

# Pacific Journal of Mathematics

**ON A CONJECTURE OF R. J. KOCH**

L. E. WARD

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L. E. WARD, JR.<sup>1</sup>

*Dedicated to Professor Alexander Doniphan Wallace  
 on the occasion of his sixtieth birthday*

**R. J. Koch proved that if  $X$  is a compact, continuously partially ordered space and if  $W$  is an open subset of  $X$  which has no local minima, then each point of  $W$  is the supremum of an order arc which meets  $X - W$ . More recently he extended this result to quasi ordered spaces in which the sets  $E(x) = \{y: x \leq y \leq x\}$  are assumed to be totally disconnected and  $W$  is a chain. He conjectured that the latter hypothesis is superfluous, and we show here that Koch's conjecture is correct.**

**As a corollary it follows that if  $X$  is a compact, continuously quasi ordered space with zero (i.e., a unique minimal element), if each set  $E(x)$  is totally disconnected, and if each set  $L(x) = \{y: y \leq x\}$  is connected, then  $X$  is arcwise connected.**

We begin by recalling a few definitions (see [1], [2], [3] and [4]). We say that  $X = (X, \Gamma)$  is a continuously quasi ordered space provided  $X$  is a Hausdorff space,  $\Gamma$  is a quasi order (= reflexive, transitive relation) on  $X$  and the graph of  $\Gamma$  is a closed subset of  $X \times X$ . We identify  $\Gamma$  with its graph and regard the symbols  $x \leq y$ , and  $x \Gamma y$  and  $(x, y) \in \Gamma$  as synonyms.

A *chain* of a quasi ordered space  $X$  is a subset  $C$  of  $X$  such that  $a \leq b$  or  $b \leq a$  holds for each  $a$  and  $b$  in  $C$ . We also define

$$\begin{aligned} L(a, \Gamma) &= \{x \in X: (x, a) \in \Gamma\}, \\ M(a, \Gamma) &= \{x \in X: (a, x) \in \Gamma\}, \\ E(a, \Gamma) &= L(a, \Gamma) \cap M(a, \Gamma), \end{aligned}$$

for each  $a \in X$ . It is also convenient to define

$$I(a, b, \Gamma) = M(a, \Gamma) \cap L(b, \Gamma),$$

the closed "interval" from  $a$  to  $b$ . Where there is no ambiguity we shall write  $(L(a)$  (resp.,  $M(a)$ ,  $E(a)$ ,  $I(a, b)$ ) for  $L(a, \Gamma)$ , (resp.,  $M(a, \Gamma)$ ,  $E(a, \Gamma)$ ,  $I(a, b, \Gamma)$ ). It is well known [3] that if  $X$  is a continuously quasi ordered space then the sets  $L(a)$ ,  $M(a)$ ,  $E(a)$  and  $I(a, b)$  are closed and, if  $X$  is compact, then  $X$  contains a minimal element, that is, an element  $m$  such that  $L(m) - E(m)$  is empty.

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A subset  $Y$  of the quasi ordered space  $(X, \Gamma)$  is said to *have no local  $\Gamma$ -minima* if, for each  $x \in Y$  and each neighborhood  $U$  of  $x$ , the set

$$Y \cap U \cap L(x, \Gamma) - E(x, \Gamma)$$

is nonempty. This definition is due to Koch [2].

In case the relation  $\Gamma$  is a partial order, it is known that a connected chain joining two distinct points is an arc. (Here we use the term *arc* to describe a continuum with precisely two non-cutpoints.) An arc which is also a chain is termed an *order arc*.

The following two lemmas will be of later use.

**LEMMA 1.** *Let  $X$  be a compact, continuously quasi ordered space, let  $a$  and  $b$  be members of  $X$ , and let  $K$  be a closed subset of  $X$  such that  $I(a, b) \cap K = 0$ . Then there exist open sets  $U$  and  $V$  such that  $a \in U$ ,  $b \in V$  and for each  $a' \in U$  and  $b' \in V$  it follows that  $I(a', b') \cap K = 0$ .*

*Proof.* Suppose, on the contrary, that for all neighborhoods  $U$  and  $V$  of  $a$  and  $b$ , respectively, there exists  $a' \in U$  and  $b' \in V$  such that  $I(a' b') \cap K \neq 0$ . Then

$$\Gamma \cap (\bar{U} \times K) \cap (K \times \bar{V}) \neq 0.$$

These sets form a family of nonempty closed sets with the finite intersection property and hence their intersection is nonempty:

$$\Gamma \cap (\{a\} \times K) \cap (K \times \{b\}) \neq 0,$$

that is to say,  $I(a, b) \cap K \neq 0$ , contrary to the hypothesis.

