|ε(z)|-CLOSENESS OF APPROXIMATION

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For a given function $F(Q)$ defined for $Q \in S$, the connection between these questions is investigated: (1) For arbitrary $\varepsilon > 0$ (or possibly $\varepsilon_i$, where $\varepsilon_i$ corresponds to a component $S_i$ of $S$), does there exist a function $f$ of a specified class $\mathcal{F}$ such that $\sup_{Q \in S} |F(Q) - f(Q)| < \varepsilon$ on $S$ (or $\varepsilon_i$ on $S_i$)? (2) Given an admissible function $\varepsilon(Q)$, does there exist a function $f \in \mathcal{F}$ such that $|F(Q) - f(Q)| \leq |\varepsilon(Q)|$ on $S$? A continuous function $\varepsilon(Q)$ defined on $S$ is admissible if for each zero $Q_\beta$ there is a positive integer $n_\beta$ such that $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero in a deleted neighborhood of $Q_\beta$. A typical result is: Corresponding to any $F(z)$ analytic on a closed bounded set $S$ and to any admissible $\varepsilon(z)$, there exists a rational function $r(z)$ with its poles on a certain preassigned set such that $|F(z) - r(z)| \leq |\varepsilon(z)|$ on $S$.

When the sup-topology is used in approximating a given function $F$ defined on a set $S$ by a function $f$ in a certain class $\mathcal{F}$, it is required that, for arbitrary $\varepsilon > 0$, there exists $f \in \mathcal{F}$ such that

$$\sup |F(X) - f(X)| < \varepsilon \text{ for } X \in S.$$

In this paper the connection is investigated between existence of such an approximating function and existence of an approximating $g \in \mathcal{F}$ when for any admissible function $\varepsilon(X)$ it is required $|F(X) - g(X)| \leq |\varepsilon(X)|$ when $X \in S$.

The latter formulation has the advantage of automatically specifying that, at any zero $X_\varepsilon$ of $\varepsilon(X)$ on $S$, $g(X_\varepsilon) = F(X_\varepsilon)$ and at multiple zeros corresponding derivatives of $F$ and $g$ agree, provided $F$ has derivatives at these points. One interesting application, in case $F$ is continuous and is well-behaved near zeros, is that in which

$$|F(X) - f(X)| \leq p |F(X)|$$

is required, where $p$ denotes a preassigned per cent.

Approximation in the real case in which a neighborhood $N_{\varepsilon_1,\varepsilon_2}$ of $F$ consists of those $f$ such that $\xi_1(x) \leq F(x) - f(x) \leq \xi_2(x)$ has been suggested by P.C. Hammer. If $[\xi_2(x) - \xi_1(x)]/2$ is an "admissible" $\varepsilon(x)$, the problem reduces to the $|\varepsilon(x)|$-closeness of approximation.

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considered in this paper. For $\xi_{1}(x) \leq F(x) - f(x) \leq \xi_{2}(x)$ if and only if

\[ -[\xi_{1}(x) - \xi_{2}(x)]/2 \leq F(x) - [\xi_{1}(x) + \xi_{2}(x)]/2 \]
\[ -f(x) \leq [\xi_{1}(x) - \xi_{2}(x)]/2 \]

This paper is perhaps of most interest in connection with approximation in the complex plane. However, as the Weierstrass-factor Theorem, Mittag-Leffler Theorem, and Runge Theorem [2] upon which the results depend, hold also on the open Riemann surface, the theorems are stated in abstract form for the open Riemann surface: then certain specializations to the complex plane are given in the corollaries.

As is customary, "open" Riemann surface denotes a noncompact Riemann surface [1]. A point on a Riemann surface is denoted by $Q$, a point in the complex plane, in particular, by $z$, and a point on the real axis by $x$. For the sake of clarity the notation $f(Q)$ is frequently used to denote the function $f$.

When it is specified a function has poles coinciding with those of another function, it is to be understood that they have identical principal parts; likewise, if a function has zeros coinciding with those of a second function, the order of the respective zeros is the same.

