SOME RESULTS IN THE LOCATION OF ZEROS OF POLYNOMIALS

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Three out of the four theorems proved in this paper deal with the location of the zeros of a polynomial $P(z)$ whose zeros $z_i, i = 1, 2, \ldots, n$ satisfy the conditions $|z_i| \leq 1$, and $\sum_{i=1}^{n} z_i^p = 0$ for $p = 1, 2, \ldots, l$. One of those estimates is

$$\left| \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} \right| < \frac{l + 1}{|z|(|z|^{l+1} - 1)}$$

for $|z| > 1$.

The fourth result is of a different nature. It refines, in particular, a theorem due to Eneström and Kakeya. It is shown that no zero of the polynomial $h(z) = \sum_{k=0}^{r} b_k z^k$ lies in the disk

$$|z - \frac{\beta e^{-i\theta}}{(\beta + 1)}| < \frac{1}{\beta + 1},$$

where $\beta = \max_{|z|=1} |h'(z)|/\max_{|z|=1} |h(z)|$, and $\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$.

We generalize and strengthen certain well-known results due to Biernacki [1], Dieudonné [3, 5], and Kakeya [8].

We use repeatedly a recent result due to Walsh which is a generalized form of an earlier theorem of his [10]. It concerns the case in which all the zeros of a polynomial lie within a certain distance of their centroid.

**Theorem 1.** Let $h(z) = \sum_{k=0}^{r} b_k z^k (b_k \text{ complex}),$

$$\beta = \frac{\max_{|z|=1} |h'(z)|}{\max_{|z|=1} |h(z)|},$$

$\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$, and let $C_\beta$ be the disk $|z - \beta e^{-i\theta}/(\beta + 1)| < 1/(\beta + 1)$, then no zero of $h$ lies in $C_\beta$.

**Proof.** Consider the function $F(z) = e^{-i\theta} h(z e^{i\theta})/m$, where $h(e^{i\theta}) = me^{i\theta}$. Then $F$ satisfies the conditions, $|F(z)| < 1$ in $|z| < 1$, $F(1) = 1$. Let $x_n \to 1$ as $n \to \infty$, $0 < x_n < 1$, and let $\alpha = \lim_{n \to \infty} [(1 - |F(x_n)|)/(1 - x_n)]$. Then $\alpha \leq |F''(1)|$. It follows readily (see [2] p. 57) that

$$\lim_{n \to \infty} [(1 - |F(x_n)|)/(1 - x_n)] = F''(1) = e^{i(\theta - \phi)} h'(e^{i\theta})/m = |h'(e^{i\theta})|/m.$$

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We apply now the following result due to Julia [2]: If a function \( f \) is regular in the unit disc and \( |f(z)| < 1 \) for \( |z| < 1 \), and there exists a sequence of number \( z_1, \ldots, z_n, \ldots \) such that \( \lim_{n \to -\infty} z_n = 1 \), \( \lim_{n \to -\infty} (1 - |f(z_n)|)/(1 - |z_n|) = \alpha \) then

\[
\left| \frac{1 - f(z)}{1 - |f(z)|^2} \right| \leq \frac{1 - |1 - z|^2}{1 - |z|^2} \quad \text{for } |z| < 1.
\]

In (1), set \( f(z) = F(z), \alpha = |h'(e^{i\theta})|/m \). If \( F(z_0) = 0 \) and \( |z_0| < 1 \), then \( (1 - |z_0|^2)/|1 - z_0|^2 \leq \alpha \), which is equivalent to \( e^{-i\theta}z_0 \in C_\alpha \). Since \( \alpha \leq \beta \), it follows that \( C_\beta \subset C_\alpha \); hence \( e^{-i\theta}z_0 \in C_\beta \), which concludes the proof.

**Corollary 1.** Let \( h(z) = \sum_{k=0}^{n} b_k z^k, b_k > 0 \). Then \( \beta = \sum_{k=1}^{n} kb_k/\sum_{k=0}^{n} b_k \), and no zero is in the disc

\[
\left| z - \frac{\sum_{k=0}^{n} kb_k}{\sum_{k=0}^{n} (k + 1)b_k} \right| < \frac{\sum_{k=0}^{n} b_k}{\sum_{k=0}^{n} (k + 1)b_k}.
\]

In particular, if \( b_k \) is a strictly increasing sequence, then all the zeros of \( h(z) \) lie in the complement of \( C_\beta \) with respect to the unit disc. This makes more precise the theorem of Enestrom and Kakeya [8].

