Pacific Journal of Mathematics

EMBEDDING A CIRCLE OF TREES IN THE PLANE

HORACE C. WISER

Vol. 15, No. 4 December 1965

PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

EMBEDDING A CIRCLE OF TREES IN THE PLANE

H. C. WISER

Concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold, it is shown here that there is an embedding if the continuum is decomposable and the manifold is orientable. The embedding is obtained by constructing an annulus on the manifold containing the continuum; in the nonorientable case an annulus or a Möbius strip containing the continuum may be found. Similar results are obtained for continua on a 2-manifold which have a decomposition into continua with zero 1-dimensional Betti numbers such that the decomposition space is a finite planar graph.

This extends the results of [2] concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold. Definitions and a summary of other results may be found in [2].

THEOREM 1. *A decomposable homogeneous proper subcontinuum of an orientable 2-manifold can be embedded in the plane.*

Proof. Let X be a decomposable proper subcontinuum of an orientable 2-manifold M. By Theorem 11 of [2] there is a continuous collection G of disjoint continua filling X such that the decomposition space X' is a simple closed curve and the elements of G are mutually homeomorphic, homogeneous, and treelike. Consider the upper semi-continuous decomposition of M whose nondegenerate elements are the elements of G. By Theorem 1 of [1], the decomposition space M' is homeomorphic to M. Let A' be a closed annular neighborhood of the simple closed curve X' on M'. Let f be the projection map from M to M' and A be $f^{-1}(A')$. Consider A' to be filled by a continuous collection of simple closed curves $\{J_\alpha\}$, $\alpha \in [01]$, where $J_{1/2} = X'$. Then f is one-to-one on A-X and $f^{-1}(J_\alpha)$ must be compact since J_α is compact; thus $f/f^{-1}(J_\alpha)$ is a continuous one-to-one map of a compact set for $\alpha \neq 1/2$ and therefore a homeomorphism. For a closed subinterval I of $[01]$ not containing $1/2, f^{-1}(\Sigma_{\alpha \in I} J_\alpha)$ is a closed annulus. If $p \in A$-$f^{-1}(J_0 + J_1)$ then $f(p)$ is in the interior of A' and thus p is interior to A. The continuum A must then be a 2-manifold with boundary consisting of $f^{-1}(J_0) + f^{-1}(J_1)$. Fit 2-cells C_0 and C_1 to the boundary curves of A to make a 2-manifold M_1 without boundary. Considering an upper semi-continuous decomposition of M_1 with the elements of G as the nondegenerate elements, we have M_1' as merely

Received September 16, 1964. Research supported by NSF Grant GP-107.

A' with 2-cells C_0' and C_1' fitted to J_0 and J_1; i.e., M_1' is a 2-sphere. Using Theorem 1 of [1] again, M_1 must be homeomorphic to M_1' and A must be a closed annulus; X is thus planar.

In the same manner we have the following results:

THEOREM 2. *A decomposable homogeneous proper subcontinuum of a nonorientable 2-manifold is contained in an open annulus or open Möbius strip on the manifold.*

THEOREM 3. *If a subcontinuum of an orientable 2-manifold has an upper semi-continuous decomposition into continua with zero mod 2 1-dimensional Betti numbers*[1] *such that the decomposition space is a finite planar graph then the continuum can be embedded in the plane.*

In view of Theorem 1, a nonplanar homogeneous subcontinuum of an orientable 2-manifold would have to be in the class of nontreelike indecomposable continua. No planar homogeneous continuum in this class is known, although the pseudo-circle is a candidate. It would be nice to eliminate the condition of orientability in Theorem 1.

REFERENCES

1. J. H. Roberts and N. E. Steenrod, *Monotone transformations of two-dimensional manifolds,* Ann. of Math. **39** (1938), 851-863.
2. H. C. Wiser, *Decomposition and homogeneity of continua on a 2-manifold,* Pacific. J. Math. **12** (1962), 1145-1162.

UNIVERSITY OF UTAH
AND
WASHINGTON STATE UNIVERSITY

[1] As in [**1**], the statement "the continuum X on the 2-manifold M has a zero mod 2, 1-dimensional Betti number" means that for any complex K of M containing X there is a smaller complex L of M containing X such that each of its 1-cycles bounds in K.

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should by typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. No separate author's resumé is required. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 15, No. 4 December, 1965