For reference we state:

**HYPOTHESIS H.** Suppose that $S$ is a closed set on the open Riemann surface $\mathbb{R}$, Let $B^*$ consist of precisely one point of each of those components of $\mathbb{R} - S$ whose closure is compact.

Theorem 1 includes the case that $S$ is compact with no interior points. For example, if $\mathbb{R}$ is the finite complex plane, $S$ may be a bounded closed interval on the real axis; in fact, $S$ may be any closed bounded set with or without interior points.

**THEOREM 1.** Assume Hypothesis H and suppose a function $\varepsilon(Q)$ ($\neq 0$) defined on $S$. Let $R$ be an open set (which may be $\mathbb{R}$) such that $S \subset R \subset \mathbb{R}$ and suppose $\mathcal{J}$ is a collection of functions meromorphic on $R$, analytic on $R - B^*$. Then these approximation requirements (1) and (2) are equivalent.

1. Corresponding to any function $M(Q)$ analytic on $S^0$ (the interior of $S$) and continuous on $S$, there exists $k \in \mathcal{J}$ such that \[ |M(Q) - k(Q)| \leq |\varepsilon(Q)| \quad \text{when} \quad Q \in S. \]

2. Corresponding to any function $m(Q)$ meromorphic on $S^0$ and continuous on $S$ except at poles, there exists $f = h + k$, where $k \in \mathcal{J}$ and $h$ is meromorphic on $\mathbb{R}$ with its only poles coinciding with those of $m$ on $S$, such that \[ |m(Q) - f(Q)| \leq |\varepsilon(Q)| \quad \text{on} \quad S. \]

**Proof.** Clearly, (2) includes (1). We proceed to prove (1) implies (2).
The set of points at which \( m \) has poles on \( S \) is an isolated set on \( \mathbb{R} \). Hence, according to the Mittag-Leffler partial fractions theorem [2, p. 591; 7] there exists a function \( h \) meromorphic on \( \mathbb{R} \) whose only poles coincide with those of \( m \) on \( S \) and have the same principal parts. (We note that, if \( m \) has only a finite number of poles on \( S \) and if \( \mathbb{R} \) is the finite complex plane, then \( h \) may be required to be a rational function.)

The function \( m - h \) is analytic on \( S' \) and continuous on \( S \). Hence, by the conclusion in (1), there is a function \( k \in \mathscr{F} \), such that

\[
| [m(Q) - h(Q)] - k(Q) | \leq | \varepsilon(Q) |
\]

when \( Q \in S \), that is,

\[
| m(Q) - [h(Q) + k(Q)] | \leq | \varepsilon(Q) |
\]
on \( S \).

Thus, \( h + k \), which is meromorphic on \( \mathbb{R} \) and analytic on \( \mathbb{R} - B^* \) except for poles on \( S \) coinciding with those of \( m \), is a function \( f \) as required.

**Corollary 1.1.** The theorem is true if in

(1) \( M(Q) \) is assumed analytic on \( S \) and in

(2) \( m(Q) \) is assumed meromorphic on \( S \).

**Corollary 1.2.** For \( \mathbb{R} \) the finite complex plane and \( S \) a compact set on \( \mathbb{R} \), the theorem is true if in

(1) \( k \) is required to be a rational function and in

(2) \( f \) is required to be a rational function.

H. J. Landau [5] proved: If on the complex plane, \( S \) is a closed bounded set with no interior and if there exist cutting sets of \( S \) whose closures have arbitrarily small measure, then any function continuous on \( S \) may be uniformly approximated on \( S \) by a rational function whose poles lie in \( B^* \cup \infty \). It follows from Corollary 1.2 that, if \( m \) is continuous on such a set \( S \) except for a finite number of poles, \( m(z) \) can be uniformly approximated by a rational function whose poles lie in \( B^* \cup \infty \) and at the poles of \( m \) on \( S \).

