

Pacific Journal of Mathematics

**ON SIMPLE ALGEBRAS OBTAINED FROM HOMOGENEOUS
GENERAL LIE TRIPLE SYSTEMS**

ARTHUR ARGYLE SAGLE

ON SIMPLE ALGEBRAS OBTAINED FROM HOMOGENEOUS GENERAL LIE TRIPLE SYSTEMS

ARTHUR A. SAGLE

We continue the investigation of the simple anti-commutative algebras obtained from a homogeneous general L.t.s. In particular we consider the algebra which satisfies

$$(1) \quad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy).$$

The usual process of analyzing a nonassociative algebra is to decompose it relative to elements whose right and left multiplications are diagonalizable linear transformations e.g. idempotents or Cartan subalgebras. In this paper we show that such a process yields only Lie algebras and indicates the difficulty in finding any non-Lie multiplication table for a simple anticommutative algebra satisfying (1).

A general Lie triple system [2] is an extension of a Lie triple system used in differential geometry and Jordan algebras. A general L.t.s. may be regarded as an anti-commutative algebra A with a trilinear operation $[x, y, z]$ so that the mappings $D(x, y): z \rightarrow [x, y, z]$ are derivations of A which generate a Lie algebra, $I(A)$, under commutation satisfying certain natural identities. A homogeneous general L.t.s. is a general L.t.s. for which the operation $[x, y, z]$ is a homogeneous expression in the products of x, y and z ; that is, using anti-commutativity, $[x, y, z] = \alpha xy \cdot z + \beta yz \cdot x + \gamma zx \cdot y$ for some fixed α, β, γ in the base field. From [1] we see that if A is a homogeneous general L.t.s. over a field of characteristic zero which is either an irreducible general L.t.s. or $I(A)$ -irreducible or a simple algebra, then A is a Lie or Malcev algebra or satisfies

$$(1) \quad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy)$$

where $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$. The main result of this paper is the following theorem.

THEOREM. *If A is a simple finite dimensional anti-commutative algebra over a field F of characteristic zero which satisfies (1) and if A contains a nonzero element u so that right multiplication by u, R_u , is a diagonalizable linear transformation, then A is a Lie algebra.*

2. **Proof of theorem.** For any anti-commutative algebra we have the identity

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned}$$

But using (1) we also have

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = -2[J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y)] . \end{aligned}$$

Thus using the two preceding identities we have

$$(2) \quad \begin{aligned} J(w, xy, z) + J(w, yz, x) + J(w, zx, y) \\ = J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned}$$

Now let $u \neq 0$ be an element of A so that $R_u: x \rightarrow xu$ is a diagonalizable linear transformation. Then $R_u \neq 0$, for this implies that the one dimensional subspace uF is an ideal of A and therefore equals A . Thus $A^2 = 0$, a contradiction to the simplicity of A . Since R_u acts diagonally in A we may write

$$A = A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$$

where

$$A_\lambda = \{x \in A : x(R_u - \lambda I) = 0\} .$$

We shall now prove

$$(3) \quad A_\alpha A_\beta \subset A_{\alpha+\beta} .$$

For let $x \in A_\alpha, y \in A_\beta$, then from (1)

$$\begin{aligned} J(u, x, y)R_u &= J(u, u, xy) + J(u, x, yu) + J(u, y, ux) \\ &= \beta J(u, x, y) - \alpha J(u, y, x) \\ &= (\alpha + \beta)J(u, x, y) . \end{aligned}$$

Thus $J(u, x, y) \in A_{\alpha+\beta}$ and therefore

$$xy(R_u - (\alpha + \beta)I) = xy \cdot u + yu \cdot x + ux \cdot y \in A_{\alpha+\beta} .$$

From this $xy(R_u - (\alpha + \beta)I)^2 = 0$ and setting $xy = \sum z_\gamma \in A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$ we see by the diagonal action of R_u that $xy \in A_{\alpha+\beta}$. In particular (3) shows A_0 is a subalgebra of A .

