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GROUP EXTENSION REPRESENTATIONS AND THE STRUCTURE SPACE

ROBERT J. BLATTNER

Let K be a locally compact group. K^* will denote the Jacobson structure space of $C^*(K)$, the group C^*-algebra of K. For any unitary representation V of K on a Hilbert space, let E_V denote the projection valued measure on the Borel sets of K^* defined by Glimm (Pacific J. Math. 12 (1962), 885-911; Theorem 1.9). A (not necessarily Borel) subset S of K^* is called E_V-thick if $E_V(S_1) = 0$ for every Borel $S_1 \subseteqq K^* \sim S$. For any two representations V_1 and V_2, $\mathscr{R}(V_1, V_2)$ will denote the space of operators intertwining V_1 and V_2.

Suppose K is a closed normal subgroup of the locally compact group G. If V is a representation of K and $x \in G$, V^x is defined by $V_k^x = V_{xkx^{-1}}$, $k \in K$. If $z \in K^*$, $zx = \mathrm{Ker}\,(V^x)$, where V is any irreducible repesentation such that $z = \mathrm{Ker}\,(V)$. (By Ker we mean the kernel in the group C^*-algebra.) This composition turns (K^*, G) into a topological transformation group (Glimm, op. cit., Lemma 1.3). The present paper first shows that the stability subgroups of G at points $z \in K^*$ are closed. Then the following two theorems are proved:

THEOREM 1. Let $z \in K^*$ and let H be the stability subgroup of G at z. Let L be a representation of H such that $\{z\}$ is $E_{L|K}$-thick. Then $\mathscr{R}(U^L, U^L)$ is isomorphic to $\mathscr{R}(L, L)$ and $\{z\}G$ is $E_{U^L|K}$-thick.

THEOREM 2. Let M be a representation of G such that $\{z\}G$ is $E_{M|K}$-thick for some $z \in K^*$. Let H be the stability subgroup of G at z. Suppose G/H is σ-compact. Then there is a representation L of H such that $\{z\}$ is $E_{L|K}$-thick and such that $M \simeq U^L$.

In the above, U^L denotes the representation of G induced by L.

It is shown further that if $C^*(K)|z$ contains an ideal isomorphic to the algebra of all compact operators on some Hilbert space, then the representation $L|K$ of these theorems is a multiple of the (essentially unique) irreducible representation L^0 of K such that $\mathrm{Ker}\,(L^0) = z$. Finally, it is shown that if M is primary and if K^*/G is almost Hausdorff (i.e., every nonvoid closed subset contains a nonvoid relatively open Hausdorff subset), then M satisfies the hypothesis of Theorem 2.

These results generalize Mackey's Theorem 8.1 [13], in the case of the trivial multiplier. In [13], Mackey attacks the problem of

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reducing the representation theory of a locally compact group G with closed normal subgroup K to the representation theories of K and G/K. His main theorem, Theorem 8.1, supposes the following restrictions on G and K: G satisfies the second axiom of countability and K is type I (in the case of the trivial multiplier). The present paper explores what happens when these restrictions are lifted.

It turns out that a great deal of Mackey's theorem remains true in modified form. The chief modifications are these:

(a) We replace the dual space $\hat{K}$ of K by the structure space K^* of its group C^*-algebra. This is done because K^* is fairly well behaved, being a T_0 topological space, while $\hat{K}$ can be very messy when K is not type I. The first example of § 6 shows how a theory based on $\hat{K}$ cannot get off the ground.

(b) We replace the projection valued measure based on $\hat{K}$ which is canonically associated with the direct integral decomposition of a given representation of K (when K is type I) with the measure based on K^* introduced by Glimm in [10].

These modifications and the lack of separability force us to replace Mackey's highly measure theoretic arguments with arguments more in the spirit of the present author's previous work [1–3] and that of Glimm's paper [10].

After the preliminaries of § 2, we prove our analogue of Mackey's theorem in §§ 3 and 4. Section 5 is concerned with what additional hypotheses are needed to make the analogue exact. The paper closes with some examples in § 6.

The problem dealt with in the last example was the starting point for this investigation. We wish to thank James Glimm for several stimulating conversations on this problem.

2. Preliminaries. Let G be a locally compact group and let $C_0(G)$ be the space of continuous complex valued functions on G with compact support. If $f, g \in C_0(G)$ set $(f \circ g)(x) = \int_G f(y)g(xy^{-1})dy$ and $f^*(x) = \overline{f^*(x^{-1})}\delta_G(x)^{-1}$, where dy denotes right invariant Haar measure and where δ_G is its modular function. $\circ$, $*$, the usual addition of functions, and the usual inductive limit topology on $C_0(G)$ turn it into a topological $*$-algebra. Let L be a unitary representation of G. Then setting $L_f = \int_G f(x)L_x^{-1}\,dx$ (strong operator topology integral) gives us a continuous $*$-representation of $C_0(G)$. Moreover $\{L_f : f \in C_0(G)\}$ has no simultaneous null vectors (we say that L is a *nondegenerate* representation of $C_0(G)$). Conversely, if Φ is a nondegenerate continuous $*$-representation of $C_0(G)$, there is a unique unitary representation L of G such that $L_f = \Phi_f$ for $f \in C_0(G)$.

If L is unitary representation of G, then $\| L_f \| \leqq \| f \|_1$ for $f \in C_0(G)$.

Therefore $||f|| = LUB\{||L_f||: L$ unitary representation of $G\}$ exists. $||\cdot||$ is a norm on $C_0(G)$ and the completion of $C_0(G)$ with respect to $||\cdot||$ is a C^*-algebra, called the C^*-group algebra of G and denoted by $C^*(G)$. Clearly there is a one-to-one correspondence between nondegenerate $*$-representations of $C^*(G)$ and nondegenerate continuous $*$-representations of $C_0(G)$. Thus the representation theory of $C^*(G)$ is "the same" as that of G.

In what follows, G^* will denote the Jacobson structure space of $C^*(G)$; i.e. the space of kernels of irreducible nondegenerate $*$-representations of $C^*(G)$ equipped with the hull-kernel topology. G^* is a T_0-space.

Let $\mathfrak{A}$ be a C^*-algebra, Z its structure space, and Φ a nondegenerate $*$-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. Glimm [10] has shown that there is unique projection valued measure E on the Borel field generated by the topology of Z with the following property: if S is a closed subset of Z, then $E(S)$ is the projection on the manifold of $v \in \mathfrak{H}$ such that $\Phi(a)v = 0$ for all $a \in \cap S$. E takes its values in the center of the von Neumann algebra generated by $\Phi(\mathfrak{A})$. In our case, if L is a unitary representation of G, E_L will denote the Glimm measure on G^* associated with the representation of $C^*(G)$ determined by L.

For the formulation of induced representations used in this paper the reader is referred to [1]. If L is a representation of the closed subgroup H of G, we define a regular Borel projection valued measure E^L on G/H as follows: if S is a Borel subset of G/H, then $E^L(S)f = (\chi_S \circ \pi)\cdot f$, where χ_S is the characteristic function of S, π is the canonical projection of G onto G/H, and $f \in \mathfrak{H}(U^L)$. This E^L determines, and is determined by, the $*$-representation of $C_0(G/H)$ defined by

$$E^L(h) = \int_{G/H} h(p)dE^L(p) ,$$

$h \in C_0(G/H)$ (cf. [3]).

Finally, if E is any projection valued measure on the measurable space $(Z, \mathscr{B})$, any subset $S \subseteq Z$ (not necessarily in $\mathscr{B}$) will be called *E-thick* if $E(T) = 0$ whenever $T \cap S = \varnothing$ (cf. [11], p. 74).

Let G be a locally compact group and let K be a closed normal subgroup. For $f \in C_0(K)$ and $x \in G$, we define $xf \in C_0(K)$ by the formula $(xf)(\xi) = f(x^{-1}\xi x)\Delta(x)$ for $\xi \in K$, where $\Delta(x)$ is the (constant) Radon-Nikodym derivative $[d(x^{-1}\xi x)]/d\xi$. If L is a unitary representation of K and if L^x is defined by $L^x_\xi = L_{x\xi x^{-1}}$, $\xi \in K$, then $L^x_f = L_{xf}$. From this it follows readily that $f \to xf$ is an automorphism of $C_0(K)$ which is isometric in the C^*-norm $||\cdot||$. Therefore this map extends to an automorphism of $C^*(K)$. We define an action of G on K^* by setting $zx = \{f \in C^*(K) : xf \in z\}$ for $z \in K^*$, $x \in G$. Glimm [10] and Fell [7] have shown that the map of $K^* \times G \to K^*$ given by $(z, x) \to zx$ is continuous, giving us a topological transformation group.

LEMMA 1. *Let* (S, G) *be a topological transformation group. Suppose* S *is a* T_0*-space. Then the stability subgroups of* G *are closed.*

Proof. Let H be the stability subgroup of G at $p \in S$. Then $\{p\}H = \{p\}$, so that $\{p\}^- H^- \subseteq \{p\}^-$. If $x \in H^-$, we have $\{px\}^- = \{p\}^- x \subseteq \{p\}^-$. Since $x^{-1} \in H^-$, we have $\{p\}^- x^{-1} \subseteq \{p\}^-$ and hence $\{p\}^- \subseteq \{p\}^- x = \{px\}^-$; i.e., $\{px\}^- = \{p\}^-$. But S is T_0. Therefore $px = p$ and $x \in H$.

We may now state our main theorems.

THEOREM 1. *Let* G *be a locally compact group and let* K *be a closed normal subgroup of* G. *Let* $z \in K^*$ *and let* H *be the (closed) stability subgroup of* G *at* z. *Let* L *be a representation of* H *such that* $\{z\}$ *is* $E_{L|K}$*-thick. Then* $\mathscr{R}(U^L, U^L)$ *is isomorphic to* $\mathscr{R}(L, L)$ *and* $\{z\}G$ $E_{U^L|K}$*-thick.*

THEOREM 2. *Let* G *be a locally compact group and let* K *be a closed normal subgroup of* G. *Let* M *be a representation of* G. *Assume that* $\{z\}G$ *is* $E_{M|K}$*-thick for some* $z \in K^*$. *Let* H *be the (closed) stability subgroup of* G *at* z. *Suppose* G/H *is* σ*-compact. Then there is a representation* L *of* H *such that* $\{z\}$ *is* $E_{L|K}$*-thick and such that* $M \simeq U^L$.

3. Proof of Theorem 1.

We begin our proofs with the following lemma.

LEMMA 2. *Let* H *and* K *be closed subgroups of the locally compact group* G, K *normal, and* $K \subseteq H \subseteq G$. *Let* L *be a representation of* H. *Then for every* $f \in C^*(K)$ *and* $g \in \mathfrak{H}(U^L)$ *we have* $[(U^L \mid K)_f g](x) = (L \mid K)_f^x g(x)$ *for locally almost all* $z \in G$.

Proof. Suppose first that $f \in C_0(K)$ and g is continuous with compact support modulo H. Set

$$u(x) = (L \mid K)_f^x g(x) = \int_K f(\xi) L_{x\xi^{-1}x^{-1}} g(x) d\xi = \int_K f(\xi) g(x\xi^{-1}) d\xi \, .$$

Clearly u is continuous with compact support modulo H and belongs to $\mathfrak{H}(U^L)$. Let $v \in \mathfrak{H}(U^L)$ be continuous with compact support modulo H, and choose $h \in C_0(G)$ such that $\int_H h(\eta x) d\eta = 1$ for x in the support of v. Then

$$(u, v) = \int_G h(x)(u(x), v(x))\,dx = \int_G\int_K h(x)f(\xi)(g(x\xi^{-1}), v(x))d\xi dx$$
$$= \int_K f(\xi)(U^L_{\xi^{-1}}g, v)d\xi = ((U^L \mid K)_f g, v) .$$

Since the set of all such v is dense in $\mathfrak{H}(U^L)$, our result holds in this case.

Next suppose $f \in C_0(K)$ and $g \in \mathfrak{H}(U^L)$. Choose a sequence $g_n \in \mathfrak{H}(U^L)$, continuous with compact support modulo H, such that $\| g_n - g \| < 2^{-n}$. Then $\| (U^L \mid K)_f g_n - (U^L \mid K)_f g \| < \| f \| 2^{-n}$. As in the proof of Proposition 1 of [1], $g_n \to g$ and $(U^L \mid K)_f g_n \to (U^L \mid K)_f g$ locally almost everywhere.