By the Carleman approximation theorem [3; 4] if \( w(x) \) is continuous on the real axis, then corresponding to any \( \{ \varepsilon_i \} \), there exists an entire function \( f \) such that \( | w(x) - f(x) | < \varepsilon_i \) when \( i - 1 < | x | \leq i, \ i = 1, 2, \ldots \). Hence, Theorem 1 implies that, if \( w(x) \) is continuous on the finite real axis except for a finite or a denumerable number of poles with limit point at \( \infty \), then \( w(x) \) can be approximated in the above
sense by a meromorphic function $f$ whose poles lie on the real axis and coincide with those of $w$. According to an extension by the author [8, Theorem 3] of the Carleman Theorem, if $S$ consists of the union of closed circular disks $S_i$ tangent externally on the real axis and extending to infinity and if $w$ is analytic at interior points of $S$, continuous on $S$, then, corresponding to any $\{\varepsilon_i\}$, there exists an entire function $f$ such that $|w(z) - f(z)| < \varepsilon_i$ on $S_i$, $i = 1, 2, \ldots$.

By Theorem 1, $w$ may be allowed poles on $S^0$ provided the approximating function $f$ is allowed coincident poles.

An analogue of the type of generalization given in Theorem 1 for a $Q$-set has previously been used by the author [8; 9].

A sequential limit point of a set $S$ is a limit point of a set of points chosen one from each component of $S$. A set $S$ in the extended complex plane whose components $S_1, S_2, \ldots$, are compact and whose set of sequential limit points $B \subset \mathcal{C}(S)$ is called a $Q$-set [9]. We require, in addition, that a $Q$-set on an open Riemann surface $\mathfrak{R}$ be a closed set, that is, $\mathfrak{R}$ contains no sequential limit point of $S$. When in the complex domain $\mathfrak{R}$ is chosen as the extended plane minus $B$, the set of sequential limit points of $S$, a $Q$-set is closed.

A function $\varepsilon(Q)$ defined for $Q \in S$ is admissible on $S$ if

1. It is continuous on $S$;
2. Corresponding to each of its zeros $Q_\beta$ on $S$, there is a positive integer $n_\beta$ such that $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero in a neighborhood $N_{Q_\beta} \subset S$. The smallest positive integer $n_\beta$ satisfying the condition in (2) is called the order of the zero of $\varepsilon(Q)$ at $Q_\beta$.

**Theorem 2.** Assume Hypothesis $H$ with $S = \cup S_n$, where the $S_n$ are compact and disjoint. Let $R$ be an open set such that $S \subset R \subset \mathfrak{R}$. Suppose $M$ is any function which is analytic on $S^0$, continuous on $S$. Then (1) below implies (2); also, if $S$ is a $Q$-set or a compact set, (2) implies (1), and if $K$ is any isolated interior subset of $S$, $f(z) = M(z)$ can be required on $K$.

1. Corresponding to any $\{\varepsilon_n\}$ ($\varepsilon$ if $S$ is compact), there exists $f$ analytic on $R - B^*$, meromorphic on $R$, such that $|M(Q) - f(Q)| \leq \varepsilon_n$ when $Q \in S_n$, $n = 1, 2, \ldots$ (or $\varepsilon$ when $Q \in S$).

2. Corresponding to any $\varepsilon(Q)$ which is admissible on $S$, there exists $F$ analytic on $R - B^*$ and meromorphic on $R$ such that

$$|M(Q) - F(Q)| \leq |\varepsilon(Q)|$$

on $S$. If $f$ in (1) can be required to be a rational function and if $S$ is compact, then $F$ can be required to be a rational function.

**Proof.** We first show (1) implies (2). Admissibility requirement (2) for $\varepsilon(Q)$ implies the zeros of $\varepsilon$ on $S$ are isolated. Hence, by the
Weierstrass-factor Theorem [2, p. 591] there exists $g$ analytic on $\mathbb{R}$ whose only zeros are the zeros $Q_\beta$ of $\varepsilon(Q)$ and are of the respective orders $n_\beta$. Let $\varepsilon_n = \inf |\varepsilon(Q)/g(Q)|$ for $Q$ on $S_n$ (or $\varepsilon = \inf |\varepsilon(Q)/g(Q)|$ for $Q$ on $S$). Now, by Theorem 1 with $\varepsilon(Q) = \varepsilon_n$ on $S_n$ (or $\varepsilon$ on $S$) and (1) above, there exists a function $k$ meromorphic on $\mathbb{R}$, analytic in $\mathbb{R} - B^*$ except at zeros of $g$ on $S$, such that $|M(Q)/g(Q) - k(Q)| \leq \varepsilon_n$ (or $\varepsilon$ on $S$) where defined. Then on each $S_n$ (or $S$)