**LEMMA 2.** *If  $R$  is an open subset of the compact, continuously quasi ordered space  $X$ , then the set*

$$F = \{(a, b) \in X \times X: I(a, b) - R \neq 0\}$$

*is closed.*

*Proof.* If  $(a, b) \notin F$  then  $I(a, b) \cap (X - R) = 0$ . By Lemma 1, there are open sets  $U$  and  $V$  with  $a \in U$  and  $b \in V$  such that for each  $a' \in U$  and  $b' \in V$  it follows that  $I(a', b') \subset R$ , and hence  $(U \times V) \cap F = 0$ . Therefore,  $F$  is closed.

2. Koch's theorem for quasi ordered spaces. The crux of our proof is embodied in the following theorem.

**THEOREM.** *Let  $X = (X, \Gamma)$  be a compact, continuously quasi*

ordered space and let  $W$  be an open subset of  $X$ . If

- (i)  $E(x, \Gamma)$  is totally disconnected for each  $x \in X$ ,
- (ii)  $W$  has no local  $\Gamma$ -minima, then  $X$  admits a minimal quasi order which has a closed graph and satisfies (i) and (ii). Moreover, this minimal quasi order is a partial order.

*Proof.* Let  $\{\Gamma_\alpha\}$  be a maximal nest of quasi orders on  $X$  such that each  $\Gamma_\alpha$  has a closed graph and satisfies (i) and (ii), and let  $\Gamma = \bigcap \{\Gamma_\alpha\}$ . Clearly  $(X, \Gamma)$  is a continuously quasi ordered space and  $E(x, \Gamma)$  is totally disconnected. We will show that  $W$  has no local  $\Gamma$ -minima.

Let  $x \in W$  and let  $U$  be a neighborhood of  $x$ ; since  $W$  is open and  $E(x, \Gamma)$  is totally disconnected, we may assume that  $U \subset W$  and that  $E(x, \Gamma) \cap U$  is closed. Since  $X$  is normal there exist open sets  $V$  and  $R$  such that

$$\begin{aligned} E(x, \Gamma) \cap U &\subset V \subset \bar{V} \subset U, \\ X - U &\subset R \subset \bar{R} \subset X - \bar{V}. \end{aligned}$$

For each  $\alpha$ , the compact set  $L(x, \Gamma_\alpha) \cap \bar{V}$  has a  $\Gamma_\alpha$ -minimal element which we denote  $x_\alpha$ . And since  $W$  has no local  $\Gamma_\alpha$ -minima there exists

$$y_\alpha \in (X - \bar{R}) \cap L(x_\alpha, \Gamma_\alpha) - E(x_\alpha, \Gamma_\alpha).$$

It follows that

$$y_\alpha \in L(x, \Gamma_\alpha) - \bar{R} \cup \bar{V}$$

so that the sets  $L(x, \Gamma_\alpha) - R \cup V$  are compact, nonempty and nested. Consequently there exists

$$y \in L(x, \Gamma) - R \cup V$$

and it is clear that  $y \notin E(x, \Gamma)$ . That is,  $W$  has no local  $\Gamma$ -minima.

Now suppose that  $\Gamma$  is not a partial order; then there exists a nondegenerate set  $E(x, \Gamma)$ . Since  $E(x, \Gamma)$  is compact and totally disconnected, there exist nonempty, closed and disjoint sets  $A$  and  $B$  whose union is  $E(x, \Gamma)$ . Since  $X$  is normal there exist disjoint open sets  $P$  and  $Q$  such that  $A \subset P$  and  $B \subset Q$ . Let

$$F = \{(a, b) : I(a, b) - P \cup Q \neq \emptyset\}.$$

By Lemma 2,  $F$  is a closed subset of  $X \times X$  and hence

$$\Delta = \Gamma - ((P \times Q) - F)$$

is also closed. Since  $P$  and  $Q$  are disjoint,  $\Delta$  is a reflexive relation on  $X$ .

