For a given function $F(Q)$ defined for $Q \in S$, the connection between these questions is investigated: (1) For arbitrary $\varepsilon > 0$ (or possibly $\{\varepsilon_i\}$, where $\varepsilon_i$ corresponds to a component $S_i$ of $S$), does there exist a function $f$ of a specified class $\mathcal{F}$ such that $\sup_{Q \in S} |F(Q) - f(Q)| < \varepsilon$ on $S$ (or $\varepsilon_i$ on $S_i$)?; (2) Given an admissible function $\varepsilon(Q)$, does there exist a function $f \in \mathcal{F}$ such that $|F(Q) - f(Q)| \leq |\varepsilon(Q)|$ on $S$? A continuous function $\varepsilon(Q)$ defined on $S$ is admissible if for each zero $Q_\beta$ there is a positive integer $n_\beta$ such that $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero in a deleted neighborhood of $Q_\beta$. A typical result is: Corresponding to any $F(z)$ analytic on a closed bounded set $S$ and to any admissible $\varepsilon(z)$, there exists a rational function $r(z)$ with its poles on a certain preassigned set such that $|F(z) - r(z)| \leq |\varepsilon(z)|$ on $S$.

When the sup-topology is used in approximating a given function $F$ defined on a set $S$ by a function $f$ in a certain class $\mathcal{F}$, it is required that, for arbitrary $\varepsilon > 0$, there exists $f \in \mathcal{F}$ such that

$$\sup |F(X) - f(X)| < \varepsilon \text{ for } X \in S.$$ 

In this paper the connection is investigated between existence of such an approximating function and existence of an approximating $g \in \mathcal{F}$ when for any admissible function $\varepsilon(X)$ it is required $|F(X) - g(X)| \leq |\varepsilon(X)|$ when $X \in S$.

The latter formulation has the advantage of automatically specifying that, at any zero $X_0$ of $\varepsilon(X)$ on $S$, $g(X_0) = F(X_0)$ and at multiple zeros corresponding derivatives of $F$ and $g$ agree, provided $F$ has derivatives at these points. One interesting application, in case $F$ is continuous and is well-behaved near zeros, is that in which

$$|F(X) - f(X)| \leq p |F(X)|$$

is required, where $p$ denotes a preassigned per cent.

Approximation in the real case in which a neighborhood $N_{x_1, x_2}$ of $F$ consists of those $f$ such that $\xi(x) \leq F(x) - f(x) \leq \xi(x)$ has been suggested by P.C. Hammer.¹ If $[\xi_2(x) - \xi_1(x)]/2$ is an "admissible" $\varepsilon(x)$, the problem reduces to the $|\varepsilon(x)|$-closeness of approximation

This paper is perhaps of most interest in connection with approximation in the complex plane. However, as the Weierstrass-factor Theorem, Mittag-Leffler Theorem, and Runge Theorem [2] upon which the results depend, hold also on the open Riemann surface, the theorems are stated in abstract form for the open Riemann surface: then certain specializations to the complex plane are given in the corollaries.

As is customary, "open" Riemann surface denotes a noncompact Riemann surface [1]. A point on a Riemann surface is denoted by \( Q \), a point in the complex plane, in particular, by \( z \), and a point on the real axis by \( x \). For the sake of clarity the notation \( f(Q) \) is frequently used to denote the function \( f \).

When it is specified a function has poles coinciding with those of another function, it is to be understood that they have identical principal parts; likewise, if a function has zeros coinciding with those of a second function, the order of the respective zeros is the same.

For reference we state:

**Hypothesis H.** Suppose that \( S \) is a closed set on the open Riemann surface \( \mathcal{R} \), Let \( B^* \) consist of precisely one point of each of those components of \( \mathcal{R} - S \) whose closure is compact.

Theorem 1 includes the case that \( S \) is compact with no interior points. For example, if \( \mathcal{R} \) is the finite complex plane, \( S \) may be a bounded closed interval on the real axis; in fact, \( S \) may be any closed bounded set with or without interior points.