$$|M(Q) - g(Q)k(Q)| \leq |g(Q)| \varepsilon_n$$

(or $|g(Q)| \varepsilon$). Now $g \cdot k$, which has removable singularities at the $Q_\beta$, satisfies the requirements for $F$.

Next we consider the converse, giving the proof for the case $S$ is a $Q$-set. Since $\{\varepsilon_n\}$ defines an admissible $\varepsilon(Q)$, (1) is a special case of (2). We are to verify also that interpolation conditions can be assigned. The Weierstrass-factor theorem yields existence of a function $g$ analytic on $\mathbb{R}$ such that $g$ has zeros on $K$ of the same orders as the interpolation conditions. For $\varepsilon_n(Q) = \varepsilon_n [g(Q)/\max |g(Q)|]$ when $Q \in S_n$, and $\varepsilon(Q)$ defined by $\varepsilon_n(Q)$ on $S_n$, $\varepsilon(Q)$ is admissible on $S$. By hypothesis (2), there is $F$ analytic on $\mathbb{R} - B^*$, meromorphic on $\mathbb{R}$, such that

$$|M(Q) - F(Q)| \leq |\varepsilon(Q)|$$

on $S$. Since $|\varepsilon(Q)| \leq \varepsilon_n$ on $S_n$ and $\varepsilon(Q)$ vanishes on $K$, $F$ satisfies the interpolation conditions, in addition to the requirements for $f$ in the conclusion of (1).

**Corollary 2.1.** If $M$ is analytic on the closed bounded set $S$ in the finite complex plane, then, corresponding to any admissible $\varepsilon(z)$, there exists a rational function $r$ having its poles on $B^*$ such that $|M(z) - r(z)| \leq |\varepsilon(z)|$ when $z \in S$.

**Proof.** This follows from the Walsh formulation of the Runge Theorem [10, p. 15] and Theorem 2 with $n = 1$ and $R = \mathbb{R}$ defined as the finite complex plane.

The next corollary is obtained by applying a result of Mergelyan [6; 10, p. 367].

**Corollary 2.2.** If in the complex plane $M$ is continuous on the closed bounded set $S$, analytic on $S^C$, and if $S$ does not separate the plane, then, corresponding to any admissible $\varepsilon(z)$, there exists a polynomial $p(z)$ such that $|M(z) - p(z)| \leq |\varepsilon(z)|$ on $S$.

**Corollary 2.3.** Suppose $S$ is a $Q$-set ($\cup S_n$) and $\varepsilon(z)$ is admissible on $S \subset \mathbb{R}$, the extended plane minus the set of sequential limit points of $S$. Then, if $M$ is analytic on $S$, there exists a function
If \( M \) is meromorphic on \( S \), there exists \( f \) analytic on \( R - B^* \), except at poles of \( M \) on \( S \), and meromorphic on \( R \) such that 
\[
|M(z) - f(z)| \leq |\varepsilon(z)| \quad \text{everywhere } M \text{ is defined on } S.
\]

Proof. The first part is an immediate consequence of Theorem 2 and a previous theorem of the author [9, Theorem 3]. The latter part then follows from Corollary 1.1.