Next we shall show

$$(4) \quad J(A_\alpha, A_\beta, A_\gamma) = 0 \quad \text{or} \quad \alpha + \beta + \gamma = 0$$

for any characteristic roots α, β, γ of R_u . Let $x \in A_\alpha, y \in A_\beta, z \in A_\gamma$, then from (3) $J(x, y, z) \in A_{\alpha+\beta+\gamma}$ and therefore

$$\begin{aligned} (\alpha + \beta + \gamma)J(x, y, z) &= J(x, y, z)R_u \\ &= J(u, x, yz) + J(u, y, zx) + J(u, z, xy) \\ &= -\alpha x \cdot yz + (\alpha + \beta + \gamma)x \cdot yz + (\beta + \gamma)yz \cdot x \\ &\quad - \beta y \cdot zx + (\alpha + \beta + \gamma)y \cdot zx + (\alpha + \gamma)zx \cdot y \\ &\quad - \gamma z \cdot xy + (\alpha + \beta + \gamma)z \cdot xy + (\alpha + \beta)xy \cdot z \\ &= 0. \end{aligned}$$

and this equation proves (4).

from (1) and (3) we have

$$J(A_0, A_0, A_0)A_0 \subset J(A_0, A_0, A_0)$$

and for $\alpha \neq 0$ we have from (1), (3) and (4),

$$\begin{aligned} J(A_0, A_0, A_0)A_\alpha &\subset J(A_\alpha, A_0, A_0) \\ &= 0. \end{aligned}$$

Thus $J(A_0, A_0, A_0)A \subset J(A_0, A_0, A_0)$ and therefore $J(A_0, A_0, A_0)$ is an ideal of A which is contained in $A_0 \neq A$. Since A is a simple algebra this yields

$$(5) \quad J(A_0, A_0, A_0) = 0.$$

Next we shall prove that if α is a nonzero characteristic root so that $-\alpha$ is also a characteristic root, then

$$(6) \quad J(A_\alpha, A_{-\alpha}, A_0) = 0.$$

For using (1), (3) and (5) we obtain

$$J(A_\alpha, A_{-\alpha}, A_0)A_0 \subset J(A_\alpha, A_{-\alpha}, A_0)$$

and for any $\beta \neq 0$ we also obtain

$$\begin{aligned} J(A_\alpha, A_{-\alpha}, A_0)A_\beta &\subset J(A_\beta, A_\alpha, A_{-\alpha}A_0) \\ &\quad + J(A_\beta, A_{-\alpha}, A_0A_\alpha) \\ &\quad + J(A_\beta, A_0, A_\alpha A_{-\alpha}) \\ &\subset J(A_\beta, A_\alpha, A_{-\alpha}) + J(A_\beta, A_0, A_0) \\ &= 0, \end{aligned}$$

also using (4). Thus as in the proof of (5), $J(A_\alpha, A_{-\alpha}, A_0)$ is an ideal of A which must be zero. Adopting the usual convention that if α is a characteristic root but $-\alpha$ is not, then $A_{-\alpha} = 0$ we see that (6) holds

for any characteristic root α .

Next let

$$B = \sum_{\alpha \neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha \neq 0} A_\alpha ,$$

then if $\beta \neq 0$ we see from (3) that $BA_\beta \subset B$. If $\beta = 0$, then from (6) we obtain $(A_\alpha A_{-\alpha})A_0 \subset A_\alpha A_{-\alpha}$ and therefore $BA_0 \subset B$. Thus B is an ideal of A and therefore $B = 0$ or $B = A$. If $B = 0$, then $R_u = 0$, a contradiction. Therefore we have

$$(7) \quad A = \sum_{\alpha \neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha \neq 0} A_\alpha .$$

Now from (4) and (6) we have for any characteristic roots β and $\alpha \neq 0$, $J(A_\alpha, A_{-\alpha}, A_\beta) = 0$ and therefore

$$(8) \quad J(A_\alpha, A_{-\alpha}, A) = 0 \quad (\alpha \neq 0) .$$

We shall use (7) and (8) together with the following lemma to prove A is a Lie algebra.

LEMMA. Let $N = \{x \in A : J(x, A, A) = 0\}$, then

- (i) $J(a, b, A) = 0$ implies $ab \in N$;
- (ii) N is an ideal of A which is a Lie algebra.