**Theorem 1.** Assume Hypothesis H and suppose a function \( \varepsilon(Q) \) \((\neq 0)\) defined on \( S \). Let \( R \) be an open set (which may be \( \mathcal{R} \)) such that \( S \subset R \subset \mathcal{R} \) and suppose \( \mathcal{S} \) is a collection of functions meromorphic on \( R \), analytic on \( R - B^* \). Then these approximation requirements (1) and (2) are equivalent.

1. Corresponding to any function \( M(Q) \) analytic on \( S^0 \) (the interior of \( S \)) and continuous on \( S \), there exists \( k \in \mathcal{S} \) such that \( | M(Q) - k(Q) | \leq | \varepsilon(Q) | \) when \( Q \in S \).

2. Corresponding to any function \( m(Q) \) meromorphic on \( S^0 \) and continuous on \( S \) except at poles, there exists \( f = h + k \), where \( h \in \mathcal{S} \) and \( h \) is meromorphic on \( \mathcal{R} \) with its only poles coinciding with those of \( m \) on \( S \), such that \( | m(Q) - f(Q) | \leq | \varepsilon(Q) | \) on \( S \).

**Proof.** Clearly, (2) includes (1). We proceed to prove (1) implies (2).
The set of points at which \( m \) has poles on \( S \) is an isolated set on \( \mathbb{R} \). Hence, according to the Mittag-Leffler partial fractions theorem [2, p. 591; 7] there exists a function \( h \) meromorphic on \( \mathbb{R} \) whose only poles coincide with those of \( m \) on \( S \) and have the same principal parts. (We note that, if \( m \) has only a finite number of poles on \( S \) and if \( \mathbb{R} \) is the finite complex plane, then \( h \) may be required to be a rational function.)

The function \( m - h \) is analytic on \( S^0 \) and continuous on \( S \). Hence, by the conclusion in (1), there is a function \( k \in \mathcal{S} \), such that

\[
| [m(Q) - h(Q)] - k(Q) | \leq | \varepsilon(Q) |
\]

when \( Q \in S \), that is,

\[
| m(Q) - [h(Q) + k(Q)] | \leq | \varepsilon(Q) |
\]
on \( S \).

Thus, \( h + k \), which is meromorphic on \( R \) and analytic on \( R - B^* \) except for poles on \( S \) coinciding with those of \( m \), is a function \( f \) as required.

**Corollary 1.1.** The theorem is true if in

(1) \( M(Q) \) is assumed analytic on \( S \) and in

(2) \( m(Q) \) is assumed meromorphic on \( S \).

**Corollary 1.2.** For \( \mathbb{R} \) the finite complex plane and \( S \) a compact set on \( \mathbb{R} \), the theorem is true if in

(1) \( k \) is required to be a rational function and in

(2) \( f \) is required to be a rational function.

H. J. Landau [5] proved: If on the complex plane, \( S \) is a closed bounded set with no interior and if there exist cutting sets of \( S \) whose closures have arbitrarily small measure, then any function continuous on \( S \) may be uniformly approximated on \( S \) by a rational function whose poles lie in \( B^* \cup \infty \). It follows from Corollary 1.2 that, if \( m \) is continuous on such a set \( S \) except for a finite number of poles, \( m(z) \) can be uniformly approximated by a rational function whose poles lie in \( B^* \cup \infty \) and at the poles of \( m \) on \( S \).

By the Carleman approximation theorem [3; 4] if \( w(x) \) is continuous on the real axis, then corresponding to any \( \{ \varepsilon_i \} \), there exists an entire function \( f \) such that \( | w(x) - f(x) | < \varepsilon_i \) when \( i - 1 < | x | \leq i, \ i = 1, 2, \ldots \). Hence, Theorem 1 implies that, if \( w(x) \) is continuous on the finite real axis except for a finite or a denumerable number of poles with limit point at \( \infty \), then \( w(x) \) can be approximated in the above
sense by a meromorphic function $f$ whose poles lie on the real axis and coincide with those of $w$. According to an extension by the author [8, Theorem 3] of the Carleman Theorem, if $S$ consists of the union of closed circular disks $S_i$ tangent externally on the real axis and extending to infinity and if $w$ is analytic at interior points of $S$, continuous on $S$, then, corresponding to any $\{\varepsilon_i\}$, there exists an entire function $f$ such that $|w(z) - f(z)| < \varepsilon_i$ on $S_i$, $i = 1, 2, \ldots$. By Theorem 1, $w$ may be allowed poles on $S^o$ provided the approximating function $f$ is allowed coincident poles.

