} \subseteq \mathcal{H}_g$.

We first consider the case in which $I \in \mathcal{B}$, so that the theory developed in § 3 applies to show that \mathcal{H}_g is a C*-algebra. We remark that each element f of the pure state space $\mathcal{P}(\mathcal{H}_g)$ can be extended to an element \bar{f} of $\mathcal{P}(\mathcal{A})$. For there is a net (f_i) of pure states of \mathcal{H}_g , converging to f in the weak * topology. Each f_i can be extended to a pure state \bar{f}_i of \mathcal{A} (see, for example, [2] 304). Since $\mathcal{P}(\mathcal{A})$ is compact, the net (\bar{f}_i) has at least one weak * limit point $\bar{f} \in \mathcal{P}(\mathcal{A})$, and \bar{f} is an extension of f .

Suppose that $\mathcal{B} \neq \mathcal{H}_g$. Then by Glimm's theorem there exist distinct $g, h \in \mathcal{P}(\mathcal{H}_g)$ such that $g(B) = h(B)$ ($B \in \mathcal{B}$). We may extend g, h to elements, \bar{g}, \bar{h} respectively of $\mathcal{P}(\mathcal{A})$. Clearly $(\bar{g}, \bar{h}) \in \mathcal{E}$. Thus, by the definition of \mathcal{H}_g , $\bar{g}(T) = \bar{h}(T)$ whenever $T \in \mathcal{H}_g$; that is, $g = h$, contrary to hypothesis. This proves Theorem 2 for the case in which $I \in \mathcal{B}$.

If $I \notin \mathcal{B}$, let $\mathcal{B}_1 = \mathcal{B} + CI$ be the C*-algebra generated by I, \mathcal{B} (C denotes the complex field). With an obvious modification of the notation introduced in Theorem 2, it is clear that $\mathcal{N}(\mathcal{B}_1)$ is empty and that $\mathcal{E}(\mathcal{B}_1) = \mathcal{E}(\mathcal{B})$. Thus $\mathcal{H}_g \subseteq \mathcal{H}_{g_1}$; since $I \in \mathcal{B}_1$, the first part of this proof shows that $\mathcal{B}_1 = \mathcal{H}_{g_1}$, so $\mathcal{H}_g \subseteq \mathcal{B}_1$.

Now let f be the pure state of \mathcal{B}_1 defined by $f(\lambda I + B) = \lambda$ ($\lambda \in C, B \in \mathcal{B}$), and let g be any extension of f to a pure state of \mathcal{A} . Clearly $g \in \mathcal{N}(\mathcal{B})$. Hence $g(\mathcal{H}_g) = (0)$, and

$$\mathcal{H}_g \subseteq \mathcal{B}_1 \cap g^{-1}(0) = f^{-1}(0) ;$$

that is, $\mathcal{H}_a \subseteq \mathcal{B}$. The reverse inclusion has already been noted, so $\mathcal{B} = \mathcal{H}_a$.

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