Finally, if $f \in C^*(K)$ and $g \in \mathfrak{H}(U^L)$, we may choose a sequence $f_n \in C_0(K)$ such that $\| f_n - f \| < 2^{-n}$. Then $\| (L \mid K)^x_f - (L \mid K)^x_f \| < 2^{-n}$ uniformly for all $x \in G$, so that $(L \mid K)^x_{f_n} g(x) \to (L \mid K)^x_f g(x)$ uniformly on G. Moreover $\| (U^L \mid K)_{f_n} g - (U^L \mid K)_f g \| < 2^{-n} \| g \|$, from which it follows that $(U^L \mid K)_{f_n} g \to (U^L \mid K)_f g$ locally almost everywhere. Our lemma is thereby proved.

We now assume all the hypotheses of Theorem 1. If π is the natural projection of G into G/H, we define $\alpha : G/H \to K^*$ by $\alpha(\pi(x)) = zx$ for $x \in G$. α is continuous and one-to-one.

LEMMA 3. *$E_{U^L|K}(S) = E^L(\alpha^{-1}(S))$ for every Borel set $S \subseteqq K^*$.*

Proof. In the first place, we note that $(L \mid K)_f = 0$ for $f \in z$. In fact, $\{z\} \cap \complement\{z\}^- = \emptyset$ implies that $E_{L|K}(\complement\{z\}^-) = 0$ from which we get $E_{L|K}(\{z\}^-) = I$. But this says that $(L \mid K)_f v = 0$ for all $v \in \mathfrak{H}(L)$ and all $f \in \cap \{z\}^- = z$, as desired.

Let S be a closed subset of K^*. Then $\pi(x) \in \alpha^{-1}(S)$ if and only if $zx \in S$, that is, if and only if $zx \supseteqq \cap S$. Therefore $f \in \cap S$ implies $xf \in z$ and hence $(L \mid K)^x_f = 0$. Let $g \in \mathfrak{H}(U^L)$. Let $f \in \cap S$. By Lemma 2, we have $[(U^L \mid K)_f g](x) = 0$ for locally almost all $x \in \pi^{-1}(\alpha^{-1}(S))$. If, moreover, $g \in$ Range $(E^L(\alpha^{-1}(S)))$ then $g(x) = 0$ for locally almost all $x \notin \pi^{-1}(\alpha^{-1}(S))$, so by Lemma 2, $[(U^L \mid K)_f g](x) = 0$ for locally almost all $x \notin \pi^{-1}(\alpha^{-1}(S))$. We conclude that Range $(E^L(\alpha^{-1}(S))) \subseteqq$ Range $(E_{U^L|K}(S))$.

Suppose now that $g \notin$ Range $(E^L(\alpha^{-1}(S)))$. Then g does not vanish for locally almost all $x \notin \pi^{-1}(\alpha^{-1}(S))$. Since g is Bourbaki measurable ([4], p. 180), there exists a compact set $C \subseteqq \complement\pi^{-1}(\alpha^{-1}(S))$ of positive Haar measure upon which g is continuous and does not vanish. Let $x \in C$. Then $z \notin Sx^{-1}$ so that $E_{L|K}(Sx^{-1}) = 0$. Hence there exists $f \in \cap \{Sx^{-1}\}$ such that $(L \mid K)_f g(x) \neq 0$. Setting $f_{(x)} = x^{-1}f$, we have $f_{(x)} \in S$ and $(L \mid K)^x_{f_{(x)}} g(x) \neq 0$. By continuity we have $(L \mid K)^y_{f_{(x)}} g(y) \neq 0$

for y in some neighborhood N_x of x in C. Since C is compact and of positive measure, N_x has positive measure for some $x \in C$. It follows from Lemma 2 that for that x, $(U^L \mid K)_{f(x)} g \neq 0$. Therefore $g \notin \text{Range } (E_{U^L|K}(S))$.

We have proved Lemma 3 for closed S. The general case then follows from the fact that a projection valued Borel measure on a topological space is uniquely determined by its values on closed sets.

LEMMA 4. *Let H be a closed subgroup of the locally compact group G and let E be a regular Borel projection valued measure on G/H. Let $\mathscr{T}$ be a T_0 topology on G/H weaker than the natural topology such that $((G/H)_{\mathscr{T}}, G)$ is a topological transformation group. Then E takes its values in the von Neumann algebra generated by $\{E(S) : S \in \mathscr{T}\}$.*

Proof. Let B be a Borel set in G/H and let T be a self-adjoint bounded operator commuting with all $E(S)$, $S \in \mathscr{T}$. We must show that $E(\complement B)TE(B) = 0$. Since E is regular, it will suffice to show that $E(C_1)TE(C_2) = 0$ for every disjoint compact pair C_1, $C_2 \subseteqq G/H$. A standard compactness argument reduces the problem to the following: if $p_1, p_2 \in G/H$, $p_1 \neq p_2$, find disjoint neighborhoods N_1 of p_1 and N_2 of p_2 such that $E(N_1)TE(N_2) = 0$. To do this we find a $\mathscr{T}$-closed S which separates p_1 and p_2; say, $p_1 \notin S$, $p_2 \in S$. Since S is closed in the natural topology of G/H, we can find a compact neighborhood N of e in G such that $p_1 NN^{-1} \cap S = \varnothing$. Set $N_1 = p_1 N$, $N_2 = p_2 N$. Clearly $N_1 \subseteqq \complement SN$ and $N_2 \subseteqq SN$. Since S is $\mathscr{T}$-closed and N is compact, SN is $\mathscr{T}$-closed. By hypothesis $E(\complement SN)TE(SN) = 0$, and our result follows.

Proof of Theorem 1. Let $\mathscr{T} = \alpha^{-1}$ (topology of K^*). $\mathscr{T}$ and E^L satisfy the hypotheses of Lemma 4. According to Lemma 3 $\{E^L(S) : S \in \mathscr{T}\} \subseteqq \{\text{values of } E_{U^L|K}\}$. This, in turn, is contained in the center $\mathscr{C}$ of the von Neumann algebra generated by $U^L \mid K$. By Lemma 4, $\{\text{values of } E^L\} \subseteqq \mathscr{C}$. Therefore $\mathscr{R}(U^L, U^L) = \mathscr{R}((E^L, U^L), (E^L, U^L) \cong \mathscr{R}(L, L)$ by [2]. Finally, $\{z\}G$ is $E_{U^L|K}$-thick by Lemma 3.

4. Proof of Theorem 2.

For the proof of Theorem 2 we need the following lemmas;

LEMMA 5. *Let H be a closed subgroup of the locally compact group G such that G/H is σ-compact. Let $\mathscr{T}$ be a T_0 topology on G/H weaker than the natural topology such that $((G/H)_{\mathscr{T}}, G)$ is a topological transformation group. Let $\mathscr{B}$ be the Borel field generated by $\mathscr{T}$. Let $f \in C_0(G/H)$. Then f is $\mathscr{B}$-measurable.*

Proof. As is well known, it is enough to show that $C \in \mathscr{B}$, where $C = \bigcap_i^\infty O_i$, C is compact and the O_i are open in G/H. (See [11], p. 220. Such a set is called a compact G_δ.) Since $\complement O_i$ is closed, it is σ-compact, whence $\complement C$ is σ-compact. Therefore $\complement C$ has the Lindelöf property. Let $p \in C$, $q \in \complement C$. As in the proof of Lemma 4, there exist open neighborhoods N_{pq} of p and M_{pq} of q and a set $S_{pq} \subseteqq G/H$ such that either S_{pq} or $\complement S_{pq} \in \mathscr{T}$, $N_{pq} \subseteqq S_{pq}$, $M_{pq} \subseteqq \complement S_{pq}$. Since C is compact, we find $p_1, \cdots, p_n$ such that $C \subseteqq \bigcup_{i\,1}^n N_{p_iq}$ and set $S_q = \bigcup_{i\,1}^n S_{p_iq}$ and $M_q = \bigcap_{i\,1}^n M_{p_iq}$. Then $S_q \in \mathscr{B}$, $C \subseteqq S_q$, $M_q \subseteqq \complement S_q$, and M_q is an open neighborhood of q. Since $\complement C$ is Lindelöfian, we find $q_1, q_2, \cdots$ such that $\complement C \subseteqq \bigcup_{i\,1}^\infty M_{q_i}$ and set $S = \bigcap_{i\,1}^\infty S_{q_i}$. Then $S \in \mathscr{B}$. $C \subseteqq S$, $\complement C \subseteqq \complement S$; that is, $C \in \mathscr{B}$.

LEMMA 6. *Let G be a locally compact group, K a closed normal subgroup, and U a representation of G. Let G act on K^* as above. Then, for any Borel set S in K^* and $x \in G$, $E_{U|K}(Sx) = U_x^{-1}E_{U|K}(S)U_x$.*

Proof. By the uniqueness of Glimm measure, it suffices to prove this for S closed in K^*. We note that $(U\,|\,K)^{x^{-1}} = U_x^{-1}(U\,|\,K)U_x$. Therefore $v \in \text{Range}\, E_{U|K}(Sx)$ if and only if $(U\,|\,K)_f v = 0$ for all $f \in \cap Sx$, if and only if $(U\,|\,K)_f^{x^{-1}} v = 0$ for all $f \in \cap S$, if and only if $U_x^{-1}(U\,|\,K)_f U_x v = 0$ for all $f \in \cap S$. But this is true if and only if $U_x v \in \text{Range}\; E_{U|K}(S)$, if and only if $v \in \text{Range}\; U_x^{-1} E_{U|K}(S) U_x$.

LEMMA 7. *Let U be a representation of the locally compact group K. Let $\mathscr{S}$ be a collection of closed subsets of K^*. Then* Range $E_U(\cap \mathscr{S}) = \cap\{\text{Range}\; E_U(S)\,;\; S \in \mathscr{S}\}$.

Proof. Let $S_0 = \cap \mathscr{S}$. Then $\cap S_0 =$ the closed ideal of $C^*(K)$ generated by $\cup\{\cap S : S \in \mathscr{S}\} =$ the closed linear span in $C^*(K)$ of $\cup\{\cap S : S \in \mathscr{S}\}$. Now let $v \in \cap\{\text{Range}\, E_U(S) : S \in \mathscr{S}\}$. Then, for every $S \in \mathscr{S}$ and every $f \in \cap S$, we have $U_f v = 0$; that is, $U_f v = 0$ for every $f \in \cup\{\cap S : S \in \mathscr{S}\}$. By linearity and continuity, $U_f v = 0$ for every $f \in \cap S$. Therefore $v \in \text{Range}\; E_U(S_0)$. The opposite inclusion is clear, since E_U is monotonic.

Proof of Theorem 2. Let α be as above, let $\mathscr{T} = \alpha^{-1}$ (topology of K^*), and let $\mathscr{B}$ be the Borel field generated by $\mathscr{T}$. Then $\mathscr{B} = \alpha^{-1}$ (Borel field of K^*). As in [11], p. 75, $E_0(\alpha^{-1}(S)) = E_{M|K}(S)$ for all Borel S in K^* defines a projection valued measure E_0 on $\mathscr{B}$. According to Lemma 5, every function in $C_0(G/H)$ is $\mathscr{B}$-measurable. Define $E(f) = \int_{G/H} f dE_0$. Clearly E is a *-representation of $C_0(G/H)$ in the sense of [3]. We assert that (E, M) is a representation of the locally compact transformation group $(G/H, G)$ as defined in [3].

(1) $E(C_0(G/H))\mathfrak{H}(M)$ is dense in $\mathfrak{H}(M)$. In fact, since G/H is σ-compact, there exists a sequence of functions $f_n \in C_0(G/H)$ such that $0 \leqq f_n \uparrow 1$. By the monotone convergence theorem, $E(f_n) \rightarrow I$ weakly, and (1) is established.

(2) $M_x E(f) M_x^{-1} = E(R_x f)$ for all $f \in C_0(G/H)$ and all $x \in G$. Here $(R_x f)(p) = f(px)$. For this, it suffices to show that $M_x E_0(B) M_x^{-1} = E_0(Bx^{-1})$ for all $B \in \mathscr{B}$ and $x \in G$. But this follows immediately from Lemma 6 and the definition of E_0.

According to the Corollary of Theorem 2 in [3], (E, M) is unitarily equivalent to an induced representation of $(G/H, G)$; that is, there is a representation L of H such that (E, M) is unitarily equivalent to (E^L, U^L). We shall henceforth assume $E = E^L$ and $M = U^L$. In particular, we have $E_0 \circ \alpha^{-1} = E_{U^{L|K}}$ and also $E_0(C) = E^L(C)$ for every compact G_δ C in G/H.

We must now show that $\{z\}$ is $E_{L|K}$-thick.