We claim that  $\Delta$  is a quasi order. For suppose  $p \Delta q$  and  $q \Delta r$  but  $(p, r) \in (X \times X) - \Delta$ . Now  $(p, r) \in \Gamma$  so that  $(p, r) \in (P \times Q) - F$  and hence  $q \in P$  or  $q \in Q$ . If  $q \in P$  then, since  $r \in Q$  and  $(q, r) \in \Delta$  we infer that  $(q, r) \in F$  and thus  $I(q, r) - P \cup Q \neq 0$ . But  $I(q, r) \subset I(p, r)$  and hence  $I(p, r) - P \cup Q \neq 0$ , contrary to the fact that  $(p, r) \in (P \times Q) - F$ . A similar contradiction ensues if  $q \in Q$ , and thus  $\Delta$  is a quasi order.

Since  $\Delta \subset \Gamma$  it is obvious that each set  $E(x, \Delta)$  is totally disconnected. Now suppose  $z \in W$  and that  $O$  is a neighborhood of  $z$ ,  $O \subset W$ . If  $z \in W - Q$  then

$$L(z, \Delta) = L(z, \Gamma)$$

and hence there exists

$$y \in O \cap L(z, \Delta) - E(z, \Delta).$$

And if  $z \in Q$ , the fact that  $W$  has no local  $\Gamma$ -minima insures the existence of

$$y \in O \cap Q \cap L(z, \Gamma) - E(z, \Gamma).$$

But  $y \notin P$  implies  $y \in L(z, \Delta)$ , so that in any event  $W$  has no local  $\Delta$ -minima.

Finally we note that  $\Delta$  contradicts the minimality of  $\Gamma$ , for if  $a \in A$  and  $b \in B$  then  $(a, b) \in \Gamma - \Delta$ . Therefore  $\Gamma$  is a partial order.

**COROLLARY 1.** *Let  $X$  be a compact, continuously quasi ordered space and let  $W$  be an open subset of  $X$ . If conditions (i) and (ii) of the theorem are satisfied, then each point of  $W$  is the supremum of an order arc which meets  $X - W$ .*

*Proof.* By the preceding theorem we may assume that the quasi order is a partial order. Thus Koch's theorem for partially ordered spaces applies.

An element  $0$  of the quasi ordered space  $X$  is a zero of  $X$  provided

$$0 = E(0) = \cap \{L(x) : x \in X\}.$$

**COROLLARY 2.** *If  $X$  is a compact, continuously quasi ordered space with zero, if each set  $E(x)$  is totally disconnected and if each set  $L(x)$  is connected, then  $X$  is arcwise connected.*

*Proof.* Let  $W = X - \{0\}$ ; the connectedness of the sets  $L(x)$  guarantees that  $W$  has no local minima and therefore each point of  $W$  lies in arc containing  $0$ .

Following Koch we say that a subset  $C$  of the quasi ordered space  $X$  is *biconnected* if  $C$  is connected and if each of the sets  $E(x) \cap C$  is

connected.

**COROLLARY 3.** *Let  $X$  be a compact, continuously quasi ordered space and suppose there exists  $a \in X$  such that*

$$E(a) = \bigcap \{L(x) : x \in X\} .$$

*If  $X - E(a)$  has no local minima then each element of  $X$  can be joined to  $E(a)$  by a biconnected chain.*

*Proof.* Let  $Z$  denote the compact, continuously partially ordered space which is obtained when  $E(x)$  is identified with a point, for each  $x \in X$ . Let  $\phi(X) = Z$  be the canonical quotient map and let

$$X \xrightarrow{m} Y \xrightarrow{l} Z$$

be the monotone-light factorization of  $\phi$ . It is easy to see that  $Y$  inherits a quasi order from  $Z$  which has a closed graph and is such that  $E(y)$  is totally disconnected, for each  $y \in Y$ . Moreover,  $Y - m(E(a))$  has no local minima and hence, by the theorem, there are order arcs joining points of  $Y$  to  $m(E(a))$ . Since  $m$  is monotone, the corollary follows at once.