For \( \varepsilon(Q) \) continuous on \( S \), in order that (2) of Theorem 2 hold, the admissibility restriction (2) on \( \varepsilon \) is necessary at any interior zero of \( \varepsilon \) at which \( M \) is analytic. For, if 
\[
|M(Q) - F(Q)| \leq |\varepsilon(Q)| \quad \text{on } S,
\]
then, at a zero \( Q_\beta \) of \( \varepsilon \), 
\[
M(Q_\beta) = F(Q_\beta).
\]
If (as is the case if \( M \) is analytic at \( Q_\beta \) and \( F(Q) \neq M(Q) \)) 
\[
M(Q) - F(Q) = (Q - Q_\beta)^{\gamma} g(Q),
\]
where, in some neighborhood \( N_{Q_\beta} \subset S \), \( g \) is bounded from zero, then
\[
|M(Q) - F(Q)| \leq |\varepsilon(Q)|
\]
on \( S \) implies 
\[
|(Q - Q_\beta)^{\gamma} \varepsilon(Q) - g(Q)| \leq 1 \quad \text{on } N_{Q_\beta},
\]
where defined. The last inequality is possible only if the first factor is bounded on \( N_{Q_\beta} \), that is, 
\[
\varepsilon(Q)/(Q - Q_\beta)^{\gamma} \text{ is bounded from zero on } N_{Q_\beta}.
\]
At an interior point of \( S \), \( M \) is necessarily analytic if Hypothesis (1) of Theorem 2 is satisfied; hence, if the conclusion of Theorem 2 is to hold, continuous \( \varepsilon(Q) \) must satisfy admissibility requirement (2) at any interior zero of \( \varepsilon \). An example is next given to illustrate an application of Theorem 2 for the case \( n = 1 \). Let 
\[
R = \mathbb{R} = \{z/ \mid z < \infty\}; \quad M(z) = z \sin 1/z \quad \text{for } z \neq 0, M(0) = 0; \quad \varepsilon(z) = (z - 1)^{\gamma}(z - 3/4)(z - 1/2)g(z),
\]
where \( g \) is any function continuous and nonvanishing on \( S \); 
\[
S = \{x/0 \leq x \leq 1\} \cup \gamma \gamma \gamma
\]
where the \( \gamma \) are nonintersecting closed disks with centers at the zeros of \( \varepsilon(z) \). Now, by a Walsh approximation theorem [10, p. 47], \( M(z) \) can be uniformly approximated by a polynomial, that is, (1) in Theorem 2 is satisfied with \( f(z) \) a polynomial in \( z \). Hence, Theorem 2 implies that for any admissible \( \varepsilon(z) \), in particular as defined above, there is a polynomial \( F(z) \) such that 
\[
|M(z) - F(z)| \leq |\varepsilon(z)| \quad \text{on } S.
\]
The next theorem yields degree of convergence in the \( O(\varepsilon_n(Q)) \)-sense by setting 
\[
S = S_1 = S_2 = \cdots,
\]
also other special results as stated in the corollaries.

Corresponding to given \( \{\varepsilon_n\} \) with \( \varepsilon_n(Q) \), defined on \( S_n \) and nonvanishing on \( \partial S_n \), \( n = 1, 2, \cdots \), will be called \( \varepsilon_n \)-admissible on \( S = \bigcup S_n \) if there exists \( g(Q) \) analytic on \( \mathbb{R} \) such that, for each \( n, \varepsilon_n(Q) = g(Q)\phi_n(Q) \) and 
\[
\varepsilon_n \leq \inf |\phi_n(Q)|, \quad n = 1, 2, \cdots, \quad \text{for } Q \in S_n.
\]

**Theorem 3.** Assume Hypothesis \( H \), with 
\[
S = \bigcup_{n=1}^\infty S_n, \quad \text{where the } S_n \text{ are compact, but not necessarily disjoint. Let } S_n \text{ be a collection}
\]

of functions each meromorphic on an open set $R_n$ and analytic on $R_n - B^*$, where $S_n \subset R_n \subset \mathbb{R}$. ($R_n$ may be $\mathbb{R}$.) Suppose a certain sequence of positive constants $\{\varepsilon_n\}$ assigned. Then (1) below implies (2).

(1) Corresponding to any $\{m_n\}$, with $m_n$ analytic on $S_n$, continuous on $S_n$, and such that $m_n(Q) = m_j(Q)$ on $S_n \cap S_j$ (if this is not the null set), there exists $f_n, f_n \in \mathcal{S}_n$, and $M$ (independent of $n$) such that $| m_n(Q) - f_n(Q) | < M\varepsilon_n$ on $S_n$.