Let S be K^* closed. First suppose $z \in S$. We assert that $E_{L|K}(S) = I$. By Lemma 7, it suffices to show that $E_{L|K}(SN) = I$ for every compact neighborhood N of e in G. Let $g \in C_0(G)$, $v \in \mathfrak{H}(L)$. As in [1], we define $\varepsilon(g, v) \in \mathfrak{H}(U^L)$ by

$$\varepsilon(g, v)(x) = \int_H g(\xi x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}L_\xi^{-1}v\,d\xi \ .$$

$\varepsilon(g, v)$ is continuous and has compact support modulo H. Let C be a compact G_δ neighborhood of $\pi(e) \subseteqq \pi(N)$. Suppose Support $(g) \subseteqq N \cap \pi^{-1}(C)$. Then

$$\begin{aligned}\varepsilon(g, v) \in \text{Range } E^L(C) &= \text{Range } E_0(C) \subseteqq \text{Range } E_0(\alpha^{-1}(SN)) \\ &= \text{Range } E_{U^{L|K}}(SN) \ .\end{aligned}$$

According to Lemma 2, $f \in \cap SN$ implies $(L \mid K)_f^x \varepsilon(g, v)(x) = 0$ for locally almost all $x \in G$, hence for all $x \in G$ by continuity. In particular, $(L \mid K)_f \varepsilon(g, v)(e) = 0$. Letting $g\delta_H^{-1/2}\delta_G^{1/2} \mid H$ approach the Dirac δ function on H, we get $(L \mid K)_f v = 0$. Since v is arbitrary, $E_{L|K}(SN) = I$.

Now suppose $z \notin S$. We assert that $E_{L|K}(S) = 0$. Let $v \in \text{Range}\, E_{L|K}(S)$. Choose a compact neighborhood N of e in G such that $\pi(NN^{-1}) \cap \alpha^{-1}(S) = \varnothing$. Then $\alpha^{-1}(SN) \cap \pi(N) = \varnothing$. Let $f \in \cap SN$. Let $x \in G$. Then $xf \in \cap SNx^{-1}$. Hence if $\xi \in Nx^{-1} \cap H$, we have $xf \in \cap S\xi$, so that $\xi xf \in \cap S$. Let C be a compact G_δ neighborhood of $\pi(e)$ in $\pi(N)$. Suppose Support $(g) \subseteqq N \cap \pi^{-1}(C)$. Then

$$(L \mid K)_f^x \varepsilon(g, v)(x) = \int_{N_x^{-1} \cap H} g(\xi x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}L_\xi^{-1}(L \mid K)_{\xi xf}v\,d\xi = 0$$

(compare the proof of Lemma 6). From Lemma 2,

$$\varepsilon(g, v) \in \text{Range } E_{UL|K}(SN) = \text{Range } E_0(\alpha^{-1}(SN)) .$$

On the other hand $\varepsilon(g, v) \in$ Range $E^L(C) =$ Range $E_0(C)$. Since $\alpha^{-1}(SN) \cap C = \varnothing$, $\varepsilon(g, v) = 0$. Therefore $\varepsilon(g, v)(e) = 0$ because $\varepsilon(g, v)$ is continuous. Again letting $g\delta_H^{-1/2}\delta_G^{1/2} \mid H$ approach the Dirac δ function on H, we get $v = 0$. Therefore $E_{L|K}(S) = 0$.

Finally let $\mathscr{C}$ be the class of all Borel sets S in K^* such that either $z \in S$ and $E_{L|K}(S) = I$ or $z \notin S$ and $E_{L|K}(S) = 0$. Clearly $\mathscr{C}$ is a σ-field, and by the foregoing $\mathscr{C}$ contains all the closed sets. Therefore $\mathscr{C}$ consists of all the Borel sets of K^*; that is, $\{z\}$ is $E_{L|K}$-thick.

5. **Connections with Mackey's work.** In the original forms of Theorems 1 and 2 due to Mackey ([13], Theorem 8.1), it is assumed that G satisfies the second axiom of countability and that K is a type I group. K^* is replaced there by $\hat{K}$, the set of all unitary equivalence classes of irreducible representations of K, equipped with the Mackey Borel structure. According to Glimm ([8], Theorem 1) the natural mapping of $\hat{K}$ onto K^*, which sends every irreducible representation into its kernel in $C^*(K)$, is one-to-one if K is of type I and second countable. Moreover, Fell has shown [6] that in this case the Mackey Borel structure is just the σ-field generated by the topology of $\hat{K}(= K^*)$. Our result then specializes to give Mackey's result, except for the following: Mackey shows that $L \mid K$ must be a multiple of the (unique up to unitary equivalence) representation (whose kernel is) z. To get this, in our general setting, seems to require a type restriction on K (or at least on z). The form of our restriction is suggested by Glimm's theorem ([8], Theorem 1) that a separable C^* algebra is of type I if and only if its image under every irreducible representation contains the compact operators. We are led to make the following definition:

DEFINITION. Let $\mathfrak{A}$ be a C^*-algebra and let L be an irreducible representation of $\mathfrak{A}$. L is called *semi-compact* if $L_{\mathfrak{A}}$ contains the compact operators on $\mathfrak{H}(L)$. Ker L will also be called semi-compact.

We know (see Glimm [1], p. 583) that if L is semi-compact and if M is irreducible with Ker $L =$ Ker M, then L and M are unitarily equivalent.

LEMMA 8. *Let U be a representation of the C^*-algebra $\mathfrak{A}$ with structure space Z. Let z be semi-compact in Z. Suppose $\{z\}$ is E_U-thick. Then U is a multiple of the (essentially unique) irreducible representation L^0 of K such that* Ker $L^0 = z$.

Proof. By hypothesis $\mathfrak{A}$ contains an ideal $\mathscr{I} \supseteq z$ such that $\mathscr{I}/z$

is isomorphic to the algebra of all compact operators on $\mathfrak{H}(L^0)$. As in the proof of Lemma 3, $E_U(\{z\}^-) = I$ implies that $U_a = 0$ for all $a \in z$. Dividing out by z, we may therefore assume $z = \{0\}$. Let $S = \{w \in Z : w \supseteq \mathscr{J}\}$. S is closed. $\mathscr{J} \neq \{0\}$ implies $\{0\} \notin S$. Therfore $E_U(S) = 0$. Since $\mathscr{J} = \cap S$, this says that $U | \mathscr{J}$ is a nondegenerate representation of $\mathscr{J}$. From the known representation theory of the algebra of compact operators on a Hilbert space, we obtain an orthogonal decomposition of $\mathfrak{H}(U)$ into $U | \mathscr{J}$ invariant subspaces $\mathfrak{H}^\gamma$, the restriction of $U | \mathscr{J}$ to each of which is unitarily equivalent to the irreducible representation $L^0 | \mathscr{J}$. Let $a \in \mathfrak{A}$, $v \in \mathfrak{H}^\gamma$. Since $U | \mathscr{J}$ is nondegenerate, we can choose a sequence $b_n \in \mathscr{J}$ such that $U_{b_n} U_a v \rightarrow U_a v$. But $b_n a \in \mathscr{J}$ and hence $U_{b_n a} v \in \mathfrak{H}^\gamma$. Therefore $U_a v \in \mathfrak{H}^\gamma$; that is, the $\mathfrak{H}^\gamma$ are invariant under U. Let $L^\gamma = U$ restricted to act on $\mathfrak{H}^\gamma$. L^γ is irreducible since $L^\gamma | \mathscr{J}$ is. Now Range $E_U(\{\text{Ker } L^\gamma\}^-) \supseteq \mathfrak{H}^\gamma \neq \{0\}$. Since $\{0\}$ is E_U-thick, $\{0\} \in \{\text{Ker } L^\gamma\}^-$; that is, $\text{Ker } L^\gamma = \{0\}$. Since $\text{Ker } L^\gamma = \text{Ker } L^0$, $L^\gamma \simeq L^0$. Therefore $U \simeq$ a multiple of L^0.

As regards Theorem 2, Mackey shows that if, in addition to the hypotheses on G and K mentioned above, one assumes that K is regularly embedded in G (see [13], p. 302 for the definition), then $E_{M|K}$ is concentrated in an orbit if M is primary. Glimm ([9], Theorem 1) has proved that, in Mackey's case, the assumption of regular embeddedness is equivalent to the topology of $\hat{K}/G$ being almost Hausdorff, in the following sense:

DEFINITION. Let X be a topological space. X is said to be *almost Hausdorff* if every nonvoid closed subset contains a nonvoid relatively open Hausdorff subset.

We propose to turn Glimm's theorem into a definition, even when K is not of type I and second countable.

DEFINITION. Let K be a closed normal subgroup of the locally compact group G. K is *regularly embedded* in G if K^*/G is almost Hausdorff.

REMARK. It follows from [9], p. 133, that K regularly embedded in G implies that every G-orbit in K^* is a Borel set and in fact is relatively open in its closure.

LEMMA 9. *Let K be a regularly embedded closed normal subgroup of the locally compact group G. Let M be a primary representation of G. Then there is a G-orbit of K^* which is $E_{M|K}$-thick.*

Proof. If S is a G-invariant Borel set in K^*, then

$$E_{M|K}(S) \in \mathscr{R}(M, M)$$

by Lemma 6, and hence $E_{M|K}(S)$ belongs to the center of the von Neumann algebra generated by M. Therefore $E_{M|K}(S) = 0$ or I. Let $\mathscr{S}$ be the collection of all closed G-invariant $S \subseteqq K^*$ such that $E_{M|K}(S) = I$. $S_0 = \cap \mathscr{S} \in \mathscr{S}$ by Lemma 7. Let θ be the natural projection of K^* onto K^*/G. If R is any nonvoid relatively open subset of $\theta(S_0)$, then $S_0 - \theta^{-1}(R)$ is a proper closed G-invariant subset of S_0. Hence $E_{M|K}(S_0 - \theta^{-1}(R)) = 0$ so that $E_{M|K}(\theta^{-1}(R)) = I$. Now $\theta(S_0)$ is a nonvoid closed subset of K^*/G. There exists a nonvoid relatively open Hausdorff subset R_0 of $\theta(S_0)$. We assert that R_0 reduces to a point. If not, then R_0 contains two nonvoid disjoint subsets R_1 and R_2 which are open relative to R_0 and hence to $\theta(S_0)$. Then $E_{M|K}(\theta^{-1}(R_i)) = I$ for $i = 1, 2$, an impossibility. So R_0 reduces to a point, $\theta^{-1}(R_0)$ is a G-orbit in K^*, and our lemma is proved.

REMARK. This is not the only reasonable definition of regular embeddedness. Indeed, if G satisfies the second axiom of countability, we could simply require that K^*/G be T_0 or, more generally, be countably separated. The conclusion of Lemma 9 would then follow (cf. [9], p. 126). If K is not type I, the relations between these properties and the almost Hausdorff property is obscure.

6. Three examples. Our first example shows that, despite Lemma 1, the stability subgroup of G at a point in $\hat{K}$ may be very bad. Let G be the group whose underlying topological space is $\boldsymbol{T} \times \boldsymbol{Z} \times \boldsymbol{C}$, where $\boldsymbol{T} = \{\xi \in \boldsymbol{C} : |\xi| = 1\}$ and $\boldsymbol{Z}$ is the discrete integers. The group multiplication is given by

$$(\xi, m, a)(\zeta, n, b) = (\xi\zeta, m + n, a\zeta e^{in} + b) .$$

Let $K = \{1\} \times \boldsymbol{Z} \times \boldsymbol{C}$ and $N = \{1\} \times \{0\} \times \boldsymbol{C}$. N and K are normal subgroups of G, and N is abelian. We identify $\hat{N} = N^*$ with $\boldsymbol{C}$ as follows: each $\lambda \in \boldsymbol{C}$ corresponds to the character $\chi_{(\lambda)} : (1, 0, a) \longrightarrow e^{i \operatorname{Re}(a\lambda)}$. In terms of this identification, the action of K on N^* is given by $\lambda^{(1,m,a)} = \lambda e^{-im}$. By Theorems 1 and 2, ${}^{\lambda}L = {}_K U^{\chi_{(\lambda)}}$ is irreducible if $\lambda \neq 0$; moreover ${}^{\lambda}L \simeq {}^{\mu}L$ if and only if λ and μ belong to the same K-orbit in N^*.

