

# Pacific Journal of Mathematics

**ISOMETRIC IMMERSIONS OF MANIFOLDS OF  
NONNEGATIVE CONSTANT SECTIONAL CURVATURE**

EDSEL FORD STIEL

## ISOMETRIC IMMERSIONS OF MANIFOLDS OF NONNEGATIVE CONSTANT SECTIONAL CURVATURE

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Let  $M^d$  denote a  $C^\infty$  Riemannian manifold which is  $d$ -dimensional and complete. Our first result states that an isometric immersion of a flat  $M^d$  into  $(d+k)$ -dimensional Euclidean space,  $k < d$ , is  $n$ -cylindrical if the relative nullity of the immersion has constant value  $n$ . This result was obtained by O'Neill with the additional hypothesis of vanishing relative curvature. We next consider the case in which  $M^d$  and  $\bar{M}^{d+k}$ ,  $k < d$ , are manifolds of the same constant positive sectional curvature. In this case we show that an isometric immersion of  $M^d$  into  $\bar{M}^{d+k}$  is totally geodesic if the relative curvature of the immersion is zero on a certain subset of  $M^d$ .

Let  $M^a$  and  $\bar{M}^{a+k}$  be  $C^\infty$  Riemannian manifolds of the same constant sectional curvature  $C$ ,  $M^a$  being assumed complete and  $k < d$ . Let  $\psi: M^a \rightarrow \bar{M}^{a+k}$  be an isometric immersion. The character of such immersions has been studied in [4] and [5] in terms of what Chern and Kuiper call the *index of relative nullity* of  $\psi$  [2]. This function,  $\nu$ , assigns to each  $m \in M$  the dimension of  $\mathcal{N}(m)$ , the subspace of vectors  $x$  in the tangent space  $M_m$  such that  $T_x = 0$ . The linear *difference operators*  $T_x$  act on  $\bar{M}_{\psi(m)}$  and contain the same information as the classical second fundamental form operators  $S_z$  where  $z$  is a tangent vector to  $\bar{M}$  orthogonal to  $d\psi(M_m)$  [1]. In fact  $T_x$  is characterized by its skew-symmetry and the equation  $T_x(z) = d\psi(S_z(x))$ . Our first theorem concerns the case in which  $M^a$  is flat and  $\bar{M}^{a+k} = R^{a+k}$ ,  $d+k$  dimensional Euclidean space. It states that when  $\nu$  is constant on  $M^a$  the immersion  $\psi$  is 'cylindrical'. We next investigate the corresponding situation for  $C > 0$ .

We use essentially the notation in [4]. In particular we identify  $M^a$  with  $\psi(M^a)$  when it seems safe to do so. Let  $N$  denote the bundle of normal  $k$ -frames of  $M$  relative to  $\psi$ ; that is

$$N = \{(m, E) \mid m \in M \text{ and } E \text{ is a } k\text{-frame (orthonormal set of } k \text{ vectors) of } \bar{M}_{\psi(m)} \text{ orthogonal to } d\psi(M_m)\}.$$

The Riemannian connection of  $\bar{M}^{a+k}$  induces a natural connection on  $N$ . The curvature form of this connection is called the *relative curvature* of  $\psi$ . We say that  $\psi: M^a \rightarrow R^{a+k}$  is  $n$ -cylindrical provided

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$M$  and  $\psi$  can be expressed as Riemannian products  $M^a = B^{a-n} \times R^n$  and  $\psi = \bar{\psi} \times 1$  where  $\bar{\psi}$  is an isometric immersion of  $B^{a-n}$  in  $R^{d+k-n}$  and  $1$  is the identity map of  $R^n$ . We can now state our first theorem precisely. This result was obtained by O'Neill as Theorem 2 of [4] but with an additional hypothesis, namely, the assumption of zero relative curvature. We shall use a similar assumption in our Theorem 3.