An analogue of the type of generalization given in Theorem 1 for a Q-set has previously been used by the author [8; 9].

A sequential limit point of a set $S$ is a limit point of a set of points chosen one from each component of $S$. A set $S$ in the extended complex plane whose components $S_1, S_2, \cdots$, are compact and whose set of sequential limit points $B \subset \subset S$ is called a Q-set [9]. We require, in addition, that a Q-set on an open Riemann surface $\mathcal{R}$ be a closed set, that is, $\mathcal{R}$ contains no sequential limit point of $S$. When in the complex domain $\mathcal{R}$ is chosen as the extended plane minus $B$, the set of sequential limit points of $S$, a Q-set is closed.

A function $\varepsilon(Q)$ defined for $Q \in S$ is admissible on $S$ if

1. It is continuous on $S$;
2. Corresponding to each of its zeros $Q_\beta$ on $S$, there is a positive integer $n_\beta$ such that $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero in a neighborhood $N_{Q_\beta} \subset S$. The smallest positive integer $n_\beta$ satisfying the condition in (2) is called the order of the zero of $\varepsilon(Q)$ at $Q_\beta$.

**Theorem 2.** Assume Hypothesis $H$ with $S = \bigcup S_n$, where the $S_n$ are compact and disjoint. Let $R$ be an open set such that $S \subset R \subset \mathcal{R}$. Suppose $M$ is any function which is analytic on $S^o$, continuous on $S$. Then (1) below implies (2); also, if $S$ is a Q-set or a compact set, (2) implies (1), and if $K$ is any isolated interior subset of $S$, $f(z) = M(z)$ can be required on $K$.

1. Corresponding to any $\{\varepsilon_n\}$ (if $S$ is compact), there exists $f$ analytic on $R - B^*$, meromorphic on $R$, such that $|M(Q) - f(Q)| \leq \varepsilon_n$ when $Q \in S_n$, $n = 1, 2, \cdots$ (or $\varepsilon$ when $Q \in S$).

2. Corresponding to any $\varepsilon(Q)$ which is admissible on $S$, there exists $F$ analytic on $R - B^*$ and meromorphic on $R$ such that

$$|M(Q) - F(Q)| \leq |\varepsilon(Q)|$$

on $S$. If $f$ in (1) can be required to be a rational function and if $S$ is compact, then $F$ can be required to be a rational function.

**Proof.** We first show (1) implies (2). Admissibility requirement (2) for $\varepsilon(Q)$ implies the zeros of $\varepsilon$ on $S$ are isolated. Hence, by the
Weierstrass-factor Theorem [2, p. 591] there exists \( g \) analytic on \( \mathbb{R} \) whose only zeros are the zeros \( Q_\beta \) of \( \varepsilon(Q) \) and are of the respective orders \( n_\beta \). Let \( \varepsilon_n = \inf |\varepsilon(Q)/g(Q)| \) for \( Q \) on \( S_n \) (or \( \varepsilon = \inf |\varepsilon(Q)/g(Q)| \) for \( Q \) on \( S \)). Now, by Theorem 1 with \( \varepsilon(Q) = \varepsilon_n \) on \( S_n \) (or \( \varepsilon \) on \( S \)) and (1) above, there exists a function \( k \) meromorphic on \( R \), analytic in \( R - B^* \) except at zeros of \( g \) on \( S \), such that \( |M(Q)/g(Q) - k(Q)| \leq \varepsilon_n \) (or \( \varepsilon \) on \( S \)) where defined. Then on each \( S_n \) (or \( S \))

\[
|M(Q) - g(Q)k(Q)| \leq |g(Q)| \varepsilon_n
\]

(or \( |g(Q)| \varepsilon \)). Now \( g \cdot k \), which has removable singularities at the \( Q_\beta \), satisfies the requirements for \( F \).