We next calculate ${}^{\lambda}L^{(\xi,m,a)}$. To this end, we realize ${}^{\lambda}L$ in the Hilbert space of all square summable functions f on $\boldsymbol{Z}$ according to the rule: $({}^{\lambda}L_{(l,n,b)}f)(k) = \exp(i \operatorname{Re}(\lambda b e^{-i(n+k)}))f(n + k)$. Then $({}^{\lambda}L^{(\xi,m,a)}_{(l,n,b)}f)(k) = ({}^{\lambda}L_{(l,n,b\xi^{-1}e^{-im})}f)(k) = \exp(i \operatorname{Re}(\lambda b \xi^{-1} e^{-im} e^{-i(n+k)}))f(n + k)$. It follows that ${}^{\lambda}L^{(\xi,m,a)} = {}^{\lambda\xi^{-1}e^{-im}}L$. Therefore ${}^{\lambda}L^{(\xi,m,a)} \simeq {}^{\lambda}L$ if and only if λ and $\lambda\xi^{-1}$ are in the same K-orbit. Supposing $\lambda \neq 0$, we see that the stability subgroup of G at $({}^{\lambda}L)\hat{}$ in $\hat{K}$ is $\{(\xi, m, a) : \xi = e^{in}$ for some

$n \in \boldsymbol{Z}\}$, a proper dense subgroup of G.

As a consequence of Lemma 1, we see that ${}^{\lambda}L$ and ${}^{\mu}L$ have the same kernel in $C^*(K)$ if $|\lambda| = |\mu|$. (That this sufficient condition is also necessary may be seen by applying the structure theory of Glimm [10].) We also see that Theorems 1 and 2 are useless in this case in analyzing the irreducible representations M of G for which $\{\mathrm{Ker}\, {}^{\lambda}L\}G$ is $E_{M|K}$-thick. This is precisely because the stability subgroup of G at $\mathrm{Ker}\, {}^{\lambda}L$ in K^* is G itself.

Our second example shows that in Theorem 2 some restriction on G/H, such as σ-compactness, is necessary. Let G_1 be the "$ax + b$ group"; that is, the group whose underlying topological space is $\boldsymbol{R} \times \boldsymbol{R}$ and whose group multiplication is given by $(a, b)(c, d) = (a + c, be^c + d)$. Let G be the same group, except that the topology is modified by making the first factor discrete. Let $K_1 = \{0\} \times \boldsymbol{R} \subseteqq G_1$ and $K = \{0\} \times \boldsymbol{R} \subseteqq G$. Let φ be the (continuous) identity map of G onto G_1. Let χ be a nontrivial character of K_1. By Theorem 1, $M_1 = {}_{G_1}U^{\chi}$ is an irreducible representation of G_1 and $\{\chi\}G_1$ is $E_{M_1|K_1}$-thick in $\hat{K}_1 = K_1^*$. Let $M = M_1 \circ \varphi$, an irreducible representation of G. Because $\varphi | K$ is an isomorphism of K onto K_1 which is equivariant with respect to G, when G and G_1 are identified as abstract groups under φ, $\{\chi \circ (\varphi | K)\}G$ is $E_{M|K}$-thick in $\hat{K} = K^*$. The stability subgroup of G at $\{\chi \circ (\varphi | K)\}$ is K. But M is not induced from any representation of K, because $\dim \mathfrak{H}(M) = \aleph_0$ while $\dim \mathfrak{H}({}_GU^L) \geqq 2^{\aleph_0}$ for any representation L of K. One may see that, in this case, the proof of Theorem 2 breaks down right at the beginning: the representation E of $C_0(G/K)$ defined there is identically zero.

Finally, we show that the first part of Theorem 6.2 of [5] is an easy consequence of our Theorem 1. In fact, we have the following generalization: Let L be an irreducible representation of the closed normal subgroup K of the locally compact group G and let H be the stability subgroup of G at $\hat{L}$ in $\hat{K}$. If ${}_GU^L$ is irreducible, then $H = K$; if L is semi-compact, the converse holds. In fact, suppose $H \neq K$ and choose $x \in H$, $x \notin K$. Define T on $\mathfrak{H}(U^L)$ by setting $(Tf)(y) = Vf(xy)$, where V implements the equivalence of L^x and L; i.e., $L^x_\xi = V^{-1}L_\xi V$, $\xi \in K$. Then $(Tf)(\xi y) = Vf(x\xi y) = VL_{x\xi x^{-1}}V^{-1}Vf(xy) = L_\xi(Tf)(y)$ for $f \in \mathfrak{H}(U^L)$, $\xi \in K$, $y \in G$, and it follows easily that $Tf \in \mathfrak{H}(U^L)$ and that T is bounded. T clearly intertwines U^L but is not a scalar multiple of I, so the first assertion is established. The converse assertion follows from Theorem 1 together with the observation that H is the stability subgroup of G at $\mathrm{Ker}\, L$ in K^* by virtue of the comment following the definition of semi-compactness.

There remains the question of whether the converse is true without the semi-compactness condition. If L is not semi-compact, but if G/K

is discrete, the converse is true ([12], Theorem 3′). However. the general case is open.

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ON THE CONTINUOUS IMAGE OF A SINGULAR CHAIN COMPLEX

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A continuous surjection $\pi : X \to Y$ between topological spaces is called "ductile" if, for each $y \in Y$ and neighborhood U of y there is a neighborhood V of y which contracts to y through U in such a way that this contraction can be covered by a homotopy of $\pi^{-1}(V)$. It is shown, in this note, that if $\pi : X \to Y$ is ductile and Y is paracompact then the inclusion of the image $\pi_* C_*(X)$ of the singular chain complex of X in the singular chain complex $C_*(Y)$ of Y induces an isomorphism in homology. Thus $H_*(Y)$ can be computed from those singular simplices of Y which are images of singular simplices of X.

This result does not hold, in general, when π is not ductile. This question was brought to our attention (for a specific case) by Klingenberg who plans to use our result in a study of geodesics on a Riemannian manifold. We shall now rephrase the condition that a map be ductile in a more convenient language.

Let $\mathscr{M}$ be the category whose objects are surjective maps $\pi : X \to Y$ between topological spaces and whose morphisms are commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ Y & \longrightarrow & Y' \end{array}$$

of continuous maps (where $\pi, \pi' \in \mathscr{M}$). This contains an analogue of homotopy, that is a commutative diagram

$$\begin{array}{ccc} X \times I & \longrightarrow & X' \\ \downarrow{\scriptstyle\pi \times 1} & & \downarrow{\scriptstyle\pi'} \\ Y \times I & \longrightarrow & Y' \end{array}$$

For $\pi : X \to Y$ and $A \subset Y$ we let π_A denote the restriction $\pi^{-1}(A) \to A$ of π.

We will say that $\pi : X \to Y$ (in $\mathscr{M}$) is *ductile* if, for each point $y \in Y$ and neighborhood U of y, there is a neighborhood V of y with $V \subset U$ such that the inclusion $\pi_V \to \pi_U$ is homotopic (in $\mathscr{M}$) to a map into $\pi_{\{y\}}$. (Thus V contracts, through U, to $\{y\}$ and this contraction is covered by a homotopy of $\pi^{-1}(V)$.)

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Most nice mappings are ductile. The following are all examples of ductile maps:

(a) Simplicial maps.

(b) Let $A \subset X$ both be *ANR*'s (compact metric) and π the map of identifying A to a point.

(c) $Y = Y_1 \cup Y_2$ where Y_1, Y_2 $Y_1 \cap Y_2$ are *ANR's*, $X = Y_1 + Y_2$ (disjoint union) and where π is the natural map.

(d) π is the map of a differentiable manifold X onto its orbit space under some compact Lie group acting differentiably on X (see [1, Chapter VIII, 3.8]). According to Smale this also holds when X is an infinite dimensional manifold.

(e) Let M be a compact Riemannian manifold and X the space of mappings $S^1 \to M$ in the uniform metric. Regarding S^1 as the unit circle in the complex plane, S^1 acts on X by $(zf)(z') = f(zz')$. Let Y be the orbit space of this action. According to Švarc [4], this is ductile. According to Smale it falls under the infinite dimensional case of example (*d*). It is this example that Klingenberg uses in studying geodesics on M.

THEOREM. *Let π be a ductile map of the space X onto the paracompact space Y. Then the inclusion $\pi_* C_*(X) \subset C_*(Y)$ of chain complexes induces an isomorphism in homology.*

For the proof, it is convenient to introduce some notation. For $\pi : X \to Y$ in $\mathscr{M}$, let $C_*(\pi) = \pi^* C_*(X)$, $H_p(\pi) = H_p(C_*(\pi))$, $C^*(\pi) = \operatorname{Hom}(C_*(\pi), Z)$, and $H^p(\pi) = H^p(C^*(\pi))$. These are functors on $\mathscr{M}$. It is clear that homotopies in $\mathscr{M}$ induce chain homotopies, and therefore that homotopic maps $\pi \to \pi'$ induce identical homomorphisms

$$H_p(\pi) \to H_p(\pi') \quad \text{and} \quad H^p(\pi') \to H^p(\pi) .$$

Note that, as a subcomplex of $C_*(Y)$, $C_*(\pi)$ admits the operation of subdivision, and that standard methods show that this operation induces an isomorphism in homology.

Also note that if π is ductile, then, with $y \in V \subset U$ as in the definition of ductile, the restriction $H^p(\pi_U) \to H^p(\pi_V)$ factors through $H^p(\pi_{\{y\}}) = H^p(y)$ and hence is trivial for $p \neq 0$ and has image Z for $p = 0$. ($H^p(\pi) \to H^p(\pi_{\{y\}})$ is clearly surjective). Thus, when π is ductile, the natural map

$$\lim_{\to} H^p(\pi_U) \to H^p(\pi_{\{y\}}) = H^p(y) = \begin{cases} Z, & p = 0 \\ 0, & p \neq 0 \end{cases}$$

is an isomorphism, where U ranges over the neighborhoods of y.

Now, for $\pi : X \to Y$ fixed, let S^* be the (differential) presheaf on

Y defined by $S^*(U) = C^*(\pi_U) = \operatorname{Hom}(\pi_* C_*(\pi^{-1}(U)), Z)$. S^* clearly satisfies the axiom $(F2)$ of Godement [2]. Let $\mathscr{S}^*$ be the sheaf generated by S^*. The kernel $S_0^*(Y)$ of the natural map $C^*(\pi) = S^*(Y) \to \mathscr{S}^*(Y)$ consists of those cochains with empty support (that is, which vanish on "small" simplices of $C_*(\pi)$).

LEMMA. $H^*(S_0^*(Y)) = 0$.

Proof. For an open covering $\mathfrak{U}$ of Y let $C_*^{\mathfrak{U}}(\pi)$ be the subcomplex of $C_*(\pi)$ generated by those singular simplices which are contained in some member of $\mathfrak{U}$. A standard argument using subdivision shows that $H_*(C_*^{\mathfrak{U}}(\pi)) \to H_*(C_*(\pi))$ is an isomorphism. If $C_{\mathfrak{U}}^*(\pi) = \operatorname{Hom}(C_*^{\mathfrak{U}}(\pi), Z)$ it follows that $H^*(C^*(\pi)) \to H^*(C_{\mathfrak{U}}^*(\pi))$ is an isomorphism. Thus if $K_{\mathfrak{U}}^* = \ker\{C^*(\pi) \to C_{\mathfrak{U}}^*(\pi)\}$ then $H^*(K_{\mathfrak{U}}^*) = 0$. But clearly $S_0^*(Y) = \bigcup_{\mathfrak{U}} K_{\mathfrak{U}}^* = \varinjlim K_{\mathfrak{U}}^*$. Thus $H^*(S_0^*(Y)) = H^*(\varinjlim K_{\mathfrak{U}}^*) = \varinjlim H^*(K_{\mathfrak{U}}^*) = 0$.

Now suppose that Y is paracompact. Then by [2; 3,9.1, p. 159], the sequence

$$0 \to S_0^*(Y) \to S^*(Y) \to \mathscr{S}^*(Y) \to 0$$

is exact, so that $H^*(\pi) = H^*(S^*(Y)) \approx H^*(\mathscr{S}^*(Y))$.

Since each S^p is an S^0-module, it follows that each $\mathscr{S}^p$ is an $\mathscr{S}^0$-module. $\mathscr{S}^0$ is just the ordinary singular cochain sheaf of Y in degree zero and hence it is flabby. Since Y is paracompact it follows that each $\mathscr{S}^p$ is soft.

Let $\mathscr{H}^*(\mathscr{S}^*)$ be the derived sheaf of $\mathscr{S}^*$. By standard facts, the stalk of this sheaf at $y \in Y$ is $\mathscr{H}^*(\mathscr{S}^*)_y = \varinjlim_{y \in U} H^*(S^*(U)) = \varinjlim_{y \in U} H^*(\pi_U)$ (U ranging over the neighborhoods of y). We have seen that, when π is ductile, this is identified with $H^*(y)$. Thus, when π is ductile, $\mathscr{S}^*$ is a resolution of the constant sheaf Z.