**THEOREM 1.** *Let  $M^a$  be a complete, flat,  $C^\infty$  Riemannian manifold. An isometric immersion  $\psi: M^a \rightarrow R^{d+k}$  is  $n$ -cylindrical if the relative nullity has constant value  $n$ .*

We summarize some results applicable to an isometric immersion between two manifolds of constant curvature  $C$ . Let  $\mathcal{N}^\perp(m)$  be the orthogonal complement of  $\mathcal{N}(m)$  in  $M_m$ . From [5] we have: *If  $n$  denotes the minimum value of  $\nu$ , then  $n \geq d - k$  and  $G$ , the open subset of  $M^a$  on which  $\nu = n$ , is foliated by complete totally geodesic subspaces (the leaves of  $\mathcal{N}$ ) which are also totally geodesic relative to  $\psi$ . Also there exists for any  $m \in G$  an  $x \in \mathcal{N}^\perp(m)$  such that  $T_x$  is injective on  $\mathcal{N}^\perp(m)$ .* The two cases of interest to us are:

*Case 1.*  $G = M^d$  (i.e.,  $\nu$  is constant),  $\bar{M}^{d+k} = R^{d+k}$  ( $C = 0$ ) and  $a = \infty$  (see below).

*Case 2.*  $C > 0$  and  $0 < a < \pi/4\sqrt{C}$ .

The parameter  $a$  appears in the following lemma. *Let  $\gamma: (-a, a) \rightarrow L$  be a unit speed geodesic in a leaf  $L$  of  $\mathcal{N}$  in  $G$ . Then there exists a frame field  $E = (E_1, \dots, E_{d+k})$  on a neighborhood of  $\gamma$  in  $G$  such that:*

1. *The geodesic  $\gamma$  is an integral curve of  $E_1$ ;*
2. *Each integral curve of  $E_1$  is a geodesic of  $M$ ;*
3. *The vector fields  $E_1, \dots, E_n$  are contained in  $\mathcal{N}$ ,  $E_{n+1}, \dots, E_d$  in  $\mathcal{N}^\perp$ , and  $E_{d+1}, \dots, E_{d+k}$  are contained in the orthogonal complement of  $\psi(M_m)$  in  $\bar{M}_{\psi(m)}$ ;*

4. *The frame  $E$  is parallel on  $\gamma$ .* The construction for this lemma is contained in Lemma 1 of [5], except we use the additional fact that the leaves of  $\mathcal{N}$  are  $R^n$  planes in Case 1 for  $a = \infty$ . We pull the connection form  $\bar{\phi}$  of the frame bundle of  $\bar{M}^{d+k}$  down to  $G$  by way of the frame field  $E$ . Using the following index convention,

$$\begin{aligned} 1 \leq a, b \leq n; & \quad n + 1 \leq q, r, s \leq d; \\ 1 \leq i, j \leq d; & \quad d + 1 \leq \alpha, \beta \leq d + k, \end{aligned}$$

we get

$$\begin{aligned} \phi_{ij} &= \bar{\phi}_{ij} \circ dE && \text{(connection forms of } M), \\ \tau_{i\alpha} &= \bar{\phi}_{i\alpha} \circ dE && \text{(Codazzi forms),} \\ \theta_{\alpha\beta} &= \bar{\phi}_{\alpha\beta} \circ dE && \text{(normal connection forms).} \end{aligned}$$

A set of linear operators on  $\mathcal{N}^\perp$  dependent on the frame field  $E$  can be defined by

$$P_{E_s}(E_s) = \Sigma_r \phi_{ra}(E_s) E_r .$$

From the second structural equation and the properties of the frame field  $E$  one can show that *the matrix  $P(t)$  of  $P_{\gamma'(t)}$  satisfies the differential equation  $P' = -P^2 - CI$  on  $(-a, a)$  where  $I$  denotes the  $(d - n) \times (d - n)$  identity matrix. See Lemma 3 of [5]. Our proof of Theorem 1 hinges on the central result from [4] which states that *if for all  $m \in M^a$  and  $x \in \mathcal{N}^{(m)}$  we have that  $P_x = 0$  then the immersion is  $n$ -cylindrical.* Theorem 1 can now be easily proved with the help of the following lemma which is applicable in both Case 1 and Case 2.*

**LEMMA 1.** *Let  $m \in L$ . If  $x \in \mathcal{N}(m)$  and  $y \in \mathcal{N}^\perp(m)$  then  $T_{P_x(y)} = T_y \circ P_x$  on  $\mathcal{N}^\perp(m)$ .*

*Proof.* Since  $L$  is complete there exists a geodesic  $\gamma: (-a, a) \rightarrow L$  with  $\gamma(0) = m$  and a frame field  $E$  as defined above in a neighborhood of  $\gamma$ . From  $T_{E_i}(E_j) = \Sigma_\alpha \tau_{\alpha j}(E_i) E_\alpha$  and the definition of  $\mathcal{N}$  we get that  $\tau_{\alpha\alpha} = 0$ . Using this fact with the Codazzi equation for  $\tau_{\alpha\alpha}$  we have

$$0 = d\tau_{\alpha\alpha} = -\Sigma_i \phi_{ai} \wedge \tau_{i\alpha} - \Sigma_\beta \tau_{\alpha\beta} \wedge \theta_{\beta\alpha} = \Sigma_q \phi_{\alpha q} \wedge \tau_{q\alpha} .$$

