RING-LOGICS AND RESIDUE CLASS RINGS

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Let \((R, \times, +)\) be a commutative ring with unit 1, and let \(K = \{p_1, p_2, \cdots\}\) be a transformation group in \(R\). \((R, \times, +)\) is called a ring-logic, mod \(K\) essentially if the “+” of \(R\) is equationally definable in terms of the “\(K\)-logic” \((R, \times, \rho_1, \rho_2, \cdots)\). The Boolean theory results by choosing \(K\) to be the group generated by \(x^* = 1 - x\) (order 2, \(x^{**} = x\)). The following result is proved: Let \(n = p_1 \cdots p_t\) be square-free, and let \(R_n\) be the residue class ring, mod \(n\). Let \(\circlearrowright\) be any transitive \(0 \rightarrow 1\) permutation of \(R_{p_i}(i = 1, \cdots, t)\). Let \(\circlearrowright\) be the induced permutation of \(R_n\) defined by \((x_1, \cdots, x_t)\circlearrowright = (x_1^{\circlearrowright}, \cdots, x_t^{\circlearrowright})\), \(x_i \in R_{p_i}(i = 1, \cdots, t)\), and let \(K\) be the transformation group in \(R_n\) generated by \(\circlearrowright\). Then \((R_n, \times, +)\) is a ring-logic, mod \(K\). An extension of this theorem to the case where \(n\) is arbitrary is also considered. The present proofs use the Fermat-Euler Theorem as well as a generalized form of the Chinese Residue Theorem.

The motivation for the study of ring-logics stems from the familiar equational interdefinability of Boolean rings \((R, \times, +)\) and Boolean logics (Boolean algebras) \((R, \cap, *)\) [5]. In a series of recent publications ([1]–[4]), Foster raised this equational interdefinability, as well as the entire Boolean theory, to a more general level. In particular, Foster showed [2; 3] that any \(p\)-ring with unit (and more generally, any \(p^k\)-ring with unit) is a ring-logic, modulo certain suitably chosen groups. Furthermore, the author proved [6] that \(R_n\), the residue class ring, mod \(n\), is a ring-logic, modulo the “natural group” (generated by \(x^\circlearrowright = 1 + x\)). Our present object is to further extend these results by considering certain transformation groups in \(R_n\) of rather general nature, and with respect to which \((R_n, \times, +)\) is a ring-logic (see Theorem 5).

1. The ring of residues mod \(p^k\). Let \((R_{p^k}, \times, +)\) be the residue class ring, mod \(p^k\), where \(p\) is prime and \(k \geq 1\). Let \(G\) denote the group of units in \(R_{p^k}\). Then, as is well known, the order of \(G\) is \(\varphi(p^k) = p^k - p^{k-1}\), where \(\varphi(n)\) is the familiar Euler \(\varphi\)-function (=number of positive integers which do not exceed \(n\) and which are relatively prime to \(n\)). Let, \(\circlearrowright\), be a permutation of \(R_{p^k}\). We call, \(\circlearrowright\), a transitive \(0 \rightarrow 1\) permutation if (i) \(0^\circlearrowright = 1\), and (ii) for any elements \(\alpha, \beta\) in \(R_{p^k}\), there exists an integer \(r\) such that \(\alpha^{\circlearrowright r} = \beta\), where \(\alpha^{\circlearrowright r} = (\cdots((\alpha^{\circlearrowright})^{\circlearrowright})^{\circlearrowright})^{\circlearrowright}\) (\(r\)-iterations).
We recall from [4] the characteristic function \( \delta_\mu(x) \), defined as follows: for any given \( \mu \in R_{pk} \), \( \delta_\mu(x) = 1 \) if \( x = \mu \) and \( \delta_\mu(x) = 0 \) if \( x \neq \mu \). Following [4], we also define: \( a \times b = \prec (a^{-1} \times b) \succ \), where, \( \prec \), is the inverse of the \( 0 \rightarrow 1 \) permutation, \( \succ \). One readily verifies that \( a \times \prec 0 = 0 \times \succ a = a \). Hence, we have the following “normal expansion formula” [4]:

\[
(1.1) \quad f(x, y, \cdots) = \sum_{\alpha, \beta, \cdots \in R_{pk}} f(\alpha, \beta, \cdots)(\delta_\alpha(x)\delta_\beta(y)\cdots).
\]