If $\mathscr{C}^*$ is the ordinary singular sheaf of Y, the diagram

$$\begin{array}{ccc} H^*(C^*(Y)) & \to & H^*(S^*(Y)) = H^*(C^*(\pi)) \\ \downarrow & & \downarrow \\ H^*(\mathscr{C}^*(Y) & \to & H^*(\mathscr{S}^*(Y)) \end{array}$$

commutes (note that $\mathscr{C}^* = \mathscr{S}^*$ when π is the identity). If Y is paracompact, the vertical maps are isomorphisms and so is the lower map when π is ductile (see [2, 4.6.2, p. 178]).

We wish to obtain this isomorphism on the homology level. Note that $C_*(Y)/C_*(\pi)$ is a free chain complex (generated by those singular simplices not in the image of π). We wish to show that $H_*(C_*(Y)/C_*(\pi)) = 0$, under the hypotheses of the theorem. We know that the cohomology of this chain complex is trivial. Thus, by the universal coefficient theorem, it suffices to show that, for any abelian group A, Hom $(A, Z) = 0 = \text{Ext}(A, Z)$ implies that $A = 0$. This is proved in [3, Theorem 8.5] and completes the proof of our theorem.

In conclusion we give an example of a map $\pi ; X \rightarrow Y$ which is not ductile even though each point $y \in Y$ has a neighborhood U such that $\pi^{-1}(U)$ can be deformed into $\pi^{-1}(y)$. Indeed the conclusion of the theorem does not hold for this example.

Let Y_1 be the interval $[0, 1]$ on the x-axis of the $x - y$ plane and for $n > 1$ let Y_n be the upper semicircle $(y \geqq 0)$ with radius $1/n$ and center at $(1/n, 0)$. Let $Y = \bigcup_{n=1}^{\infty} Y_n$ and let X be the disjoint union of the Y_n with $\pi : X \rightarrow Y$ the natural map.

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ON REAL EIGENVALUES OF COMPLEX MATRICES

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This paper contains many inter-related results dealing with the general question of determination of real eigenvalues of complex matrices. We first discuss the relationship between the number of elementary divisors associated with real eigenvalues of a matrix A and the signature of a Hermitian matrix H when AH is also Hermitian. We then obtain sets of equivalent conditions for a matrix to be similar to a real matrix; for a matrix to be symmetrizable; and for a matrix to be similar to a real diagonal matrix. As corollaries we obtain results on the eigenvalues and elementary divisors of products of two Hermitian matrices. Some of the results are not new; these are included to give a more complete survey of what is known in these particular areas.

Recently a theorem on the stability of complex matrices, due to Lyapunov, has been generalized by Taussky [15, 16], and independently, by Ostrowski and Schneider [12]. Their result may be stated as follows: Given a complex matrix A, there exists a Hermitian H for which $AH + HA^* > 0$ (positive definite) if and only if A has no imaginary eigenvalues. Further, if $AH + HA^* > 0$, the numbers of eigenvalues of A with positive and negative real parts equal respectively the numbers of eigenvalues of H which are positive and negative.

Further generalizations of these results have been obtained by Schneider and this author [4, 6], under the condition that $AH + HA^* \geqq 0$ (positive semi-definite). This paper will use these results and methods to prove the theorems mentioned in the synopsis above.

I wish to acknowledge with thanks the contribution of Professor Emilie Haynsworth, who pointed out to me the connection between [7] and my results, and thus sparked this investigation. I also wish to thank the referee for many helpful comments, and for references to several related papers, especially [13], [14], and [17], with which I had not been familiar.

2. **Definitions.** We define the inertia of a complex matrix A to be $InA = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$, $\nu(A)$, $\delta(A)$ are respectively the number of eigenvalues of A with positive, negative, and zero real parts. We shall always let G, H and K represent Hermitian matrices; we denote the signature of H by $\sigma(H) = \pi(H) - \nu(H)$. We shall define, as in [12], $R(AH) = \frac{1}{2}(AH + HA^*)$.

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We shall use several simple propositions throughout the paper. The first two assume a matrix of the form (we let $\cdot$ denote a zero matrix)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \cdot & A_{22} \end{bmatrix} \left(\text{or } A = \begin{bmatrix} A_{11} & \cdot \\ A_{21} & A_{22} \end{bmatrix} \right).$$

We let λ be an eigenvalue of A, and let

$$a_1 \geqq a_2 \geqq \cdots \geqq a_r,\ b_1 \geqq b_2 \geqq \cdots \geqq b_s,\ c_1 \geqq c_2 \cdots \geqq c_t$$

be the degrees of the elementary divisors associated with λ in respectively A, A_{11}, and A_{22} (let $a_i = 0$ for $i > r$, etc.).

PROPOSITION 1. *$a_i \leqq b_1 + c_1$ for all i.*

PROPOSITION 2. *$b_i \leqq a_i$ for all i, and $s \leqq r$; if $c_1 = c_2 = \cdots = 0$, then $b_i = a_i$ for all i.*

Proofs. Proposition 2 follows immediately from the more precise inequality

$$a_{t+1} \leqq b_i \leqq a_i \text{ for all } i$$

proved in Appendix A of [5].

To prove Proposition 1, we first assume $\lambda = 0$ (if $\lambda \neq 0$, we consider $A - \lambda I$ in place of A). Let $\{x_j, j = 1, \cdots, a_i\}$ be a lower Jordan chain (see [8], p. 201) associated with $\lambda = 0$ and define $x_0 = 0$; we have $Ax_j = x_{j-1}$, $j = 1, \cdots, a_i$. Let

$$x_j = \begin{bmatrix} y_j \\ z_j \end{bmatrix};\ \text{then } Ax_j = \begin{bmatrix} A_{11}y_j + A_{12}z_j \\ A_{22}z_j \end{bmatrix} = \begin{bmatrix} y_{j-1} \\ z_{j-1} \end{bmatrix}.$$

Suppose k is the minimal j for which $z_j \neq 0$; then it follows from the above calculation that $A_{11}y_j = y_{j-1}$, $j = 1, \cdots, k-1$, and $A_{22}z_j = z_{j-1}$, $j = k+1, \cdots, a_i$ and $A_{22}z_k = 0$. Thus $\{y_j, j = 1, \cdots, k-1\}$ and $\{z_j, j = k, \cdots, a_i\}$ form parts of Jordan chains for A_{11} and A_{22} respectively. As b_1 and c_1 are respectively the maximal lengths of such chains, $k - 1 \leqq b_1$ and $a_i - k + 1 \leqq c_1$, so that $a_i \leqq b_1 + c_1$.

Our third proposition is quite different. If $AH = K$, and S is nonsingular, let $B = SAS^{-1}$, $H_0 = SHS^*$, and $K_0 = SKS^*$. Then $BH_0 = K_0$ and B is similar to A; $InH_0 = InH$ and $InK_0 = InK$ (by Sylvester's Law of Inertia). Thus we have

PROPOSITION 3. *If $AH = K$, we may replace A by a matrix similar to it and either H or K by a matrix complex-congruent to it, leaving invariant the eigenvalues and elementary divisors of A*

and the inertias of H and K. In particular, we could assume A to be in some variant of Jordan canonical form or $H = H_{11} \oplus 0$, where H_{11} is nonsingular. We shall also always assume that A, H, and K are partitioned conformably into submatrices.

It will be convenient to make the following definitions. Let $\gamma(A)$ be the number of a_i which are associated with real eigenvalues λ. Let $\gamma_0(A)$ be the number of a_i which are odd and associated with real λ. Let $\gamma_1(A)$ be the number of a_i which are either odd and associated with real nonzero λ, or even and associated with $\lambda = 0$. Let $a(A)$ be the sum of all a_i associated with real λ.

One last comment: throughout the paper we shall discuss matrices of the form AH or H_1H_2, where (say) H_2 has some special property. All our results will remain true if we replace AH by HA, or assume H_1 has the desired special property instead of H_2.

3. Elementary divisors associated with real eigenvalues.

Drazin and Haynsworth in [7] proved that a necessary and sufficient condition that A have (at least) m elementary divisors associated with real eigenvalues is that there exists an $H \geqq 0$, of rank m, for which AH is Hermitian. Our first theorem generalizes the conditions on H.

THEOREM 1. *Let A be a complex matrix. A necessary and sufficient condition that $\gamma(A) \geqq m$ is that there exists a Hermitian H for which*

$$|\sigma(H)| = m \tag{1}$$

and AH is Hermitian.

If AH is Hermitian and H is nonsingular, then

$$|\sigma(H)| \leqq \gamma_0(A), \tag{2}$$

$$|\sigma(AH)| \leqq \gamma_1(A). \tag{3}$$

Proof. The necessity of the first assertion is contained in the Drazin-Haynsworth theorem (If $H \geqq 0$, of rank m, then $|\sigma(H)| = m$). To prove sufficiency, we assume that AH is Hermitian (i.e., $AH - HA^* = 0$) for some H satisfying (1). Then if $A = iB$, $\gamma(A)$ is also the number of elementary divisors associated with imaginary eigenvalues of B. We have $R(BH) = \frac{1}{2}(BH + HB^*) = 0$. We may apply Theorem 3 of [4], which gives a set of bounds on the inertia of H when $R(BH) \geqq 0$, of rank r (obviously here $r = 0$). For $r = 0$, bound (14) of [4] becomes $|\sigma(H)| \leqq \gamma(A)$. As $|\sigma(H)| = m$ was assumed, the sufficiency is proved.

If H is nonsingular, then (2) is merely a restatement of the second display of § 9 of [4]. Also, (3) is a restatement of a theorem by

Loewy [10, p. 69] as given by Bromwich [3, p. 349]. This completes the proof.

4. Complex matrices similar to real matrices. The next result contains a conjugate-transpose analogue for complex matrices of a theorem proved by Taussky and Zassenhaus [18]: That every matrix with elements in a field F may be taken into its transpose by similarity transformation by a matrix symmetric in F.

THEOREM 2. *Let A be a complex matrix. The following four conditions are equivalent:*

(i) *A is similar to a real matrix.*

(ii) *There exists a Hermitian H for which $H^{-1}AH = A^*$.*

(iii) *There exists a nonsingular Hermitian H for which AH is Hermitian.*

(iv) *There exist Hermitian matrices G and H, with H nonsingular, for which $A = GH$.*

Proof. Suppose there exists a matrix T so that $B = T^{-1}AT$, where B is real. By the Taussky-Zassenhaus theorem, there exists a real symmetric S so that $S^{-1}BS = B' = B^*$. Calculation shows that $(TST^*)^{-1}A(TST^*) = A^*$; clearly TST^* is Hermitian. Conversely, if $H^{-1}AH = A^*$, A is similar to A^*, and conjugate eigenvalues of A must have elementary divisors with identical degrees. Thus A must be similar to a real matrix.

The equivalence of (ii), (iii), and (iv) is obvious.

5. Products of Hermitian matrices. As corollaries of the Drazin-Haynsworth theorem and our previous theorems, we obtain results on the eigenvalues and elementary divisors of products of two Hermitian matrices. Some are not new; some are easily proved independently. They are all presented, however, as, taken together they give a fairly complete description of the eigenvalues of a product of two Hermitian matrices.

Corollary 1 extends a result credited by MacDuffee [11, p. 65] to Klein [9].

COROLLARY 1. *If H_1 and H_2 are Hermitian, then*

$$\gamma(H_1H_2) \geqq |\sigma(H_2)| , \tag{4}$$

$$\alpha(H_1H_2) \geqq |\sigma(H_2)| + \delta(H_2) . \tag{5}$$

If H_2 is nonsingular, then

$$\gamma_0(H_1H_2) \geqq |\sigma(H_2)|,\ \gamma_1(H_1H_2) \geqq |\sigma(H_1)| . \tag{6}$$

Proof. We first suppose H_2 is nonsingular. Then for $A = H_1H_2$, and $H = H_2^{-1}$, $AH = H_1$ is Hermitian and (4) and (6) follow from Theorem 1 (as $InH_2 = InH_2^{-1}$). This completes the proof of (6).