(2) Corresponding to any $\varepsilon_n$-admissible $\{\varepsilon_n(Q)\}(\varepsilon_n(Q) = g(Q)\phi_n(Q))$ and to $\{m_n\}$ defined as in (1), there exists $h$ meromorphic on $\mathbb{R}$ whose only poles lie on $B^*$ or coincide with those of $m_n(Q)/g(Q)$ on $S$ and there exists $f_n \in \mathcal{S}_n$ such that

$$| m_n(Q) - g(Q)[h(Q) + f_n(Q)] | \leq M_1 | \varepsilon_n(Q) |$$

on $S_n$, $n = 1, 2, \ldots$. If in (1) the $f_n$ can be chosen as the same function for all $n$, the same is true for the $f_n$ in (2). If, in (1), $M$ is independent of $\{m_n(Q)\}$, then, in (2), $M_1 = M$.

Proof. By the Mittag-Leffler theorem there exists $h$ meromorphic on $\mathbb{R}$ whose only poles coincide with those of $m_n/g$ on $S_n$, $n = 1, 2, \ldots$. Now $(m_n(z)/g(z)) - h(z)$ is analytic on $S_n$, continuous on $S_n$. Hence, by hypothesis (1), there exists $f_n \in \mathcal{S}_n$ such that on $S_n$

$$| m_n(Q)|g(Q) - h(Q) | - f_n(Q) | < M_1 \varepsilon_n \leq M_1 | \phi_n(Q) | .$$

This yields the required result.

If in both (1) and (2) the $m_n$ are assumed analytic on $S_n$, the theorem remains true.

**COROLLARY 3.1.** Let $m$ be analytic on the bounded closed set $S$ which does not separate the complex plane. Suppose $\{\varepsilon_n\}$ is a certain sequence of positive constants such that there exist polynomials $\{p_n(z)\}$ of respective degrees $n$ and some $M$ such that $| m(z) - p_n(z) | < M\varepsilon_n$ on $S$. Then, for $\varepsilon_n$-admissible $\{\varepsilon_n(z)\}$ with $\varepsilon_n(z) = P_n(z)\phi_n(z)$, where $P_n(z)$ is a polynomial of degree $N$, there exist polynomials $P_{n+N}(z)$ of degrees $N + n$ such that $| m(z) - P_{n+N}(z) | \leq M_1 | \varepsilon_n(z) |$ on $S$.

Proof. In the theorem set $S = S_1 = S_2 = \cdots$ and $m(z) = m_n(z) = m_1(z) = \cdots$, and let $\mathcal{S}_n$ denote the set of all polynomials of degree $n$. Since, by the hypothesis, (1) is satisfied, the conclusion of the theorem yields the result when it is noted that $h$ can be chosen as an appropriate rational function.

**EXAMPLE.** If $m(z)$ is analytic on $S$, $| z | \leq 1$, $m$ is analytic in a larger region $D_\rho: | z | < \rho$ [10, p. 79]. Fix $R, 1 < R < \rho$, and set $\varepsilon_n = 1/R^n$. Let $\phi$ be any function which is continuous and nonvanishing on
and let \( P_n(z) \) be a polynomial of degree \( N \), nonvanishing on \( \partial S \).

Then \( K \) can be chosen so that, for \( \varepsilon_n(z) \) defined as \( KP_n(z)\phi(z)/(z^n + R^n) \), and \( \phi_n(z) = K\phi(z)/(z^n + R^n) \), \( \{\varepsilon_n(z)\} \) is \( \varepsilon \)-admissible on \( S \). There are known to be polynomials \( p_n \) of respective degrees \( n \) such that, for some \( M \), \( |m(z) - p_n(z)| < M/R^n \) on \( S \) [10, p. 79], whence, by Corollary 3.1, there exist polynomials \( q_{n+N} \) of degrees \( n + N \) such that

\[
|m(z) - q_{n+N}(z)| \leq M |\varepsilon_n(z)|
\]

on \( S \), for some \( M \), independent of \( n \).

