ON SMALL MAPS OF MANIFOLDS

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A result announced by R. F. Brown in 1963, and completed by Brown and Fadell, generalizing classical results of H. Hopf for differentiable manifolds, is the following:

**Theorem:** Let $M$ be a compact connected topological manifold; then

(a) $M$ admits arbitrarily small maps with a single fixed point;

(b) If the Euler characteristic $\chi_M$ of $M$ is zero, then $M$ admits arbitrarily small maps without fixed points (and conversely). Here a map is small if it is close to the identity map. We propose to give a short proof of this theorem.

We will use the recent result of J. Kister (also Mazur and Stallings) that any microbundle over a complex is a bundle [4]. We note that according to [2] the result (b) holds also for manifolds with boundary.

2. Characteristic class. We consider the tangent microbundle $\tau_M: M \xrightarrow{d} M \times M \xrightarrow{p_1};$ here $d$ is the diagonal map, and $p_1$ the first projection (cf. [5]). Attached to $\tau_M$ is the Thom class $u$, a well-defined element of $H^n(M \times M, M \times M - d(M))$ (here $n = \dim M$); the coefficients used are the integers $\mathbb{Z}$, if $M$ is orientable, and twisted integers, determined by the orientations of the horizontal factor $M$ at the points of $M \times M$, in the nonorientable case. (Cf. [6] for details in the orientable case.) We write $\tilde{u}$ for the image of $u$ in the absolute group $H^*(M \times M)$; the Euler class $e_M$ is the image of $\tilde{u}$ in $H^*(M)$ under the diagonal map $d^*$ (twisted coefficients in the nonorientable case). Furthermore, $M$ has a fundamental cycle $\mu$ (again twisted coefficients for nonorientable $M$). It is a well-known fact that the value $\langle e_M, \mu \rangle$ of $e_M$ on $\mu$ equals the Euler-Poincaré characteristic $\chi_M$ of $M$.

[Since this is not easy to find in the literature, we sketch a proof: First assume $M$ orientable. Let $\{x_i\}$ be a basis for $H^*(M)$ modulo torsion, and let $\{\alpha_i\}$ be the basis of $H_*(M)$ modulo torsion, dual to $\{x_i\}$ under $\langle \cdot, \cdot \rangle$; put $r_i = \dim \alpha_i$. Define $\{x'_i\}$ by $b x'_i = \alpha_i$, where $b$ is the Poincaré duality operator $b x = x \cap \mu$; then $\{x'_i\}$ is again a basis for $H^*(M)$ modulo torsion. Finally let $\{\alpha'_i\}$ be dual to $\{x'_i\}$ under $\langle \cdot, \cdot \rangle$.

One verifies that $d_* \mu = \Sigma \alpha_i \times \alpha'_i$ modulo torsion (use $\langle x \times y, d_* \mu \rangle = \langle x \cup y, \mu \rangle$). Now $\tilde{u}$ satisfies the relation $\langle x, \alpha \rangle = (-1)^n \langle \tilde{u}, b x \times \alpha \rangle$ for $x \in H^*(M)$ (cf. [6]). Therefore we have

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For nonorientable $M$ let $\tilde{M}$ be the orientable double covering, and use the facts that the Thom class is preserved under the covering map, that the fundamental cycle of $\tilde{M}$ maps onto twice the (twisted) fundamental cycle of $M$, and that $\chi_{\tilde{u}} = 2\chi_M$ (as one can see, e.g., from the Smith sequence).

In particular, if $\chi_M = 0$, then also the Euler class $e_M$ vanishes. Furthermore, in all this discussion we may, by Kister’s result, replace the tangent microbundle by an actual bundle (in the local product sense) whose fibre is $\mathbb{R}^n$ with a well-defined origin and which therefore has a well-defined 0-section. We denote this bundle by $\tau_M$.

### 3. Proof of theorem

We begin with part (b); thus assume $\chi_M = 0$. Embed $M$ in a number space $\mathbb{R}^k$ with $k \geq 2n + 1$, and let $V$ be a (closed) polyhedral neighborhood of $M$ that retracts onto $M$, via the map $r$. We consider the bundle $r^*\tau_M$, induced from the bundle $\tau_M$ (see end of §2) by $r$. By naturality the Euler class of $r^*\tau_M$ vanishes. Therefore, if $K$ is any polyhedron of dimension $\leq n$ contained in $V$, the restriction of $r^*\tau_M$ to $K$ admits a nonvanishing section (i.e., one that does not meet the 0-section of $r^*\tau_M$); to prove this one uses the interpretation of the Euler class as obstruction. Let $\mathscr{N}$ be a finite, open covering of $M$, of dimension $n$, such that (a) the nerve $N_\mathscr{N}$ can be realized in $V$ and (b) an associated barycentric map $f: M \to N_\mathscr{N}$ (cf. [3], p. 69) is homotopic to the identity $1_M$ of $M$ in $V$; this exists of course. Let $s$ be a nonvanishing section of $r^*\tau_M$ | $N_\mathscr{N}$. Applying the covering homotopy theorem to the map $s \circ f$ of $M$ into the bundle formed by the complement of the 0-section of $r^*\tau_M$ and to the homotopy between $f$ and $1_M$, one gets a nonvanishing section of $r^*\tau_M$ | $M$, i.e. of $\tau_M$. This section amounts of course to a fixed-point-free map of $M$ into itself. Again according to Kister, $\tau_M$ can be assumed to lie in any preassigned neighborhood of the diagonal of $M \times M$, which means that the map can be constructed as close to the identity as one pleases.

The converse is classical (Lefschetz fixed point theorem).

### 4. Proof of theorem continued

We come to part (a). As before we imbed $M$ in a Euclidean space $\mathbb{R}^k$, and $r$ is a retraction of some neighborhood of $M$ onto $M$. Let $A$ be a coordinate system in $M$ (i.e., an open subset homeomorphic to $\mathbb{R}^n$), and let $B$, respectively $C$, be the subsets of $A$ corresponding to the set of points in $\mathbb{R}^n$ of norm $< 1$, respectively $< \frac{1}{2}$. There exists a polyhedral neighborhood $W$ of $M - B$ in $\mathbb{R}^k$, whose $r$-image lies in $M - C$. Since $H^*(M - C)$ (twisted coefficients if needed) vanishes ($M - C$ being a manifold with
nonempty boundary), the characteristic class of \( r^*\tilde{\tau}_M | W \) is zero. By the same argument as before, the bundle \( \tilde{\tau}_M | M - B \) has a nonvanishing section, which can be interpreted as a map \( f \) of \( M - B \) into \( M \), without fixed points. We may assume that the \( f \)-image of the boundary of \( M - B \) lies in \( A \) (by taking \( \tilde{\tau}_M \) small enough), and it is then clear, using \( A \approx R^* \), how to extend \( f \) to a map of \( M \) into itself whose only fixed point is the point of \( A \) corresponding to the origin of \( R^* \).

If \( f \) is homotopic to the identity map of \( M \) (as it will be if it is small enough: apply \( r \) to the linear homotopy in \( R^k \)), then the index of the fixed point is \( \chi_M \): the index equals \( \pm \) the intersection number of the graph of \( f \) in \( M \times M \) and the diagonal, and it is well known that this is \( \chi_M \) under the present circumstances. In fact, this last remark yields another version of the proof of (a): if \( \chi_M = 0 \), one can extend \( f \) over \( B \) without any fixed point.

**Bibliography**

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