In (1.1), \( \alpha, \beta, \cdots \) range independently over all the elements of \( R_{pk} \) while \( x, y, \cdots \) are indeterminates over \( R_{pk} \). Also, \( \sum_{i \in R} \alpha_i \) denotes \( \alpha_1 \times_\alpha \alpha_2 \times_\alpha \cdots \), where \( \alpha_1, \alpha_2, \cdots \) are all the elements of \( R \).

We now have the following

\textbf{Lemma 1.} Let, \( \prec \), be any transitive permutation of \( R_{pk} \), and let \( K \) be the transformation group in \( R_{pk} \) generated by, \( \prec \). Then all the elements of \( R_{pk} \) are equationally definable in terms of the \( K \)-logic \((R_{pk}, \times, \prec)\).

\textbf{Proof.} Since, \( \prec \), is a transitive permutation of \( R_{pk} \), therefore, \( R_{pk} = \{0, 0^-2, 0^-3, \cdots, 0^{-pk-1}\} \). Similarly, we have, \( xx^-1 \times_\prec x^-2 \times_\prec \cdots x^-r = 0 \), for all \( x \) in \( R_{pk} \). The last equation shows that \( 0 \) (and with it \( 0^-2, 0^-3, \cdots, 0^{-pk-1}\)) is expressible in terms of the \( K \)-logic, and the lemma is proved.

\textbf{Lemma 2.} Let \( G = \{1, \zeta_2, \zeta_3, \cdots, \zeta_p\} \) be the group of units in the residue class ring \((R_{pk}, \times, +)\). Let, \( \prec \), be a transitive \( 0 \rightarrow 1 \) permutation of \( R_{pk} \) satisfying \( 1^- = \zeta_2, \zeta_2^- = \zeta_3, \cdots, \zeta_p^- = \zeta_p \), but otherwise, \( \prec \), is entirely arbitrary. Let \( K \) be the transformation group in \( R_{pk} \) generated by, \( \prec \). Then each characteristic function \( \delta_\mu(x), \mu \in R_{pk} \), is equationally definable in terms of the \( K \)-logic \((R_{pk}, \times, \prec)\).

\textbf{Proof.} Since, \( \prec \), is transitive, therefore, there exists an integer \( r \) such that \( \mu^{-r} = 0 \). Now, one readily verifies that

\[
\tilde{\delta}_\mu(x) = (x^{-r+1}x^{-r+3}x^{-r+3} \cdots x^{-r+p})^{pk-pk-1},
\]

since, by the Fermat-Euler Theorem, \( a^{pk-pk-1} = 1 \) for all \( a \) in \( G \). This proves the lemma.

\textbf{Theorem 3.} Let \( K, \prec \), be as in Lemma 2. Then the residue class ring \((R_{pk}, \times, +)\) is a ring-logic, mod \( K \).

\textbf{Proof.} By (1.1), \( x + y = \sum_{\alpha, \beta \in R_{pk}} (\alpha + \beta)(\delta_\alpha(x)\delta_\beta(y)) \). By Lemma 1 and Lemma 2, each of \( \alpha + \beta, \delta_\alpha(x), \) and \( \delta_\beta(y) \), is expressible in terms
of the $K$-logic. Hence, the "$+$" of $R_{p^k}$ is equationally definable in terms of the $K$-logic. Next, we show that $(R_{p^k}, \times, +)$ is fixed by its $K$-logic. Suppose that $(R_{p^k}, \times, +')$ is another ring with the same class of elements $R_{p^k}$ and the same "$\times$" as $(R_{p^k}, \times, +)$ and which has the same logic as $(R_{p^k}, \times, +)$. To prove that $+ = +'$. But this follows, since, up to isomorphism, there is only one cyclic group of order $p^k$.

2. The general case. In attempting to generalize Theorem 3 to the residue class ring $(R_n, \times, +)$, $n$ arbitrary, we need the following concept of independence, introduced by Foster [4].