This implies that

$$\Sigma_{\alpha,q} \phi_{q\alpha}(E_s) \tau_{\alpha q}(E_r) E_\alpha = \Sigma_{\alpha,q} \phi_{q\alpha}(E_r) \tau_{\alpha q}(E_s) E_\alpha$$

or that

$$T_{E_r}(P_{E_s}(E_s)) = T_{E_s}(P_{E_r}(E_r)) .$$

Hence for  $x \in \mathcal{N}(m)$  and  $y, z \in \mathcal{N}^\perp(m)$  we have

$$T_y(P_x(z)) = T_z(P_x(y)) = T_{P_x(y)}(z) ,$$

the last equality above following from the symmetry of the second fundamental form operators.

**2. Proof of Theorem 1.** We shall show that  $P_x = 0$  for  $x \in \mathcal{N}(m)$ ,  $m \in M^d$ . We may assume  $x$  is a unit vector and  $\gamma$  is a unit speed com-

plete geodesic of the leaf through  $m$  with  $\gamma'(0) = x$ . By a previous remark we may pick  $y \in \mathcal{N}^\perp(m)$  such that  $T_y$  is injective on  $\mathcal{N}^\perp(m)$ . Then  $\mathcal{N}^\perp + T_y(\mathcal{N}^\perp)$  is invariant under both  $T_y$  and  $T_{P_x(y)}$ . Hence the  $2(d - n) \times 2(d - n)$  matrix of  $T_y | (\mathcal{N}^\perp + T_y(\mathcal{N}^\perp))$  can be represented by a  $(d - n) \times (d - n)$  matrix  $A$  in the upper right hand corner,  $-A^t$  in the lower left hand corner and zeros elsewhere. If  $B$  is the analogous block for  $T_{P_x(y)}$  then  $Q = -AB^t$  will be the matrix of  $T_y \circ T_{P_x(y)} | \mathcal{N}^\perp$ . The difference operators  $T_y$  and  $T_{P_x(y)}$  commute on  $M_m$  since  $M$  is flat and hence we have  $AB^t = BA^t$ . By Lemma 1,  $P_x = T_y^{-1} \circ T_{P_x(y)} | \mathcal{N}^\perp$  and hence  $P(0) = (A^{-1})^t B^t$ . Let

$$R = -A^{-1}Q(A^{-1})^t = B^t(A^{-1})^t .$$

Since  $Q$  is symmetric so is  $R$  and therefore  $P(0)$  has the same (real) eigenvalues as  $R$ . These eigenvalues satisfy  $\lambda'_k = -\lambda''_k$  on the real line (since  $P$  satisfies this equation by a result stated above) and hence each  $\lambda_k = 0$ . Thus  $R = 0$  and this implies  $P(0) = 0$  which is the desired result.

3. Positive curvature case. For completeness we include Corollary 1 of [5] as

**THEOREM 2.** *Let  $M^a$  and  $\bar{M}^{a+k}$  be  $C^\infty$  manifolds with the same constant positive curvature  $C$ ,  $M^a$  being assumed complete. Let  $\psi: M^a \rightarrow \bar{M}^{a+k}$  be an isometric immersion with  $2k \leq d$ . Then  $\psi$  is totally geodesic.*

As above let  $n$  denote the minimum value of  $\nu$  and let  $G$  consist of the  $m \in M^d$  for which  $\nu(m) = n$ .

**THEOREM 3.** *Let  $M^a$  and  $\bar{M}^{a+k}$  be  $C^\infty$  manifolds with the same constant positive curvature  $C$ ,  $M^a$  being assumed complete. Let  $\psi: M^a \rightarrow \bar{M}^{a+k}$  be an isometric immersion with  $k < d$ . Then  $\psi$  is totally geodesic if the relative curvature of  $\psi$  is zero on  $G$ .*

