

# Pacific Journal of Mathematics

**ON SMALL MAPS OF MANIFOLDS**

HANS SAMELSON

## ON SMALL MAPS OF MANIFOLDS

HANS SAMELSON

**A result announced by R. F. Brown in 1963, and completed by Brown and Fadell, generalizing classical results of H. Hopf for differentiable manifolds, is the following:**

**THEOREM:** Let  $M$  be a compact connected topological manifold; then

(a)  $M$  admits arbitrarily small maps with a single fixed point;

(b) If the Euler characteristic  $\chi_M$  of  $M$  is zero, then  $M$  admits arbitrarily small maps without fixed points (and conversely). Here a map is small if it is close to the identity map. We propose to give a short proof of this theorem.

We will use the recent result of J. Kister (also Mazur and Stallings) that any microbundle over a complex is a bundle [4]. We note that according to [2] the result (b) holds also for manifolds with boundary.

**2. Characteristic class.** We consider the tangent microbundle  $\tau_M: M \xrightarrow{d} M \times M \xrightarrow{p_1}$ ; here  $d$  is the diagonal map, and  $p_1$  the first projection (cf. [5]). Attached to  $\tau_M$  is the Thom class  $u$ , a well-defined element of  $H^n(M \times M, M \times M - d(M))$  (here  $n = \dim M$ ); the coefficients used are the integers  $\mathbf{Z}$ , if  $M$  is orientable, and twisted integers, determined by the orientations of the horizontal factor  $M$  at the points of  $M \times M$ , in the nonorientable case. (Cf. [6] for details in the orientable case.) We write  $\tilde{u}$  for the image of  $u$  in the absolute group  $H^n(M \times M)$ ; the Euler class  $e_M$  is the image of  $\tilde{u}$  in  $H^n(M)$  under the diagonal map  $d^*$  (twisted coefficients in the nonorientable case). Furthermore,  $M$  has a fundamental cycle  $\mu$  (again twisted coefficients for nonorientable  $M$ ). It is a well-known fact that the value  $\langle e_M, \mu \rangle$  of  $e_M$  on  $\mu$  equals the Euler-Poincaré characteristic  $\chi_M$  of  $M$ .

[Since this is not easy to find in the literature, we sketch a proof: First assume  $M$  orientable. Let  $\{x_i\}$  be a basis for  $H^*(M)$  modulo torsion, and let  $\{\alpha_i\}$  be the basis of  $H_*(M)$  modulo torsion, dual to  $\{x_i\}$  under  $\langle \ , \ \rangle$ ; put  $r_i = \dim \alpha_i$ . Define  $\{x'_i\}$  by  $\delta x'_i = \alpha_i$ , where  $\delta$  is the Poincaré duality operator  $\delta x = x \cap \mu$ ; then  $\{x'_i\}$  is again a basis for  $H^*(M)$  modulo torsion. Finally let  $\{\alpha'_i\}$  be dual to  $\{x'_i\}$  under  $\langle \ , \ \rangle$ . One verifies that  $d_*\mu = \sum \alpha_i \times \alpha'_i$  modulo torsion (use  $\langle x \times y, d_*\mu \rangle = \langle x \cup y, \mu \rangle$ ). Now  $\tilde{u}$  satisfies the relation  $\langle x, \alpha \rangle = (-1)^{n-r} \langle \tilde{u}, \delta x \times \alpha \rangle$  for  $x \in H^r(M)$  (cf. [6]). Therefore we have  $\langle e_M, \mu \rangle = \langle \tilde{u}, d_*\mu \rangle =$

---

Received September 8, 1964. This work was supported by NSF Grant GP-2510.

$\langle \tilde{u}, \Sigma \alpha_i \times \alpha'_i \rangle = \Sigma (-1)^{r_i} \langle x'_i, \alpha'_i \rangle = \Sigma (-1)^{r_i} = \chi_M$ . For nonorientable  $M$  let  $\hat{M}$  be the orientable double covering, and use the facts that the Thom class is preserved under the covering map, that the fundamental cycle of  $\hat{M}$  maps onto twice the (twisted) fundamental cycle of  $M$ , and that  $\chi_{\hat{M}} = 2\chi_M$  (as one can see, e.g., from the Smith sequence).]