Next we consider the converse, giving the proof for the case \( S \) is a \( Q \)-set. Since \( \{\varepsilon_n\} \) defines an admissible \( \varepsilon(Q) \), (1) is a special case of (2). We are to verify also that interpolation conditions can be assigned. The Weierstrass-factor theorem yields existence of a function \( g \) analytic on \( \mathbb{R} \) such that \( g \) has zeros on \( K \) of the same orders as the interpolation conditions. For \( \varepsilon_n(Q) = \varepsilon_n[g(Q)/\max |g(Q)|] \) when \( Q \in S_n \), and \( \varepsilon(Q) \) defined by \( \varepsilon_n(Q) \) on \( S_n \), \( \varepsilon(Q) \) is admissible on \( S \). By hypothesis (2), there is \( F \) analytic on \( R - B^* \), meromorphic on \( R \), such that

\[
|M(Q) - F(Q)| \leq |\varepsilon(Q)|
\]
on \( S \). Since \( |\varepsilon(Q)| \leq \varepsilon_n \) on \( S_n \) and \( \varepsilon(Q) \) vanishes on \( K \), \( F \) satisfies the interpolation conditions, in addition to the requirements for \( f \) in the conclusion of (1).

**Corollary 2.1.** If \( M \) is analytic on the closed bounded set \( S \) in the finite complex plane, then, corresponding to any admissible \( \varepsilon(z) \), there exists a rational function \( r \) having its poles on \( B^* \) such that \( |M(z) - r(z)| \leq |\varepsilon(z)| \) when \( z \in S \).

**Proof.** This follows from the Walsh formulation of the Runge Theorem [10, p. 15] and Theorem 2 with \( n = 1 \) and \( R = \mathbb{R} \) defined as the finite complex plane.

The next corollary is obtained by applying a result of Mergelyan [6; 10, p. 367].

**Corollary 2.2.** If in the complex plane \( M \) is continuous on the closed bounded set \( S \), analytic on \( S^0 \), and if \( S \) does not separate the plane, then, corresponding to any admissible \( \varepsilon(z) \), there exists a polynomial \( p(z) \) such that \( |M(z) - p(z)| \leq |\varepsilon(z)| \) on \( S \).

**Corollary 2.3.** Suppose \( S \) is a \( Q \)-set (= \( \cup S_n \)) and \( \varepsilon(z) \) is admissible on \( S \subset \mathbb{R} \), the extended plane minus the set of sequential limit points of \( S \). Then, if \( M \) is analytic on \( S \), there exists a function
If $M$ is meromorphic on $S$, there exists $f$ analytic on $R - B^*$, except at poles of $M$ on $S$, and meromorphic on $R$ such that $|M(z) - f(z)| \leq |\varepsilon(z)|$ everywhere $M$ is defined on $S$.

Proof. The first part is an immediate consequence of Theorem 2 and a previous theorem of the author [9, Theorem 3]. The latter part then follows from Corollary 1.1.

For $\varepsilon(Q)$ continuous on $S$, in order that (2) of Theorem 2 hold, the admissibility restriction (2) on $\varepsilon$ is necessary at any interior zero of $\varepsilon$ at which $M$ is analytic. For, if $|M(Q) - F(Q)| \leq |\varepsilon(Q)|$ on $S$, then, at a zero $Q_\beta$ of $\varepsilon$, $M(Q_\beta) = F(Q_\beta)$. If (as is the case if $M$ is analytic at $Q_\beta$ and $F(Q) \neq M(Q)$) $M(Q) - F(Q) = (Q - Q_\beta)^{n_\beta} g(Q)$, where, in some neighborhood $N_{Q_\beta} \subset S$, $g$ is bounded from zero, then