**Definition.** Let $\{U_1, \ldots, U_t\}$ be a finite set of algebras of the same species $S$. We say that the algebras $U_1, \ldots, U_t$ are independent or satisfy the Chinese Residue Theorem, if, corresponding to each set $\{\Psi_i\}$ of expressions of species $S$, there exists a single expression $X$ such that $\Psi_i = X \pmod{U_i}$ ($i = 1, \ldots, t$). By an expression we mean some composition of one or more indeterminate-symbols $x, \ldots$ is terms of the primitive operations of $U_1, \ldots, U_t$; $\Psi_i = X \pmod{U_i}$ means that this is an identity of the algebra $U_i$.

As usual, we shall use the same symbols to denote the operation symbols of the algebras $U_1, \ldots, U_t$ when these algebras are of the same species. We now have the following

**Lemma 4.** Let $p_1, \ldots, p_t$ be distinct primes. Let, $\tau$, be any transitive $0 \rightarrow 1$ permutation of $R_{p_i^j}$, and let $K_i$ be the transformation group in $R_{p_i^j}$ generated by $\tau$, ($i = 1, \ldots, t$). Then the $K_i$-logics $(R_{p_i^j}, \times, \tau)(i = 1, \ldots, t)$ are independent.

**Proof.** Let $n = p_1^{i_1} \cdots p_t^{i_t}$ and let $E = x_1^{\tau}x_2^{\tau^2} \cdots x_1^{\tau^{n-1}}$. Let $p_i^{e_i}n_i = n$. Since $(p_i^{e_i}, n_i) = 1$, therefore, there exist integers $r_i, s_i$ such that $r_in_i - s_ip_i^{e_i} = 1$. Now, one readily verifies that

$$
\omega_i = \text{def} = E^{-ri_i} = \begin{cases} 1 \pmod{R_{p_i^{e_i}}}, \\ 0 \pmod{R_{p_j^{e_j}}} \end{cases} (j \neq i).
$$

To prove the independence of the logics $(R_{p_i^{e_i}}, \times, \tau)$, let $\{\Psi_i\}$ be a set of $t$ expressions of species $\times, \tau$; i.e., primitive composition of indeterminate-symbols in terms of the operations $\times, \tau$. Define

$$X = \Psi_1\omega_1 \times_\tau \cdots \times_\tau \Psi_t\omega_t.$$

It is readily verified that $\Psi_i = X \pmod{R_{p_i^{e_i}}}(i = 1, \ldots, t)$, since $a \times_\tau 0 = 0 \times_\tau a = a$. This proves the lemma.
We are now in a position to consider \((R_n, \times, +)\) in regard to the concept of ring-logic. Indeed, let \(n = p_1^{a_1} \cdots p_t^{a_t}\), where the \(p_i\) are distinct primes \((i = 1, \cdots, t)\), and let \(G = \{1, \zeta_{a_1}, \zeta_{a_2}, \cdots, \zeta_{a_t}\}\) be the group of units in the residue class ring \((R_{p_i^{a_i}}, \times, +)\). For each \(i\), define, \(\sim\), to be a transitive \(0 \rightarrow 1\) permutation of \(R_{p_i^{a_i}}\) satisfying \(1 \sim = \zeta_{a_1}, \zeta_{a_2} \sim = \zeta_{a_3}, \cdots, (\zeta_{i, \varphi - 1} \sim = \zeta_{i, \varphi}, \) but otherwise, \(\sim\), is entirely arbitrary, and let \(K_i\) be the transformation group in \(R_{p_i^{a_i}}\) generated by \(\sim\). Now, it is well known that the residue class ring \(R_n\) is isomorphic to the direct product of \(R_{p_i^{a_i}}\), \(i = 1, \cdots, t\):

\[ R_n \cong R_{p_1^{a_1}} \times \cdots \times R_{p_t^{a_t}} \text{ (direct product), } n = p_1^{a_1} \cdots p_t^{a_t}. \]

Furthermore, it is easily seen that by defining \((x_1, \cdots, x_t) \sim = (x_1^-, \cdots, x_t^-)\), \((x_1, \cdots, x_t) \in R_n\), we obtain a transitive \(0 \rightarrow 1\) permutation of \(R_n\). Let \(K\) be the transformation group in \(R_n\) generated by the above permutation, \(\sim\). We now have the following

**THEOREM 5.** The residue class ring \((R_n, \times, +), n\) arbitrary, is a ring-logic, mod \(K\), where \(K\) is the transformation group in \(R_n\) above.