Proof. Clearly (ii) follows from (i). So let $a, b \in A$ be such that $J(a, b, A) = 0$ and let $w, z \in A$. Then from (1) and (2) we have

$$(9) \quad \begin{aligned} 0 &= wJ(a, b, z) \\ &= J(w, ab, z) + J(w, bz, a) + J(w, za, b) \\ &= J(wa, b, z) + J(wb, z, a), \text{ using (2) .} \end{aligned}$$

Now interchanging z and w in this last equation we obtain $0 = J(za, b, w) + J(zb, w, a) = J(w, bz, a) + J(w, za, b)$ and using this in (9) yields $J(ab, w, z) = 0$; that is, $ab \in N$.

To show that A is a Lie algebra, suppose it is not. Then from the lemma $N = 0$ and from (8) $A_\alpha A_{-\alpha} \subset N = 0$. Thus from (7) $A = \sum_{\alpha \neq 0} A_\alpha$ and therefore $A_0 = 0$; this contradicts $0 \neq u \in A_0$.

BIBLIOGRAPHY

1. A. Sagle, *On anti-commutative algebras and general Lie triple systems*, to appear in Pacific J. Math.
2. K. Yamaguti, *On the Lie triple system and its generalization*, J. Sci. Hiroshima University, **21** (1958), 155-160.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California

R. M. BLUMENTHAL

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California
Los Angeles, California 90007

*RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Basil Gordon, Acting Managing Editor until February 1, 1966.

Robert James Blattner, <i>Group extension representations and the structure space</i>	1101
Glen Eugene Bredon, <i>On the continuous image of a singular chain complex</i>	1115
David Hilding Carlson, <i>On real eigenvalues of complex matrices</i>	1119
Hsin Chu, <i>Fixed points in a transformation group</i>	1131
Howard Benton Curtis, Jr., <i>The uniformizing function for certain simply connected Riemann surfaces</i>	1137
George Wesley Day, <i>Free complete extensions of Boolean algebras</i>	1145
Edward George Effros, <i>The Borel space of von Neumann algebras on a separable Hilbert space</i>	1153
Michel Mendès France, <i>A set of nonnormal numbers</i>	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i>	1171
Frederick Paul Greenleaf, <i>Norm decreasing homomorphisms of group algebras</i>	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i>	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i>	1239
James Patrick Jans, <i>Some aspects of torsion</i>	1249
Laura Ketchum Kodama, <i>Boundary measures of analytic differentials and uniform approximation on a Riemann surface</i>	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic functions</i>	1279
Ronald John Larsen, <i>Almost invariant measures</i>	1295
You-Feng Lin, <i>Generalized character semigroups: The Schwarz decomposition</i>	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i>	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i>	1319
Mitsuru Nakai, <i>Φ-bounded harmonic functions and classification of Riemann surfaces</i>	1329
L. G. Nova, <i>On n-ordered sets and order completeness</i>	1337
Fredos Papan gelou, <i>Some considerations on convergence in abelian lattice-groups</i>	1347
Frank Albert Raymond, <i>Some remarks on the coefficients used in the theory of homology manifolds</i>	1365
John R. Ringrose, <i>On sub-algebras of a C^*-algebra</i>	1377
Jack Max Robertson, <i>Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres</i>	1383
Zalman Rubinstein, <i>Some results in the location of zeros of polynomials</i>	1391
Arthur Argyle Sagle, <i>On simple algebras obtained from homogeneous general Lie triple systems</i>	1397
Hans Samelson, <i>On small maps of manifolds</i>	1401
Annette Sinclair, <i>$\varepsilon(z)$-closeness of approximation</i>	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional curvature</i>	1415
Earl J. Taft, <i>Invariant splitting in Jordan and alternative algebras</i>	1421
L. E. Ward, <i>On a conjecture of R. J. Koch</i>	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i>	1435
Horace C. Wiser, <i>Embedding a circle of trees in the plane</i>	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i>	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their application to learning theory</i>	1471
Charles Vernon Coffman, <i>Correction to: Non-linear differential equations on cones in Banach spaces</i>	1472
P. H. Doyle, III, <i>Correction to: A sufficient condition that an arc in S^n be cellular</i>	1474
P. P. Saworotnow, <i>Correction to: On continuity of multiplication in a complemented algebra</i>	1474
Basil Gordon, <i>Correction to: A generalization of the coset decomposition of a finite group</i>	1474