$$|M(Q) - F(Q)| \leq |\varepsilon(Q)|$$

on $S$ implies $|(Q - Q_\beta)^{n_\beta}/\varepsilon(Q)| \cdot |g(Q)| \leq 1$ on $N_{Q_\beta}$, where defined. The last inequality is possible only if the first factor is bounded on $N_{Q_\beta}$, that is, $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero on $N_{Q_\beta}$. At an interior point of $S$, $M$ is necessarily analytic if Hypothesis (1) of Theorem 2 is satisfied; hence, if the conclusion of Theorem 2 is to hold, continuous $\varepsilon(Q)$ must satisfy admissibility requirement (2) at any interior zero of $\varepsilon$.

An example is next given to illustrate an application of Theorem 2 for the case $n = 1$. Let $R = \mathbb{R} = \{z/|z| < \infty\}; M(z) = z \sin 1/z$ for $z \neq 0$, $M(0) = 0$; $\varepsilon(z) = (z - 1)^4(z - 3/4)(z - 1/2)g(z)$, where $g$ is any function continuous and nonvanishing on $S$; $S = \{x/0 \leq x \leq 1\} \cup \bigcup_{j=1}^3 \gamma_j$ where the $\gamma_j$ are nonintersecting closed disks with centers at the zeros of $\varepsilon(z)$. Now, by a Walsh approximation theorem [10, p. 47], $M(z)$ can be uniformly approximated by a polynomial, that is, (1) in Theorem 2 is satisfied with $f(z)$ a polynomial in $z$. Hence, Theorem 2 implies that for any admissible $\varepsilon(z)$, in particular as defined above, there is a polynomial $F(z)$ such that $|M(z) - F(z)| \leq |\varepsilon(z)|$ on $S$.

The next theorem yields degree of convergence in the $O(\varepsilon_n(Q))$-sense by setting $S = S_1 = S_2 = \cdots$, also other special results as stated in the corollaries.

Corresponding to given $\{\varepsilon_n\}, \{\varepsilon_n(Q)\}$ with $\varepsilon_n(Q)$, defined on $S_n$ and nonvanishing on $\partial S_n$, $n = 1, 2, \cdots$, will be called $\varepsilon_n$-admissible on $S = \bigcup S_n$ if there exists $g(Q)$ analytic on $\mathbb{R}$ such that, for each $n$, $\varepsilon_n(Q) = g(Q)\phi_n(Q)$ and $\varepsilon_n \leq \inf |\phi_n(Q)|$, $n = 1, 2, \cdots$, for $Q \in S_n$.

THEOREM 3. Assume Hypothesis $H$, with $S = \bigcup_{n=1}^\infty S_n$, where the $S_n$ are compact, but not necessarily disjoint. Let $\mathcal{S}_n$ be a collection
of functions each meromorphic on an open set \( R_n \) and analytic on \( R_n - B^* \), where \( S_n \subset R_n \subset \mathbb{R} \). (\( R_n \) may be \( \mathbb{R} \)). Suppose a certain sequence of positive constants \( \{\varepsilon_n\} \) assigned. Then (1) below implies (2).

(1) Corresponding to any \( \{m_n\} \), with \( m_n \) analytic on \( S_n \), continuous on \( S_n \), and such that \( m_n(Q) = m_i(Q) \) on \( S_n \cap S_i \) (if this is not the null set), there exists \( f_n, f_n \in \mathcal{S}_n \), and \( M \) (independent of \( n \)) such that
\[
| m_n(Q) - f_n(Q) | < M\varepsilon_n \quad \text{on } S_n.
\]

(2) Corresponding to any \( \varepsilon_n \)-admissible \( \{\varepsilon_n(Q)\} \) and to \( \{m_n\} \) defined as in (1), there exists \( h \) meromorphic on \( \mathbb{R} \) whose only poles lie on \( B^* \) or coincide with those of \( m_n(Q)/g(Q) \) on \( S \) and there exists \( f_n \in \mathcal{S}_n \) such that
\[
| m_n(Q) - g(Q)[h(Q) + f_n(Q)] | \leq M_1 | \varepsilon_n(Q) | \quad \text{on } S_n, \quad n \in \mathbb{N}.
\]

If in (1) the \( f_n \) can be chosen as the same function for all \( n \), the same is true for the \( f_n \) in (2). If, in (1), \( M \) is independent of \( \{m_n(Q)\} \), then, in (2), \( M_1 = M \).