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# Pacific Journal of Mathematics

Vol. 15, No. 4

December, 1965

Robert James Blattner, <i>Group extension representations and the structure space</i> .....	1101
Glen Eugene Bredon, <i>On the continuous image of a singular chain complex</i> .....	1115
David Hilding Carlson, <i>On real eigenvalues of complex matrices</i> .....	1119
Hsin Chu, <i>Fixed points in a transformation group</i> .....	1131
Howard Benton Curtis, Jr., <i>The uniformizing function for certain simply connected Riemann surfaces</i> .....	1137
George Wesley Day, <i>Free complete extensions of Boolean algebras</i> .....	1145
Edward George Effros, <i>The Borel space of von Neumann algebras on a separable Hilbert space</i> .....	1153
Michel Mendès France, <i>A set of nonnormal numbers</i> .....	1165
Jack L. Goldberg, <i>Polynomials orthogonal over a denumerable set</i> .....	1171
Frederick Paul Greenleaf, <i>Norm decreasing homomorphisms of group algebras</i> .....	1187
Fletcher Gross, <i>The 2-length of a finite solvable group</i> .....	1221
Kenneth Myron Hoffman and Arlan Bruce Ramsay, <i>Algebras of bounded sequences</i> .....	1239
James Patrick Jans, <i>Some aspects of torsion</i> .....	1249
Laura Ketchum Kodama, <i>Boundary measures of analytic differentials and uniform approximation on a Riemann surface</i> .....	1261
Alan G. Konheim and Benjamin Weiss, <i>Functions which operate on characteristic functions</i> .....	1279
Ronald John Larsen, <i>Almost invariant measures</i> .....	1295
You-Feng Lin, <i>Generalized character semigroups: The Schwarz decomposition</i> .....	1307
Justin Thomas Lloyd, <i>Representations of lattice-ordered groups having a basis</i> .....	1313
Thomas Graham McLaughlin, <i>On relative coimmunity</i> .....	1319
Mitsuru Nakai, <i><math>\Phi</math>-bounded harmonic functions and classification of Riemann surfaces</i> .....	1329
L. G. Novoa, <i>On <math>n</math>-ordered sets and order completeness</i> .....	1337
Fredos Papangelou, <i>Some considerations on convergence in abelian lattice-groups</i> .....	1347
Frank Albert Raymond, <i>Some remarks on the coefficients used in the theory of homology manifolds</i> .....	1365
John R. Ringrose, <i>On sub-algebras of a <math>C^*</math>-algebra</i> .....	1377
Jack Max Robertson, <i>Some topological properties of certain spaces of differentiable homeomorphisms of disks and spheres</i> .....	1383
Zalman Rubinstein, <i>Some results in the location of zeros of polynomials</i> .....	1391
Arthur Argyle Sagle, <i>On simple algebras obtained from homogeneous general Lie triple systems</i> .....	1397
Hans Samelson, <i>On small maps of manifolds</i> .....	1401
Annette Sinclair, <i><math> \varepsilon(z) </math>-closeness of approximation</i> .....	1405
Edsel Ford Stiel, <i>Isometric immersions of manifolds of nonnegative constant sectional curvature</i> .....	1415
Earl J. Taft, <i>Invariant splitting in Jordan and alternative algebras</i> .....	1421
L. E. Ward, <i>On a conjecture of R. J. Koch</i> .....	1429
Neil Marchand Wigley, <i>Development of the mapping function at a corner</i> .....	1435
Horace C. Wisner, <i>Embedding a circle of trees in the plane</i> .....	1463
Adil Mohamed Yaqub, <i>Ring-logics and residue class rings</i> .....	1465
John W. Lamperti and Patrick Colonel Suppes, <i>Correction to: Chains of infinite order and their application to learning theory</i> .....	1471
Charles Vernon Coffman, <i>Correction to: Non-linear differential equations on cones in Banach spaces</i> .....	1472
P. H. Doyle, III, <i>Correction to: A sufficient condition that an arc in <math>S^n</math> be cellular</i> .....	1474
P. P. Saworotnow, <i>Correction to: On continuity of multiplication in a complemented algebra</i> .....	1474
Basil Gordon, <i>Correction to: A generalization of the coset decomposition of a finite group</i> .....	1474