**Proof.** Let \(n = p_1^{a_1} \cdots p_t^{a_t}\), where the \(p_i\) are distinct primes \((i = 1, \cdots, t)\). By Theorem 3, each \((R_{p_i^{a_i}}, \times, +)\) is a ring-logic, mod \(K_i\), where \(K_i\) is as defined above \((i = 1, \cdots, t)\). Hence, for each \(i\), there exists an expression \(\varphi_i\) such that

\[ x_i + y_i = \varphi_i(x_i, y_i; \times, \sim), \text{ for all } x_i, y_i \in R_{p_i^{a_i}}. \]

But, by Lemma 4, the \(K_i\)-logics \((R_{p_i^{a_i}}, \times, \sim)\) are independent \((i = 1, \cdots, t)\), and hence there exists a single expression \(X\) such that \(X = \varphi_i (mod R_{p_i^{a_i}})\) \((i = 1, \cdots, t)\). Now, let \(x = (x_1, \cdots, x_t), y = (y_1, \cdots, y_t)\) be any elements of \(R_n(\cong R_{p_1^{a_1}} \times \cdots \times R_{p_t^{a_t}})\). Since the operations are component-wise in this direct product, therefore,

\[
X(x, y; \times, \sim) = X((x_1, \cdots, x_t), (y_1, \cdots, y_t); \times, \sim)
\]

\[
= (X(x_1, y_1; \times, \sim), \cdots, X(x_t, y_t; \times, \sim))
\]

\[
= (\varphi_i(x_1, y_1; \times, \sim), \cdots, \varphi_i(x_t, y_t; \times, \sim))
\]

\[
= (x_1 + y_1, \cdots, x_t + y_t)
\]

\[
= x + y.
\]

Hence, the “+” of \(R_n\) is **equationally** definable in terms of the \(K\)-logic \((R_n, \times, \sim)\). The proof that \((R_n, \times, +)\) is fixed by its \(K\)-logic follows as in the “fixed” part of the proof of Theorem 3, since again, up to isomorphism, there is only one cyclic group of order \(n\). This completes the proof of the theorem.
We shall now take a closer look at the case where \( n = p_1 \cdots p_t \) is square-free. In this case the group \( G_i \) of units in \( R_{p_i}(=\text{field}) \) is precisely the set of all nonzero elements of \( R_{p_i}(i = 1, \cdots, t) \), and the, \( \sim \), described above (see paragraph preceding Theorem 5) for \( R_{p_i} \) is now simply any transitive \( 0 \rightarrow 1 \) permutation of \( R_{p_i} \). Hence, we have the following

**Corollary 6.** Let \( n = p_1 \cdots p_t \) be square-free, and let, \( \sim \), be any transitive \( 0 \rightarrow 1 \) permutation of \( R_{p_i}(i = 1, \cdots, t) \). Let, \( \sim \), be the induced permutation of \( R_n \) defined by \( (x_1, \cdots, x_t) \sim = (x_1^\sim, \cdots, x_t^\sim) \), \( x_i \in R_{p_i}(i = 1, \cdots, t) \), and let \( K \) be the transformation group in \( R_n \) generated by, \( \sim \). Then \((R_n, \times, +)\) is a ring-logic, mod \( K \).

Thus, if, in particular, we choose \( x^\sim = 1 + x \) in the above Corollary, we obtain the following (compare with [6]).

**Corollary 7.** Let \( n \) be square-free, and let \( N \) be the "natural group", generated by \( x^\sim = 1 + x \). Then \((R_n, \times, +)\) is a ring-logic, mod \( N \).

Upon choosing, \( \sim \), in Theorem 5 in all of the various available ways, we obtain the corresponding transformation groups \( K \) with respect to which \((R_n, \times, +)\) is a ring-logic.

**References**


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