**Proof.** By the Mittag-Leffler theorem there exists \( h \) meromorphic on \( \mathbb{R} \) whose only poles coincide with those of \( m_n/Q \) on \( S_n \), \( n = 1, 2, \ldots \). Now \( (m_n(z)/g(z)) - h(z) \) is analytic on \( S_n \), continuous on \( S_n \). Hence, by hypothesis (1), there exists \( f_n \in \mathcal{S}_n \) such that on \( S_n \)
\[
| [m_n(Q)/g(Q) - h(Q)] - f_n(Q) | < M\varepsilon_n \leq M_1 | \varepsilon_n(Q) | .
\]

This yields the required result.

If in both (1) and (2) the \( m_n \) are assumed analytic on \( S_n \), the theorem remains true.

**Corollary 3.1.** Let \( m \) be analytic on the bounded closed set \( S \) which does not separate the complex plane. Suppose \( \{\varepsilon_n\} \) is a certain sequence of positive constants such that there exist polynomials \( \{p_n(z)\} \) of respective degrees \( n \) and some \( M \) such that \( |m(z) - p_n(z)| < M\varepsilon_n \) on \( S \). Then, for \( \varepsilon_n \)-admissible \( \{\varepsilon_n(z)\} \) with \( \varepsilon_n(z) = P_n(z)\phi_n(z) \), where \( P_n(z) \) is a polynomial of degree \( N \), there exist polynomials \( P_{n+N}(z) \) of degrees \( N + n \) such that \( |m(z) - P_{n+N}(z)| \leq M_1 | \varepsilon_n(z) | \) on \( S \).

**Proof.** In the theorem set \( S = S_1 = S_2 = \ldots \) and \( m(z) = m_1(z) = m_2(z) = \ldots \), and let \( \mathcal{S}_n \) denote the set of all polynomials of degree \( n \). Since, by the hypothesis, (1) is satisfied, the conclusion of the theorem yields the result when it is noted that \( h \) can be chosen as an appropriate rational function.

**Example.** If \( m(z) \) is analytic on \( S, |z| \leq 1, m \) is analytic in a larger region \( D_\rho: |z| < \rho [10, p. 79] \). Fix \( R, 1 < R < \rho \), and set \( \varepsilon_n = 1/R^n \). Let \( \phi \) be any function which is continuous and nonvanishing on
and let \( P_N(z) \) be a polynomial of degree \( N \), nonvanishing on \( \partial S \). Then \( K \) can be chosen so that, for \( \varepsilon_n(z) \) defined as \( KP_N(z)\phi(z)/(z^n + R^n) \), and \( \phi_n(z) = K\phi(z)/(z^n + R^n) \), \( \{ \varepsilon_n(z) \} \) is \( \varepsilon \)-admissible on \( S \). There are known to be polynomials \( p_n \) of respective degrees \( n \) such that, for some \( M, |m(z) - p_n(z)| < M/R^n \) on \( S \) [10, p. 79], whence, by Corollary 3.1, there exist polynomials \( q_{n+N} \) of degrees \( n + N \) such that

\[
|m(z) - q_{n+N}(z)| \leq M_1 \varepsilon_n(z)
\]
on \( S \), for some \( M_1 \) independent of \( n \).