In a recent paper, Tchakaloff [9] (see also [7]) has proved that if all the zeros of the polynomials

\[
P_k(z) = a_n^{(k)} z^n + \cdots + a_0^{(k)}(a_n^{(k)}) > 0, k = 1, \ldots, m
\]

lie in the unit disc and if \( A_k > 0(k = 1, \ldots, m) \), then all the zeros of the polynomial \( \sum_{k=1}^{m} A_k P_k(z) \) lie in the disc \( |z| \leq 1/\sin(\pi/2n) \), and that this is the best possible result. We prove a more precise result in the case where there is more information about the zeros of \( P_k(z) \).

**Theorem 2.** Let the polynomials \( P_k(z)(k = 1, \ldots, m) \) of the form (2) have all their zeros \( z_{ik}(i = 1, \ldots, n; k = 1, \ldots, m) \) in the unit disc and let \( A_k > 0(k = 1, \ldots, m) \). Suppose that \( \sum_{i=1}^{n} \bar{z}_{ik}^p \neq 0 \) for \( p = 1, \ldots, l(k = 1, \ldots, m) \). Then all the zeros of the polynomial \( \sum_{k=1}^{m} A_k P_k(z) \) lie in the disc \( |z| \leq (\sin \pi/2n)^{-1/(1+l)} \). For values of the form \( n = (l+1)r \), the exact bound does not exceed \( (\sin \pi/(l+1))/2n)^{-1/(1+l)} \).

**Proof.** Without loss of generality we may assume that \( a_n^{(k)} = 1 \). By a recent result due to Walsh [11] the polynomials \( P_k \) satisfy the equality \( P_k(z) = (z - \varphi_k(z))^s \), where \( |\varphi_k(z)| < |z|^{-l} \) for \( |z| > 1 \). Let \( \zeta \) be a point outside the unit disc at which the circle \( |z| = |\zeta|^{-l} \)
subtends an angle \( \Psi \). On the circle \(|z| = |\zeta|^{-1}\) there exists a point \( a \), such that \( 0 \leq \arg ((\zeta - \varphi_k)/(\zeta - a)) \leq \Psi \), and

\[
\sum_{k=1}^{m} A_k P_k(\zeta) = (\zeta - a)^{n} \sum_{k=1}^{m} A_k \left( \frac{\zeta - \varphi_k}{\zeta - a} \right)^{n}.
\]

One deduces from equation (3) that

\[
\sum_{k=1}^{m} A_k P_k(\zeta) \neq 0 \text{ if } \Psi < \frac{\pi}{n}.
\]

For \( \Psi = \pi/n \), \( \sin (\pi/2n) = |\zeta|^{-(l+1)} \). This proves the first part of the theorem. The example \( A_1 = A_2 = 1, m = 2, P_1(z) = (z^{l+1} + \mu)^r, P_2(z) = (z^{l+1} + \overline{\mu})^r \), where \( \mu = i \exp (i\pi/2n) \), proves the second part of the theorem, since in this case the polynomial \( P_1(z) + P_2(z) \) has the zero

\[
z = \left[ \sin \frac{\pi(l + 1)}{2n} \right]^{-1/(l+1)}.
\]

Dieudonné has proved [3], (for a different proof see [4]), that if the polynomial \( P \) has all its zeros in the closed unit disc, then

\[
\left| \frac{P'(z)}{P(z)} - \frac{P''(z)}{P'(z)} \right| \leq \frac{1}{|z| - 1}, \quad \text{for } |z| > 1.
\]

We give a short proof of (4), which at the same time yields a stronger inequality in the case where the centroid of the zeros of \( P \) is at the origin.

**Theorem 3.** If all the zeros \( z_i (i = 1, \cdots, n) \) of the polynomial \( P(z) \) lie in the closed unit disc and if \( \sum_{i=1}^{n} z_i^k = 0 (k = 1, \cdots, l) \), then for \( |z| > 1 \) the following sharp estimate holds

\[
\left| \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} \right| \leq \frac{l + 1}{|z|(|z|^{l+1} - 1)}.
\]

Inequality (5) holds also for \( l = 0 \), in which case the second condition imposed on the \( z_i \) is to be omitted.