*Proof.* The proof is by contradiction. If  $\psi$  is not totally geodesic then  $n < d$ . Let  $L$  be a leaf in  $G$  and let  $m \in L$ . We first show that for any  $x \in \mathcal{N}(m)$ ,  $P_x$  is a symmetric operator and is independent of the frame field used in its definition. Let  $y \in \mathcal{N}^\perp(m)$  such that  $T_y$  is injective on  $\mathcal{N}^\perp$ . Using a geodesic  $\gamma: (-a, a) \rightarrow L$  with  $\gamma'(0) = x$  and Lemma 1 we have as in the proof of Theorem 1 that  $P(0) = (A^{-1})^t B^t$ . Since the relative curvature of  $\psi$  is zero we get from the Ricci equation of the immersion that the Codazzi forms satisfy the relation  $\Sigma_i \tau_{\alpha_i} \wedge \tau_{i\beta} = 0$ . From this we conclude that  $T_y$  and  $T_{P_x(y)}$

commute on  $(d\psi(M_m))^\perp$  or  $A^tB = B^tA$ . This equation implies that  $P(0)$  is symmetric. From the first structural equation we have that

$$[E_r, E_s] = \Sigma_i(\phi_{ri}(E_s) - \phi_{si}(E_r))E_i$$

which together with the symmetry of  $P_x$  implies  $[E_r, E_s] \in \mathcal{N}^\perp$ ; thus  $\mathcal{N}^\perp$  is integrable. For  $x \in \mathcal{N}$ ,  $P_x$  is actually a second fundamental form operator of the leaf through  $\mathcal{N}^\perp$  and thus  $P_x$  is independent of the choice of frame field used in its definition.

From the completeness of  $L$  it follows that we can find a unit speed geodesic  $\gamma$  in  $L$  defined on the real line. Since  $M$  is of constant positive curvature,  $\gamma$  is a compact immersion and  $P_\gamma$  is a periodic function on the real line. Let  $\lambda$  be one of the  $d - n$  real eigenvalue functions determined by the symmetric operator  $P_\gamma$ . We may assume  $\lambda$  attains a maximum at  $m = \gamma(0)$ . Let  $E$  be a frame field as above. Then  $\lambda$  must satisfy  $\lambda'(0) = -\lambda^2(0) - C = 0$  since  $P$  satisfies  $P' = -P^2 - CI$  on an interval containing 0. This implies  $\lambda(0)$  is not real, which is the desired contradiction. Hence  $n \geq d$  or  $\psi$  is totally geodesic on  $M$ .

As a Corollary we get a result of O'Neill's from [3]. Let  $S^{d+1}(C)$  denote the sphere of curvature  $C$ .

**COROLLARY 1.** *Let  $M^d$  and  $\bar{M}^{d+1}$  be  $C^\infty$  manifolds with the same constant positive curvature  $C$ ,  $M^d$  being assumed complete. Then any isometric immersion  $\psi: M^d \rightarrow \bar{M}^{d+1}$  is totally geodesic. In particular if  $\bar{M}^{d+1} = S^{d+1}(C)$  then any such immersion is an imbedding onto a great sphere.*

*Proof.* The vanishing of the relative curvature of  $\psi$  is trivial in the hypersurface case. In case  $\bar{M}^{d+1} = S^{d+1}(C)$  we have that  $\psi(M) = S^d(C) \subset S^{d+1}(C)$ . Letting  $\bar{S}^d(C)$  denote the universal covering manifold of  $M^d$  and  $\pi$  the natural projection, we have that  $\psi \circ \pi$  is a local isometry onto  $\psi(M)$ . Hence  $\psi \circ \pi$  and therefore  $\psi$  is injective. Thus  $\psi$  is an imbedding onto  $S^d(C)$ .

#### REFERENCES

1. W. Ambrose, *The Cartan structural equations in classical Riemannian geometry*, J. Indian Math. Soc. **24** (1960), 23-76.
2. S. S. Chern and N. H. Kuiper, *Some theorems on the isometric imbedding of compact Riemann manifolds in Euclidean space*, Ann. of Math. (2) **56** (1952), 422-430.
3. B. O'Neill, *Isometric immersions which preserve curvature operators*, Proc. Amer. Math. Soc. **13** (1962), 759-763.
4. ———, *Isometric immersion of flat Riemannian manifolds in Euclidean space*, Michigan Math. J. **9** (1962), 199-205.
5. B. O'Neill and E. Stiel, *Isometric immersions of constant curvature manifolds*, Michigan Math. J. **10** (1963), 335-339.
6. E. Stiel, *Isometric Immersions of Riemannian Manifold*, Doctoral Dissertation, University of California at Los Angeles, (1963).



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