The polynomials \( p_{n+N} \) in Corollary 3.1 cannot be required to be of degree less than \( n + N \). For \( m \) analytic on \( S \) defined as in the Example, choose \( P_N(z) \) as a polynomial whose only zeros coincide with those of \( m(z) \) on \( S \), and define \( \varepsilon_n(z) = (K/R^n)P_N(z), 1 < R < \rho \). Suppose there exist polynomials \( p_k(z) \) of degree \( k \) such that

\[
|m(z) - p_k(z)| \leq M_1 K |P_n(z)|/R^n
\]
on \( S \). Without loss of generality it can be supposed the zeros of \( p_k \) coincide with those of \( m \) on \( S \) [10, p. 310]. Now \( N = m/P_N \) is analytic on \( S \), except for removable singularities, and

\[
|N(z) - p_k(z)/P_N(z)| \leq M_1/R^n
\]
on \( S \). Since \( p_k(z)/P_N(z) \) is a polynomial of degree \( k - N \), this would yield a degree of convergence stronger than maximal convergence if \( k - N < n \) [10, p. 79].

The result stated in Corollary 2.3, which is a direct consequence of Theorem 2, is essentially that of Corollary 3.2.

**Corollary 3.2.** Suppose \( m(z) \) is analytic on \( S = \bigcup S_n \), a Q-set with components \( S_n \), and let \( B \) denote its set of sequential limit points. Let \( \mathcal{R} \) be the extended complex plane minus \( B \) and define \( B^* \) as in Hypothesis \( H \). Then, corresponding to any \( \varepsilon(z) = g(z)\phi(z) \) with \( g \) analytic on \( \mathcal{R} \) and \( \phi \) bounded from zero on each \( S_n \), there exists \( f \) analytic on \( \mathcal{R} - B^* \), meromorphic on \( \mathcal{R} \), such that

\[
|m(z) - f(z)| \leq |\varepsilon(z)| \text{ on } S.
\]

**Proof.** In the theorem, let \( R_n = \mathcal{R}, \mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = \cdots \) be the set of functions analytic on \( \mathcal{R} - B^* \), meromorphic on \( \mathcal{R} \), and define \( m_n(z) = m(z) \) on \( S_n \), \( \varepsilon_n(z) = \varepsilon(z) \) on \( S_n \), \( \phi_n(z) = \phi(z) \) on \( S_n \), \( \varepsilon_n = \inf |\phi_n(z)| \) for \( z \in S_n \). We note \( \{ \varepsilon_n(z) \} \) is \( \varepsilon \)-admissible. By a theorem of the author [9], \( M(1) \) of the theorem is satisfied, with \( n = 1 \) and \( f_1(z) = f_2(z) = \cdots \), whence the theorem implies (2), yielding the required result.
COROLLARY 3.3. Let $S = \bigcup_{n=1}^{\infty} S_n$, where the $S_n$ are closed circular disks of radii one-half tangent externally along the positive real axis and ordered by increasing distance from the origin. Suppose $m$ is analytic on each $S_n$, continuous on $S$. Then, for $\varepsilon(z) = g(z)\phi(z)$, where $g$ is an entire function (nonvanishing on $\partial S$) and $\phi$ is bounded from zero on each $S_n$, there exists an entire function $F$ such that $|m(z) - F(z)| \leq |\varepsilon(z)|$ on $S$.

Proof. Let $R = \mathbb{R}$ be the finite complex plane, $B^*$ the null set, and $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = \cdots$ the class of entire functions. Define $m_n(z) = m(z)$ on $S_n$, $n = 1, 2, \ldots$, and set $\varepsilon_n(z) = \varepsilon(z)$ on $S_n$. Then define $\phi_n(z) = \phi(z)$ on $S_n$ and $\varepsilon_n = \inf |\phi_n(z)|$ for $z \in S_n$. By a previous result [8, Theorem 3], corresponding to any $\{\varepsilon_n\}$, there exists $f(z) = f_1(z) = f_2(z) = \cdots, f \in \mathcal{S}$, such that $|m(z) - f(z)| < \varepsilon_n$ on $S_n$. Then (2) of the theorem with $F(z) = g(z)(h(z) + f(z))$ yields the required result.

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