The polynomials \( p_{n+N} \) in Corollary 3.1 cannot be required to be of degree less than \( n + N \). For \( m \) analytic on \( S \) defined as in the Example, choose \( P_n(z) \) as a polynomial whose only zeros coincide with those of \( m(z) \) on \( S \), and define \( \varepsilon_n(z) = (K/R^n)P_n(z) \), \( 1 < R < \rho \). Suppose there exist polynomials \( p_k(z) \) of degree \( k \) such that

\[
|m(z) - p_k(z)| \leq M_k |P_n(z)|/R^n
\]

on \( S \). Without loss of generality it can be supposed the zeros of \( p_k \) coincide with those of \( m \) on \( S \) [10, p. 310]. Now \( N = m/P_n \) is analytic on \( S \), except for removable singularities, and

\[
|N(z) - p_k(z)/P_n(z)| \leq M_k/R^n
\]

on \( S \). Since \( p_k(z)/P_n(z) \) is a polynomial of degree \( k - N \), this would yield a degree of convergence stronger than maximal convergence if \( k - N < n \) [10, p. 79].

The result stated in Corollary 2.3, which is a direct consequence of Theorem 2, is essentially that of Corollary 3.2.

**Corollary 3.2.** Suppose \( m(z) \) is analytic on \( S = \cup S_n \), a \( Q \)-set with components \( S_n \), and let \( B \) denote its set of sequential limit points. Let \( \Re \) be the extended complex plane minus \( B \) and define \( B^* \) as in Hypothesis \( H \). Then, corresponding to any \( \varepsilon(z) = g(z)\phi(z) \) with \( g \) analytic on \( \Re \) and \( \phi \) bounded from zero on each \( S_n \), there exists \( f \) analytic on \( \Re - B^* \), meromorphic on \( \Re \), such that

\[
|m(z) - f(z)| \leq |\varepsilon(z)| \quad \text{on} \quad S.
\]

**Proof.** In the theorem, let \( R_n = \Re, \mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = \cdots \) be the set of functions analytic on \( \Re - B^* \), meromorphic on \( \Re \), and define \( m_n(z) = m(z) \) on \( S_n \), \( \varepsilon_n(z) = \varepsilon(z) \) on \( S_n \), \( \phi_n(z) = \phi(z) \) on \( S_n \), \( \varepsilon_n = \inf |\phi_n(z)| \) for \( z \in S_n \). We note \( \{\varepsilon_n(z)\} \) is \( \varepsilon \)-admissible. By a theorem of the author [9], \( M(1) \) of the theorem is satisfied, with \( n = 1 \) and \( f_n(z) = f_1(z) = \cdots \), whence the theorem implies (2), yielding the required result.
COROLLARY 3.3. Let $S = \bigcup_{n=1}^{\infty} S_n$, where the $S_n$ are closed circular disks of radii one-half tangent externally along the positive real axis and ordered by increasing distance from the origin. Suppose $m$ is analytic on each $S_n$, continuous on $S$. Then, for $\varepsilon(z) = g(z)\phi(z)$, where $g$ is an entire function (nonvanishing on $\partial S$) and $\phi$ is bounded from zero on each $S_n$, there exists an entire function $F$ such that $|m(z) - F(z)| \leq |\varepsilon(z)|$ on $S$.

Proof. Let $R = \mathbb{R}$ be the finite complex plane, $B^*$ the null set, and $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2 = \cdots$ the class of entire functions. Define $m_n(z) = m(z)$ on $S_n$, $n = 1, 2, \cdots$, and set $\varepsilon_n(z) = \varepsilon(z)$ on $S_n$. Then define $\phi_n(z) = \phi(z)$ on $S_n$ and $\varepsilon_n = \inf |\phi_n(z)|$ for $z \in S_n$. By a previous result [8, Theorem 3], corresponding to any $\{\varepsilon_n\}$, there exists $f(z) = f_1(z) = f_2(z) = \cdots, f \in \mathcal{H}$, such that $|m(z) - f(z)| < \varepsilon_n$ on $S_n$. Then (2) of the theorem with $F(z) = g(z)[h(z) + f(z)]$ yields the required result.

REFERENCES


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