In particular, if  $\chi_M = 0$ , then also the Euler class  $e_M$  vanishes. Furthermore, in all this discussion we may, by Kister's result, replace the tangent microbundle by an actual bundle (in the local product sense) whose fibre is  $\mathbf{R}^n$  with a well-defined origin and which therefore has a well-defined 0-section. We denote this bundle by  $\bar{\tau}_M$ .

**3. Proof of theorem.** We begin with part (b); thus assume  $\chi_M = 0$ . Embed  $M$  in a number space  $\mathbf{R}^k$  with  $k \geq 2n + 1$ , and let  $V$  be a (closed) polyhedral neighborhood of  $M$  that retracts onto  $M$ , via the map  $r$ . We consider the bundle  $r^*\bar{\tau}_M$ , induced from the bundle  $\bar{\tau}_M$  (see end of § 2) by  $r$ . By naturality the Euler class of  $r^*\bar{\tau}_M$  vanishes. Therefore, if  $K$  is any polyhedron of dimension  $\leq n$  contained in  $V$ , the restriction of  $r^*\bar{\tau}_M$  to  $K$  admits a nonvanishing section (i.e., one that does not meet the 0-section of  $r^*\bar{\tau}_M$ ); to prove this one uses the interpretation of the Euler class as obstruction. Let  $\mathcal{N}$  be a finite, open covering of  $M$ , of dimension  $n$ , such that (a) the nerve  $N_{\mathcal{N}}$  can be realized in  $V$  and (b) an associated barycentric map  $f: M \rightarrow N_{\mathcal{N}}$  (cf. [3], p. 69) is homotopic to the identity  $1_M$  of  $M$  in  $V$ ; this exists of course. Let  $s$  be a nonvanishing section of  $r^*\bar{\tau}_M|N_{\mathcal{N}}$ . Applying the covering homotopy theorem to the map  $s \circ f$  of  $M$  into the bundle formed by the complement of the 0-section of  $r^*\bar{\tau}_M$  and to the homotopy between  $f$  and  $1_M$ , one gets a nonvanishing section of  $r^*\bar{\tau}_M|M$ , i.e. of  $\bar{\tau}_M$ . This section amounts of course to a fixed-point-free map of  $M$  into itself. Again according to Kister,  $\bar{\tau}_M$  can be assumed to lie in any preassigned neighborhood of the diagonal of  $M \times M$ , which means that the map can be constructed as close to the identity as one pleases.

The converse is classical (Lefschetz fixed point theorem).

**4. Proof of theorem continued.** We come to part (a). As before we imbed  $M$  in a Euclidean space  $\mathbf{R}^k$ , and  $r$  is a retraction of some neighborhood of  $M$  onto  $M$ . Let  $A$  be a coordinate system in  $M$  (i.e., an open subset homeomorphic to  $\mathbf{R}^n$ ), and let  $B$ , respectively  $C$ , be the subsets of  $A$  corresponding to the set of points in  $\mathbf{R}^n$  of norm  $< 1$ , respectively  $< \frac{1}{2}$ . There exists a polyhedral neighborhood  $W$  of  $M - B$  in  $\mathbf{R}^k$ , whose  $r$ -image lies in  $M - C$ . Since  $H^n(M - C)$  (twisted coefficients if needed) vanishes ( $M - C$  being a manifold with

nonempty boundary), the characteristic class of  $r^*\bar{c}_M|W$  is zero. By the same argument as before, the bundle  $\bar{c}_M|M-B$  has a nonvanishing section, which can be interpreted as a map  $f$  of  $M-B$  into  $M$ , without fixed points. We may assume that the  $f$ -image of the boundary of  $M-B$  lies in  $A$  (by taking  $\bar{c}_M$  small enough), and it is then clear, using  $A \approx \mathbf{R}^n$ , how to extend  $f$  to a map of  $M$  into itself whose only fixed point is the point of  $A$  corresponding to the origin of  $\mathbf{R}^n$ .

If  $f$  is homotopic to the identity map of  $M$  (as it will be if it is small enough: apply  $r$  to the linear homotopy in  $\mathbf{R}^k$ ), then the index of the fixed point is  $\chi_M$ : the index equals  $\pm$  the intersection number of the graph of  $f$  in  $M \times M$  and the diagonal, and it is well known that this is  $\chi_M$  under the present circumstances. In fact, this last remark yields another version of the proof of (a): if  $\chi_M = 0$ , one can extend  $f$  over  $B$  without any fixed point.