**Proof.** By a recent result due to Walsh [12], there exists a function \( \varphi(z), |\varphi(z)| < |z|^{-1} \), such that for \( |z| > 1 \)

\[
\frac{P'(z)}{P(z)} = \frac{n}{z - \varphi(z)}.
\]

An estimate due to Goluzin [6], applied to \( \varphi \) yields the inequality

\[
|\varphi'(z)| \leq \frac{l|z|^{l-1}}{|z|^l - 1} (1 - |\varphi(z)|^2),
\]
for $|z| > 1$. Since by (6)

$$\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} = \frac{\varphi(z) - z\varphi'(z)}{z(z - \varphi(z))}$$

is follows, using (7), that

$$\left| \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} \right| \leq \frac{1}{|z|} \left[ \frac{|\varphi(z)|}{|z| - |\varphi(z)|} + \frac{l|z|^l}{|z|^{2l} - 1} \frac{1 - |\varphi(z)|^2}{|z|} \right]$$

It remains to prove the inequality

$$\frac{x}{a - x} + \frac{la}{a^{l+1} - 1} \frac{1 - x^2}{a - x} \leq \frac{l + 1}{a^{l+1} - 1}$$

for all $0 \leq x \leq a^{-1}$, and $a > 1$.

If we denote the left hand side of (9) by $f(x)$, then $f(a^{-l}) = (l + 1)/(a^{l+1} - 1)$, and $f'(x) \geq 0$ provided the function $g(x) = a^{2l+1} - a + la(x^2 - 2ax + 1)$ is nonnegative. Since $g'(x) \leq 0$ it is enough to show that $h(a) = g(a^{-l})$ is nonnegative. Indeed one verifies that $h(1) = 0$ and $h'(a) > 0$ for all $a > 1$.

The particular case $P(z) = z^n - 1, l = n - 1$, shows that the bound (5) cannot, in general, be improved.

The result due to Dieudonné follows from (7) and (8).

Finally, we discuss a problem raised by Biernacki [1], which was also treated by Dieudonné [5], namely that of determining a region containing all but, possibly, one zero of the polynomial $aP(z) + P'(z)$ for all complex $a$. Each of the above authors has proved that if all the zeros of $P$ lie in the unit disc, then the concentric disc of radius $2^{1/2}$ is the smallest concentric disc that has the above mentioned property. Assuming additional information about the zeros of $P$, we obtain a smaller disc for all but possibly $l + 1$ zeros of the polynomial $z^lP(z) + aP'(z)$.

**Theorem 4.** If all the zeros $z_i (i = 1, \ldots, n)$ of the polynomial $P(z)$ lie in the closed unit disc and if $\sum_{i=1}^{n} z_i^k = 0 (k = 1, \ldots, l)$, then for all complex $a$ at least $n - 1$ zeros of the polynomial $z^lP(z) + aP'(z)$ lie in the disc $|z| \leq 2^{l/(2(l+1))}$.

**Proof.** Proceeding as in the proof of Theorem 3, we have

$$\frac{P'(z)}{P(z)} = -\frac{z^l}{a} = \frac{n}{z - \varphi(z)},$$
satisfied by any zero of the polynomial $z^l P + aP'$ which exceeds 1 in modulus. Set $g(z) = z^{-1} \varphi(1/z)$, $w = z^{l+1}$ and $h(w) = g(z)$. Then $|g(z)| < 1$ if $|z| < 1$ and

\begin{align}
(10) & \quad g(z) = \frac{1}{z^{l+1}} + an \\
(11) & \quad h(w) = \frac{1}{w} + an.
\end{align}

If for some $a$ the polynomial $z^l P + aP'$ has at most $n - 2$ zeros in the disc $|z| \leq 2^{l/(2(l+1))}$, then equation (10) has at least $l + 2$ roots in the disc $|z| < 2^{-l/(2(l+1))}$, and hence equation (11) has at least two roots in the disc $|w| < 2^{-l/2}$. This was proved to be impossible in [5].

Theorem 4 is sharp for all $l$ and $n$ of the form $n = 2k(l + 1), k = 1, 2, \ldots$. The upper limit is attained by the zeros of the polynomial

$$P(z) = (z^{2l+2} - 2^{l/2}z^{l+1} + 1)^{n/(2(l+1))}.$$ 

**References**


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