If H_2 is singular, we may assume $H_2 = K_{11} \oplus 0$, where K_{11} is nonsingular (there exists a unitary U so that $U^*H_2U = K_{11} \oplus 0$; we write H_1 for UH_1U^*). Then

$$H_1H_2 = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} K_{11} & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} H_{11}K_{11} & \cdot \\ H_{21}K_{11} & \cdot \end{bmatrix}. \tag{7}$$

As H_{11} and K_{11} are Hermitian and K_{11} is nonsingular, we may apply (4) to H_{11} and K_{11} to obtain

$$\gamma(H_{11}K_{11}) \geqq |\sigma(K_{11})| . \tag{8}$$

As $\pi(K_{11}) = \pi(H_2)$ and $\nu(K_{11}) = \nu(H_2)$, we have

$$|\sigma(K_{11})| = |\sigma(H_2)| . \tag{9}$$

By Proposition 2 applied to all real eigenvalues,

$$\gamma(H_1H_2) \geqq \gamma(H_{11}K_{11}) . \tag{10}$$

Combining (8), (9), and (10) we have proved (4) for all H_1 and H_2.

It is clear from our definitions that

$$\alpha(A) \geqq \gamma(A) \tag{11}$$

for any A; hence (5) follows from (4) when H_2 is nonsingular (i.e. $\delta(H_2) = 0$). When H_2 is singular we again assume, as in (7), $H_1H_2 = H_{11}(K_{11} \oplus 0)$, where the zero matrix has order $\delta(H_2)$; clearly

$$\alpha(H_1H_2) = \alpha(H_{11}K_{11}) + \delta(H_2) . \tag{12}$$

From (8), (9), and (11) (with $A = H_{11}K_{11}$), we have

$$\alpha(H_{11}K_{11}) \geqq \gamma(H_{11}K_{11}) \geqq |\sigma(K_{11})| = |\sigma(H_2)| ,$$

and substituting this in (12) we obtain (5). We have proved Corollary 1.

COROLLARY 2. *If H_1 and H_2 are Hermitian, then H_1H_2 is similar to a real matrix.*

Proof. If H_2 is nonsingular, this is part of Theorem 2. If H_2 is singular, as in Corollary 1 we may assume $H_2 = K_{11} \oplus 0$, with K_{11} nonsingular. Now H_1H_2 is given by (7).

We shall use Proposition 2 for $A = H_1H_2$ and $A_{11} = H_{11}K_{11}$. To avoid confusion, we attach a superscript (λ) to each a_i and b_i associated with the eigenvalue λ. As $H_{11}K_{11}$ is similar to a real matrix by

Theorem 2, and $A_{22}=0$, we have for all nonreal λ,

$$a_i^{(\lambda)} = b_i^{(\lambda)} = b_i^{(\bar{\lambda})} = a_i^{(\bar{\lambda})} .$$

This implies that H_1H_2 is similar to a real matrix.

REMARK 1. Corollary 2 implies that H_1H_2 is similar to H_1H_2 (we have that H_1H_2 is similar to $\overline{H_1H_2}$ and hence to $(H_1H_2)^* = H_2H_1$) for all Hermitian H_1 and H_2, a property not enjoyed by all pairs of matrices: for example take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

REMARK 2. We note that when H_2 is nonsingular the above result is trivial $((H_1H_2)^* = H_2H_1 = H_2(H_1H_2)H_2^{-1})$. However, the singular case cannot be handled in the obvious way by continuity arguments on $H_2 + \varepsilon I$, as the elementary divisors structure is not a continuous function of the elements of the matrix. The same comment applies to the two corollaries below.

COROLLARY 3. *If H_1 and H_2 are Hermitian and $H_2 > 0$, then H_1H_2 is diagonalizable, with all real eigenvalues, and In $H_1H_2 =$ In H_1.*

Proof. Let $A = H_1H_2$ and $H = H_2^{-1} > 0$; then $AH = H_1$. By the Drazin-Haynsworth theorem, for $m =$ order A, we have that H_1H_2 is diagonalizable, with all real eigenvalues. By Corollary 3 of [12], In $H_1H_2 =$ *In* H_2.

REMARK 3. Corollary 3 is well known; cf. [19, p. 108, problem 6].

Corollary 4 (below) has connections with previous work on symmetrizable operators; we shall discuss this further in § 6.

COROLLARY 4. *If H_1 is Hermitian and $H_2 \geqq 0$, then H_1H_2 has all real eigenvalues; nonzero eigenvalues have linear elementary divisors, zero eigenvalues have elementary divisors of degree less than or equal two. We have*

$$\pi(H_1H_2) \leqq \pi(H_1), \ \nu(H_1H_2) \leqq \nu(H_1) . \tag{13}$$

Proof. Let $H_2 = K_{11} \oplus 0$, where $K_{11} > 0$. Again H_1H_2 is given by (7). By Corollary 3, $H_{11}K_{11}$ is diagonalizable, with all real eigenvalues. As $(H_1H_2)_{22} = 0$, we have by Proposition 2

$$a_i^{(\lambda)} = b_i^{(\lambda)} = 1$$

for all $\lambda \neq 0$ and all nonzero $a_i^{(\lambda)}$. We also have by Proposition 1

$$a_1^{(0)} \leqq b_1^{(0)} + c_1^{(0)} \leqq 2$$

for all nonzero $a_i^{(0)}$.

All that remains to be proved is (13); but this follows from Corollary 4 of [12].

6. Symmetrizable matrices. In a recent paper [14] Silberstein discusses symmetrizable operators in unitary spaces. (Other results on symmetrizable operators may be found in Reid [13] and Zaanen [20]). We specialize his definition to our finite dimensional setting:

DEFINITION. The complex matrix A is symmetrizable if there exists a Hermitian H, $H \geqq 0$, for which

(i) $Hx = 0$ implies $Ax = 0$.

(ii) HA is Hermitian.

THEOREM 3. *Let A be a complex matrix. The following conditions are equivalent:*

(i) *A has all real eigenvalues; nonzero eigenvalues have linear elementary divisors, zero eigenvalues have elementary divisors of degree less than or equal two.*

(ii) *A is symmetrizable.*

(iii) *There exists a nonsingular H for which $AH \geqq 0$.*

(iv) *There exist Hermitian matrices H and K, with $K \geqq 0$, for which $A = HK$.*

(v) *There exist Hermitian matrices H and K, with H nonsingular and $K \geqq 0$, for which $A = HK$.*

Proof. That (i) $\Leftrightarrow$ (ii) is due to Silberstein (Theorems 3.1 and 3.2 of [14]). That (iii) $\Rightarrow$ (i) follows from our Corollary 4. We however, shall prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

Suppose A satisfies (*i*). Then for some P, $P^{-1}AP = J = D \oplus 0 \oplus (\sum_i \oplus J_i)$, where D is a real nonsingular diagonal and each

$$J_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Define $K = D \oplus I \oplus (\sum_i \oplus K_i)$, where each

$$K_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad J_i K_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then K is nonsingular, Hermitian, and $JK \geqq 0$. We define $H = PKP^*$, and then $AH = (PJP^{-1})(PKP^*) \geqq 0$.

Suppose (iii) holds; then $A = HK$, where $K = H^{-1}A = H^{-1}(AH)H^{-1} \geqq$

0. That (v) $\Rightarrow$ (iv) is obvious. Suppose (iv) holds, and $A = HK$, where $K \geqq 0$. Then $KA = KHK$ is Hermitian, and $Kx = 0 \Rightarrow Ax = HKx = 0$. The proof is complete.

As a corollary we give a slight generalization of a result due (in a more general Setting) to Reid [13].

COROLLARY 5. *If $AH \geqq 0$ for some H, rank $H = r \neq 0$, then $\alpha(A) \geqq r$. In particular, if $H \neq 0$, A has a real eigenvalue.*

Proof. We may assume by Proposition 3 that $H = H_{11} \oplus 0$, H_{11} nonsingular. Then

$$AH = \begin{bmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{bmatrix} \begin{bmatrix} H_{11} & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} A_{11}H_{11} & \cdot \\ A_{21}H_{11} & \cdot \end{bmatrix}. \tag{14}$$

As AH is Hermitian, $A_{21}H_{11} = 0$; as H_{11} nonsingular, $A_{21} = 0$. Now by Theorem 3, as $A_{11}H_{11} \geqq 0$, A_{11} has all real eigenvalues. As $A_{21} = 0$, obviously $\alpha(A) \geqq \alpha(A_{11}) = r$.

7. Elementary divisors associated with positive and negative eigenvalues. We give a corollary to Theorem 3 similar in nature to Theorem 1.

COROLLARY 6. *A necessary and sufficient condition for A to have at least p and q elementary divisors associated with, respectively, positive and negative eigenvalues is that there exist a Hermitian H for which $\pi(H) = p$, $\nu(H) = q$, and $AH \geqq 0$, of rank $p + q$.*

Proof. The proof of the necessity is modeled after the corresponding proof of the Drazin-Haynsworth theorem. Let $\beta_1, \cdots, \beta_{p+q}$ be real eigenvalues of A, of which p are positive and q are negative; let $V_1, \cdots, V_{p+q}$ be a set of linearly independent associated eigenvectors for $\beta_1, \cdots, \beta_{p+q}$. Let $D = \text{diag}\ (\beta_1, \cdots, \beta_{p+q})$ and let $V = (V_1, \cdots, V_{p+q})$. We have $AV = VD$ and

$$AVDV^* = VD^2V^* = VDV^*A^* .$$

We take $H = VDV^*$; clearly $\pi(H) = p$, $\nu(H) = q$. As $D^2 > 0$, of order $p + q$, $AH = HA^* = VD^2V^* \geqq 0$, of rank $p + q$.

To prove sufficiency, we assume by Proposition 3 that $H = H_{11} \oplus 0$, where H_{11} is nonsingular. As in Corollary 5, $A_{21} = 0$ and $AH = A_{11}H_{11} \oplus 0$. We have

$$\text{rank } H_{11} = \text{rank } H = p + q = \text{rank } AH = \text{rank } A_{11}H_{11}$$

and $A_{11}H_{11} \geqq 0$; therefore $A_{11}H_{11} > 0$.

By Theorem 1 of [12], $In A_{11} = In H_{11} = (p, q, 0)$. By our Theorem

3, as A_{11} and H_{11} are nonsingular, A_{11} has all real nonzero eigenvalues with linear elementary divisors. Thus A_{11} has p and q elementary divisors associated with respectively positive and negative eigenvalues. By Proposition 2, as $A_{21} = 0$, A has at least as many of each.

8. Matrices similar to real diagonal matrices. We next give conditions for a complex matrix to be diagonalizable, with all real eigenvalues. For the case when A is real, some of these have been given by Taussky [17].

COROLLARY 7. *The matrix A is diagonalizable, with all real nonzero eigenvalues, if and only if there exists an H for which $AH > 0$. If $AH > 0$, then $InA = InH$.*

Proof. This is Corollary 6 for $p + q =$ order A.

COROLLARY 8. *Let A be a complex matrix. The following conditions are equivalent:*

(i) *A is diagonalizable, with all real eigenvalues,*

(ii) *There exists an H for which $AH \geqq 0$, with rank A = rank H = rank AH,*

(iii) *There exists an $H > 0$ for which AH is Hermitian,*

(iv) *There exists an $H \geqq 0$ for which AH is Hermitian and rank A = rank AH,*

(v) *There exist Hermitian matrices G and H, with $H > 0$, for which $A = GH$.*

Further, if (ii) *holds, then $InA = InH$. If either* (iii) *or* (iv) *holds, then $InA = InAH$. If* (v) *holds, then $InA = InG$.*

REMARK 4. If A is a real matrix, all of the Hermitian matrices G and H of Corollaries 7 and 8 may be chosen to be real symmetric. In fact, all constructions of G and H, given in proof or referred to, may be used to obtain such real G and H. The real case of Corollary 8, (i) $\Leftrightarrow$ (v), is known; cf. [17, p. 133].

REMARK 5. That (i) $\Leftrightarrow$ (iii) is contained in Theorem 3.3 of [14]. That (i) $\Leftrightarrow$ (v) has been noted by Taussky (Amer. Math. Monthly 66 (1959), p. 427, problem 4846; published solution by Parker, Amer. Math. Monthly 67 (1960) p. 192). That (v) $\Rightarrow$ (i) is our Corollary 3. Alternate proofs for Corollary 7 and Corollary 8, (i) $\Leftrightarrow$ (iii), may be obtained using the equivalence of (i) and (v).

Proof of Corollary 8. (i) $\Rightarrow$ (ii). Let $S^{-1}AS = B = \operatorname{diag}(\beta_1, \cdots, \beta_n)$, where the β_i are real. Obviously $B^2 \geqq 0$, with rank B = rank B^2; and if we let $H = SBS^*$, $AH \geqq 0$, with rank A = rank H = rank AH.