#### BIBLIOGRAPHY

1. R. F. Brown, *Path fields on manifolds*, Notices Amer. Math. Soc. **10** (1963), p. 449, #603-61.
2. R. F. Brown, E. Fadell, *Non-singular path fields on compact topological manifolds*, Notices Amer. Math. Soc. **11** (1964), p. 533, #614-24.
3. W. Hurewicz, H. Wallman, *Dimension Theory*, Princeton 1941.
4. J. M. Kister, *Microbundles are Bundles*, BAMS **69** (1963), 854-857, and Ann. of Math. **80** (1964), 190-199.
5. J. W. Milnor, *Microbundles, Part I*, Topology 3, Suppl. **1** (1964), 53-80.
6. H. Samelson, *On Poincaré Duality*, J. d'Analyse Mathématique, **14** (1965), 323-336.



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON

Stanford University  
Stanford, California

R. M. BLUMENTHAL

University of Washington  
Seattle, Washington 98105

J. DUGUNDJI

University of Southern California  
Los Angeles, California 90007

\*RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

\* Basil Gordon, Acting Managing Editor until February 1, 1966.

# Pacific Journal of Mathematics

Vol. 15, No. 4

December, 1965

Robert James Blattner, <i>Group extension representations and the structure space</i> .....	1101
Glen Eugene Bredon, <i>On the continuous image of a singular chain complex</i> .....	1115
David Hilding Carlson, <i>On real eigenvalues of complex matrices</i> .....	1119
Hsin Chu, <i>Fixed points in a transformation group</i> .....	1131
Howard Benton Curtis, Jr., <i>The uniformizing function for certain simply connected Riemann surfaces</i> .....	1137
George Wesley Day, <i>Free complete extensions of Boolean algebras</i> .....	1145
Edward George Effros, <i>The Borel space of von Neumann algebras on a separable Hilbert space</i> .....	1153
Michel Mendès France, <i>A set of nonnormal numbers</i> .....	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i> .....	1171
Frederick Paul Greenleaf, <i>Norm decreasing homomorphisms of group algebras</i> .....	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i> .....	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i> .....	1239
James Patrick Jans, <i>Some aspects of torsion</i> .....	1249
Laura Ketchum Kodama, <i>Boundary measures of analytic differentials and uniform approximation on a Riemann surface</i> .....	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic functions</i> .....	1279
Ronald John Larsen, <i>Almost invariant measures</i> .....	1295
You-Feng Lin, <i>Generalized character semigroups: The Schwarz decomposition</i> .....	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i> .....	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i> .....	1319
Mitsuru Nakai, <i><math>\Phi</math>-bounded harmonic functions and classification of Riemann surfaces</i> .....	1329
L. G. Novoa, <i>On <math>n</math>-ordered sets and order completeness</i> .....	1337
Fredos Papangelou, <i>Some considerations on convergence in abelian lattice-groups</i> .....	1347
Frank Albert Raymond, <i>Some remarks on the coefficients used in the theory of homology manifolds</i> .....	1365
John R. Ringrose, <i>On sub-algebras of a <math>C^*</math>-algebra</i> .....	1377
Jack Max Robertson, <i>Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres</i> .....	1383
Zalman Rubinstein, <i>Some results in the location of zeros of polynomials</i> .....	1391
Arthur Argyle Sagle, <i>On simple algebras obtained from homogeneous general Lie triple systems</i> .....	1397
Hans Samelson, <i>On small maps of manifolds</i> .....	1401
Annette Sinclair, <i><math> \varepsilon(z) </math>-closeness of approximation</i> .....	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional curvature</i> .....	1415
Earl J. Taft, <i>Invariant splitting in Jordan and alternative algebras</i> .....	1421
L. E. Ward, <i>On a conjecture of R. J. Koch</i> .....	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i> .....	1435
Horace C. Wisner, <i>Embedding a circle of trees in the plane</i> .....	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i> .....	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their application to learning theory</i> .....	1471
Charles Vernon Coffman, <i>Correction to: Non-linear differential equations on cones in Banach spaces</i> .....	1472
P. H. Doyle, III, <i>Correction to: A sufficient condition that an arc in <math>S^n</math> be cellular</i> .....	1474
P. P. Saworotnow, <i>Correction to: On continuity of multiplication in a complemented algebra</i> .....	1474
Basil Gordon, <i>Correction to: A generalization of the coset decomposition of a finite group</i> .....	1474