(ii) $\Rightarrow$ (i). Assume $H = H_{11} \oplus 0$, H_{11} nonsingular. As in the proof of Corollary 5, $A_{21} = 0$ and $AH = A_{11}H_{11} \oplus 0 \geqq 0$. As rank $H =$ rank AH, $A_{11}H_{11} > 0$ and by Corollary 7, A_{11} is diagonalizable, with all real nonzero eigenvalues. As rank $A =$ rank $H =$ rank $H_{11} = =$ rank A_{11}, and $A_{21} = 0$, we have $A_{22} = 0$. The proof of (i) now follows from Proposition 1. That $InA = InH$ follows from (i) and Corollary V.2 of [6].

(i) $\Leftrightarrow$ (iii). This is the Drazin-Haynsworth theorem for $m =$ order A. That $InA = InAH$ follows from Corollary 3 of [12].

(i) $\Rightarrow$ (iv). Assume $S^{-1}AS = B = \text{diag}\ (\beta_1, \cdots, \beta_n)$, where the β_i are real. Let $H = SB^2S^* \geqq 0$.

Clearly $AH = SB^3S^*$ is Hermitian, rank $A =$ rank AH.

(iv) $\Rightarrow$ (i). Suppose now AH is Hermitian, rank $A =$ rank AH, and $H = H_{11} \oplus 0$, where $H_{11} > 0$. By (iii), as $(AH)_{11} = A_{11}H_{11}$ is Hermitian, A_{11} is diagonalizable, with all real eigenvalues, and $InA_{11} = In(A_{11}H_{11})$. As before, $A_{21} = 0$, and $AH = A_{11}H_{11} \oplus 0$. As rank $A =$ rank $AH =$ rank $A_{11}H_{11} =$ rank A_{11}, we must have $A_{22} = 0$, so that all eigenvalues of A are real. Further, all nonzero eigenvalues have linear elementary divisors by Proposition 1.

We now prove that zero eigenvalues also have linear elementary divisors. Let S_{11} be a matrix for which $S_{11}^{-1}A_{11}S_{11} = D \oplus 0$, where D is a nonsingular diagonal. Let $S = S_{11} \oplus I$. Then

$$S^{-1}AS = B = \begin{bmatrix} D & \cdot & B_{13} \\ \cdot & \cdot & B_{23} \\ \cdot & \cdot & \cdot \end{bmatrix},$$

where A_{22} corresponds to the zero matrix in the lower-right corner of B. As rank $D =$ rank $A_{11} =$ rank A, obviously $B_{23} = 0$ and all zero eigenvalues have linear elementary divisors, (again by Proposition 1).

As $A_{22} = 0$, $\pi(A) = \pi(A_{11}) = \pi(A_{11}H_{11}) = \pi(AH)$, and similarly $\nu(A) = \nu(AH)$. The proof is complete.

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FIXED POINTS IN A TRANSFORMATION GROUP

HSIN CHU

In this paper, the following result is proved: "Let (X, T, π) be a transformation group, where X is a Peano continuum with an end point fixed under T. If the group T is one of the following two types: (1) It contains a subgroup R^n such that G/R^n is compact or (2) It contains a s subgroup $Z\cdot R^n$ such that $G/(Z\cdot R^n)$ is compact, where Z is isomorphic to the discrete additive group of all integers, then T has another fixed point."

Professor A. D. Wallace, in [4], proved the following: "Let (X, Z, π) be a transformation group, where $Z=$ the discrete additive group of all integers. If X is a Peano continuum with a fixed end point under Z, then Z has another fixed point." An interesting question, (See [5]) has been raised by Wallace: "Can one reach the same conclusion about either compact groups or abelian groups"? In the case of compact groups, Professor H. C. Wang answered the question in the affirmative (See [6]). We also give an affirmative answer to the question in the case of abelian groups when the abelian group is of the type either $R^n\cdot K$ or $Z\cdot R^n\cdot K$ where R^n is a vector group of dimension n and k is a compact abelian group. Actually, we also cover the case of non-abelian groups. The same conclusion can be reached if the group, G, is one of the following two types:

(1) It contains a subgroup R^n such that G/R^n is compact or
(2) It contains a subgroup $Z\cdot R^n$ such that $G/(Z\cdot R^n)$ is compact.

2. We divide that proof of our main result into several steps.

LEMMA 1. *Let (X, T, π) be a transformation group, where X is an arcwise connected Hausdorff space with an end point e fixed under T. If X has a closed invariant set A under T which does not contain e then T has another fixed point. Let $1(t), 0 \leqq t \leqq 1$, be an arc connecting e and some point x in A such that $1(0) = e$ and $1(1) = x$. All the points which separate e and A lie on $1(t)$. Let S be the set of all those points. S is not empty. Introduce a linear ordering in $1(t), 0 \leqq t \leqq 1$, by the natural linear ordering of t. Then the upper limit point of S is a fixed point, other than e, under T.*

Proof. The first part of the lemma is an equivalent statement of a theorem, in [6], of Professor H. C. Wang. Under the same assumption

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as our lemma, Wang's conclusion is that T has no other fixed point if and only if, given any neighborhood U of e, the orbit UT under T coincides with the whole space X. We notice that if S is a closed invariant set under T which does not contain e, then $U = X - S$ is a neighborhood of e and $UT = U$ which does not coincide with the whole space X and vice versa.

The proof of the second part of this lemma can be obtained from the proof of Wang's theorem. (See [6]).

LEMMA 2. *Let (X, Z, π) be a transformation group. If X is a compact, connected, Hausdorff space which is more than a point and has a fixed end point e, then there is a closed set $H \subset X - e$, which is invariant under Z.*

Proof. This is a theorem by Wallace, See [4].

By Lemma 1 and Lemma 2, we obtain Wallace's result.

LEMMA 3. *Let (X, Z, π) be a transformation group. If X is a Peano continuum with a fixed end point e under Z, then Z has another fixed point.*

LEMMA 4. *Let (X, T, π) be a transformation group. If X is a Peano continuum with a fixed end point e under T and T contains a syndetic subgroup Z (i.e. T contains a integer group Z such that T/Z is a compact set), then T has another fixed point. If, furthermore, T is connected, then the assumption on the given end point being fixed under T is not necessary.*

Proof. Consider the transformation group (X, Z, π) induced by (X, T, π). From Lemma 3, we know that there is another fixed point p under Z. Since Z is syndetic, there is a compact subset K in T such that $T = Z \cdot K$. Consequently, $pT = (pZ)K = pK$ which is compact and therefore, is closed. It is clear that $e \notin pK$. We know pK is closed and invariant under T. By Lemma 1, X has another fixed point q under T.

If T is connected, it is easy to see that every end point is fixed under T (See [5]). Suppose e is an end point and $e \neq et$ for some $t \in T$. Then, because e is an end point and eT is connected, there is $s \in eT$ such that s separates e and et. Consequently, there exists some $t' \in T$ such that $s = et'$. It follows that as t' is a homeomorphism of X, et' is also an end point as well as a cut point. A contradiction!

As a direct consequence of Lemma 4, we have:

LEMMA 5. *Let (X, R, π) be a transformation group. If X is a Peano continuum with an end point, then R has another fixed point.*

LEMMA 6. *Let (X, R^n, π) be a transformation group where n is a positive integer. If X is a Peano continuum with an end point e, then R^n has another fixed point.*

Proof. By Lemma 4, we know that the end point e is fixed under R^n for all n. The proof of this lemma is by induction. Suppose the statement is true for $n = k$. Consider $n = k + 1$. Let $(x_1, \cdots, x_k, x_{k+1})$ be a coordinate system of R^{k+1}. Let A and B be the closed subgroups determined by $x_1 = 0$ and $x_2 = 0$ respectively. Then $A \cong B \cong R^k$. Let the transformation groups (X, A, π) and (X, B, π) both be induced by (X, R^{k+1}, π). By the inductive assumption, we know there are two points p and q such that p is invariant under A and q is invariant under B. Both p and q are distinct from e. Let C_1 be the subgroup of R^{k+1} determined by $x_2 = 0, \cdots, x_{k+1} = 0$. Let C_2 be the subgroup of R^{k+1} determined by $x_1 = 0, x_3 = 0, \cdots, x_{k+1} = 0$. Then $C_1 \cong C_2 \cong R$ and, as direct products $R^{k+1} = C_1 \cdot A = C_2 \cdot B$. Consider the orbit, $(p)R^{k+1}$, of p under R^{k+1} and the orbit, $(q)R^{k+1}$, of q under R^{k+1}. It is clear that $(p)R^{k+1} = (p)C_1$ and $(q)R^{k+1} = (q)C_2$, where $(p)C_1$ and $(q)C_2$ both are connected.

We know both $cl((p)C_1)$ and $cl((q)C_2)$ are invariant under R^{k+1}. If e is not in either $cl((p)C_1)$ or $cl((q)C_2)$, then, by Lemma 1, R^{k+1} has another fixed point. Suppose e is in both $cl((p)C_1)$ and $cl((q)C_2)$. This implies that every neighborhood of e contains points from both $(p)C_1$ and $(q)C_2$.

Let U_e be a neighborhood of e such that $\{p, q\} \cap U_e = \phi$. Since e is a fixed end point, there exists $x \in U_e$ such that $X - x = X_1 \cup X_2$ for some sets X_1 and X_2 with the properties:

$$X_1 \cap cl(X_2) = cl(X_1) \cap X_2 = \phi \quad \text{and} \quad e \in X_1 \subset U_e \,.$$

Consequently, $\{p, q\} \subset X_2$. Notice that X_1 is open in X. It follows that X_1 contains points from both $(p)C_1$ and $(q)C_2$. Since both $(p)C_1$ and $(q)C_2$ are connected, it follows that $x \in (p)C_1 \cap (q)C_2$. Since R^{k+1} is abelian, we have $p = q$ and p is a fixed point under R^{k+1} other than e. Complete the proof by Lemma 5.

LEMMA 7. *Let $(X, Z \cdot R^n, \pi)$ be a transformation group. If X is a Peano continuum with a fixed end point e under $Z \cdot R^n$, then $Z \cdot R^n$ has another fixed point.*

Proof. If $n = 0$, the statement of this lemma is the same as Lemma 3. Let $n > 0$. Let (X, A, π) be a transformation group induced by $(X, Z\cdot R^n, \pi)$ where $A = Z\cdot R^{n-1}$ is a subgroup of $Z\cdot R^n$. Let $B \cong R$ be a subgroup of $Z\cdot R^n$ such that $Z\cdot R^n = A\cdot B$. Prove this lemma by induction on n. Suppose (X, A, π) has a fixed point, p, other than e, under A. Consider the orbit $(p)(Z\cdot R^n)$. It is clear that $(p)(Z\cdot R^n) = (p)B$, which is connected. The orbit-closure $cl((p)(Z\cdot R^n))$ is a connected compact Hausdorff space. Obviously, $cl((p)(Z\cdot R^n))$ is invariant under $Z\cdot R^n$. If e is not in $cl((p)(Z\cdot R^n))$, then, by Lemma 1, $Z\cdot R^n$ has another fixed point. Suppose $e \in cl((p)(Z\cdot R^n))$. Let Z' be an integer group of B. Then e is a fixed end point of the transformation group $(cl((p)(Z\cdot R^n)), Z', \pi)$. By Lemma 2, there is a Z'-invariant closed subset H of $cl((p)(Z\cdot R^n))$ such that $e \notin H$. Consider the transformation group (X, Z', π), induced by $(X, Z\cdot R^n, \pi)$. Choose a point $q \in H$ and connect e and q by an arc $1(t)$, $0 \leqq t \leqq 1$ on which $1(0) = e$ and $1(1) = q$. Let S be the set of all points which separate e and H. By Lemma 1 the upper limit point, r, of S is a fixed point, other than e, under Z'. Since $cl((p)(Z\cdot R^n))$ is connected, we have $S \subset cl((p)(Z\cdot R^n))$. Consequently, $r \in cl((p)(Z\cdot R^n))$. Since the points in $(p)(Z\cdot R^n)$ are fixed under A, the points in $cl((p)(Z\cdot R^n))$ are also fixed under A. It follows that r is fixed under both A and Z'. Let $B = Z'K'$ for some compact set K'. Then $(r)(Z\cdot R^n) = (r)K'$ which is compact. It is obvious $e \notin (r)K'$. By Lemma 1, $(Z\cdot R^n)$ has another fixed point. Complete the proof by induction.

THEOREM. *Let (X, T, π) be a transformation group. If X is a Peano continuum with a fixed end point under T and T is one of the following two types*:

(1) *It contains a subgroup R^n such that G/R^n is compact or*

(2) *It contains a subgroup $Z\cdot R^n$ such that $G/Z\cdot R^n$ is compact.*

Proof. Complete the proof by Lemma 1, Lemma 6, Lemma 7 and a similar method used in the proof of Lemma 4.

COROLLARY 1. *Let (X, T, π) be a transformation group. If X is a Peano continuum with an end point and T is locally compact, connected, abelian group, then T has another fixed point.*

We have the following application in Topological Dynamics. (See [1]). The proof is similar to the one used for the theorem.

COROLLARY 2. *Let (X, T, π) be a transformation group. If X is arcwise connected, Hausdorff with a fixed end point e and a regularly almost periodic point p, other than e, then T has another fixed point.*

Proof. By the definition of regularly almost periodic point, for a closed neighborhood U of p such that $e \notin U$, there exists a syndetic subgroup A of T such $pA \subset U$. It follows that $cl(pA) \subset U$, and thereby, $e \notin cl(pA)$. It is clear that $cl(xA)$ is invariant under A. By Lemma 1, we have another fixed point q under A. Since A is syndetic, there exists a compact set K such that $T = A \cdot K$. From $qT = (qA)K = qK$, we know qT is compact and, therefore, is closed and $e \notin qT$. Since qT is invariant under T, by Lemma 1 we have another fixed point under T. The theorem is proved.

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THE UNIFORMIZING FUNCTION FOR CERTAIN SIMPLY CONNECTED RIEMANN SURFACES

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This paper contains a definition of a class of simply connected Riemann surfaces, the determination of the type of a surface from this class, and a representation of the uniformizing function and its derivative as infinite products of quotients as well as quotients of infinite products.

Definition of the class of surfaces. Let $\{a_{2n-1}\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that for $n \geqq 1$,

$$0 < a_{2n-1} < b_{2n-1} < b_{2n}$$

and $b_{2n+1} < b_{2n}$. A surface F of the class to be discussed consists of sheets S_n, $n = 1, 2, 3, \cdots$, over the w-sphere, where for S_n a copy of the w-sphere,

(a) S_1 is slit along the real axis from a_1 to b_1.

(b) For $n \geqq 1$, S_{2n} is slit along the real axis from a_{2n-1} to b_{2n-1} and from b_{2n} to $+\infty$.

(c) For $n \geqq 1$, S_{2n+1} is slit along the real axis from a_{2n+1} to b_{2n+1} and from b_{2n} to $+\infty$.

(d) For $n \geqq 1$, S_n is joined to S_{n+1} along the slits to make the b_n coincide and to form first order branch points at the endpoints of the slits.

The uniformizing function. Because F is simply connected and noncompact, there exists a unique function g which maps F schlichtly and conformally onto $\{|z| < R \leqq \infty\}$, where for $f(z) = g^{-1}(z)$, $f(0) = 0 \in S_1$ and $f'(0) = 1$. Two surfaces of hyperbolic type are obtained by slitting each sheet of F along the uncut parts of the real axis, and an application of the reflection principle to the uniformizing function of one of these surfaces shows that $f(z)$ is real for real z. Let $f(\alpha_{2k-1}) = a_{2k-1}$, $f(-\beta_k) = b_k$, $f(\gamma_{2k}) = \infty \in S_{2k}$ and S_{2k+1}, $f(-\gamma_1) = \infty \in S_1$, and $f(\delta_k) = 0 \in S_k$. The image of F in the z-plane satisfies the following properties. The image of S_n is a region which is symmetric about the real axis. S_1 is mapped onto a domain containing the origin and bounded by a simple closed curve C_1 which intersects the real axis at $-\beta_1$ and α_1. For $n \geqq 2$, S_n is mapped onto an annular region about the origin and bounded by two simple closed curves C_{n-1} and C_n, which

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are images of cuts. For n odd, C_n intersects the real axis at $-\beta_n$ and α_n, while for n even, C_n intersects the real axis at $-\beta_n$ and γ_n. Furthermore, for $k \geqq 1$,

$$-\beta_{k+1} < -\beta_k < -\gamma_1 < 0 < \alpha_{2k-1} < \delta_{2k} < \gamma_{2k} < \delta_{2k+1} < \alpha_{2k+1} .$$

The approximating closed surfaces. Let F_n be the surface formed from the first $2n + 2$ sheets of F with the slit in S_{2n+2} from b_{2n+2} to ∞ deleted, so that F_n is a compact, simply connected surface.

NOTATION. $\alpha^*_\varphi = 1 - z/\alpha_\varphi$, $\beta^*_\varphi = 1 + z/\beta_\varphi$,
$\gamma^*_\varphi = 1 - z/\gamma_\varphi$, $\delta^*_\varphi = 1 - z/\delta_\varphi$.

LEMMA 1. *Let R_n be the unique rational function which maps the z-sphere one-to-one onto the simply connected compact surface F_n with $R_n(0) = 0 \in S_1$, $R'_n(0) = 1$, and $R_n(\infty) = \infty \in S_{2n+2}$. Then*

$$R_n(z) = [z/(1 + z/\gamma_{1,n})]\Big[\prod_{k=2}^{2n+2} \delta^*_{k,n}\Big]\Big/\Big[\prod_{k=1}^{n} (\gamma^*_{2k,n})^2\Big]$$

and

$$R'_n(z) = [1/(1 + z/\gamma_{1,n})^2]\Big[\prod_{k=0}^{n} \alpha^*_{2k+1,n}\Big]\Big[\prod_{k=1}^{2n+1} \beta^*_{k,n}\Big]\Big/\Big[\prod_{k=1}^{n} (\gamma^*_{2k,n})^3\Big] .$$

Proof. The representations of R_n and R'_n must contain factors shown and can contain no more. The $\alpha_{2k+1,n}$, $-\beta_{k,n}$, $\gamma_{2k,n}$, and $\delta_{k,n}$, which are ordered in the same manner as the α_{2k+1}, $-\beta_k$, γ_{2k}, and δ_k, are images of a_{2k+1}, b_k, ∞, and 0, respectively, under R_n^{-1}.

LEMMA 2. *F is parabolic.*

Proof. Suppose that F is hyperbolic, and thus g maps F onto $\{|z| < R < \infty\}$. If D_n is the z-plane slit along the real axis from $-\beta_{2n+1,n}$ to $-\infty$, then $\zeta = \psi_n(z) = g[R_n(z)]$ defines a Schlicht mapping of D_n onto a simply connected region Δ_n of the ζ-plane bounded by C_{2n+2} and the segment $(-\beta_{2n+2}, -\beta_{2n+1})$. If $T_n(z) = z(1 - z/4\beta_{2n+1,n})^{-2}$, then $\zeta = \psi_n[T_n(z)]$ defines a properly normalized, Schlicht mapping of $\{|z| < 4\beta_{2n+1,n}\}$ onto Δ_n such that if the Koebe Distortion Theorem is applied to this map, then $\beta_{2n+1,n} \leqq d(0, C_{2n+2}) \leqq R < \infty$, where $d(0, C_{2n+2})$ is the distance from $\zeta = 0$ to the curve C_{2n+2}. Thus there exists a subsequence $\{\beta_{2n_j+1,n_j}\}$ such that $\beta_{2n_j+1,n_j} \to A \leqq R$ as $j \to \infty$, and ψ_{n_j} is a Schlicht mapping of D_{n_j} onto Δ_{n_j}. If D is the z-plane slit along the negative real axis from $-A$ to $-\infty$, then $\{\psi_{n_j}\}$ forms a family of functions which is normal in D, and hence there exists a subsequence $\{\psi_i\}$ such that as $i \to \infty$, $\psi_i(z) \to \psi(z)$ uniformly on any compact subset of D. Because $D_i \to D$ and $\psi_i(z) \to \psi(z)$ as $i \to \infty$, then $\Delta_i \to \{|z| < R\}$ and ψ maps D onto $\{|\zeta| < R\}$ in a one-to-one manner. ([1], p. 18). Then $R_i(z) = f[\psi_i(z)] \to f[\psi(z)] = H(z)$ uniformly on any compact subset of D as $i \to \infty$, where H is meromorphic in D, while $H(z) \not\equiv \infty$ because $R_i(0) = 0$. H maps D onto F.

Now let D^* be the z-plane slit along the real axis from $-A$ to $+\infty$. For i sufficiently large, $R_i(z)$ assumes no negative real values in any compact subset of D^*, and thus $\{R_i\}$ is a family of functions which is normal in D^*. Therefore, there exists a subsequence $\{R_m\}$ of $\{R_i\}$ such that as $m \to \infty$, $R_m(z) \to G(z)$ uniformly on any compact subset of D^*. H and G have a common domain of convergence, so that G is the analytic continuation of H. Then $w = G(z)$ defines a mapping of the z-plane punched at $z = A$ and ∞ one-to-one and conformally onto an open doubly connected Riemann surface F^* of which F is a subsurface obtained by inserting some slits in F^* over the real axis. This is impossible, as is clear from the definition of F. Hence $R = \infty$.

LEMMA 3. *$R_n(z) \to f(z)$ uniformly on any compact subset of the z-plane as $n \to \infty$.*

Proof. Because $\Delta_n \to \{|\zeta| < \infty\}$ as $n \to \infty$, it follows ([1], p. 18) that $z = R_n^{-1}[f(\zeta)] \to \zeta = g[R_n(z)]$ uniformly on any compact subset of the ζ-plane as $n \to \infty$. Also, $D_n \to \{|z| < \infty\}$ and $R_n(z) \to f(z)$ uniformly on any compact subset of the z-plane as $n \to \infty$.

LEMMA 4. *$\alpha_{2k-1,n} \to \alpha_{2k-1}$, $\beta_{k,n} \to \beta_k$, $\gamma_{2k,n} \to \gamma_{2k}$, and $\delta_{k,n} \to \delta_k$ as $n \to \infty$.*

Proof. This is a consequence of Hurwitz's Theorem.

LEMMA 5. *The infinite product*

$$\pi(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^* \delta_{2k+1}^*/(\gamma_{2k}^*)^2]$$

converges uniformly on any compact subset of the z-plane.

Proof. Since $\gamma_{2k} \to \infty$ and $\delta_k \to \infty$ as $k \to \infty$, then for any $R > 0$, there exists $n_0 = n_0(R)$ such that for $k \geqq n_0$, $\delta_k > R$ and $\gamma_{2k} > R$. Then consider

$$M_p(z) = \prod_{k=n_0}^{n_0+p} [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] .$$

M_p is holomorphic for $|z| \leqq R$ and $M_p(z) \neq 0$ for $|z| \leqq R$. A sufficient condition for the uniform convergence of $M_p(z)$ in $E = \{|z| \leqq R\}$ as $p \to \infty$ is the uniform convergence in E of

$$\sum_{k=n_0}^{n_0+p} \log [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \text{ as } p \to \infty ,$$

where each logarithm is the principal value. By the Cauchy criterion, this last sequence converges uniformly in E provided for $z \in E$ and for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for $n > N(\varepsilon)$ and $p > 0$,

$$\left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \right| < \varepsilon .$$

Now since $\delta_{2k} < \delta_{2k+1}$ and since $\gamma_{2k} < \delta_{2k+2} < \delta_{2k+3}$, then for $m \geqq 1$ and $p > 0$,

$$0 < \sum_{k=n_0+n}^{n_0+n+p} [1/(\delta_{2k})^m + 1/(\delta_{2k+1})^m - 2/(\gamma_{2k})^m] < 2/(\delta_{2n_0+2n})^m.$$

Then for all $p > 0$ and $z \in E$,

$$\left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \right| = \left| -\sum_{m=1}^{\infty} [z^m/m] \sum_{k=n_0+n}^{n_0+n+p} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right|$$
$$\leqq \sum_{m=1}^{\infty} [R^m/m][2/(\delta_{2n_0+2n})^m] \leqq 2 \sum_{m=1}^{\infty} [R/(\delta_{2n_0+2n})]^m = 2R/(\delta_{2n_0+2n} - R) .$$

Since $\delta_{2n_0+2n} \to \infty$ as $n \to \infty$, the Cauchy criterion is satisfied and M_p converges uniformly in E. Thus $\Pi(z)$ converges uniformly in any compact subset of the z-plane.

LEMMA 6. $\pi(z) = f(z)$.

Proof. As a consequence of Lemma 4, there exists $r > 0$ such that $R_n(z)/z \neq 0$ and $\pi(z)/z \neq 0$ for $|z| < r$, while each of these quotients defines a function which is holomorphic for $|z| < r$ and takes the value 1 at $z = 0$. Thus using the principal value of the logarithm, for $|z| < r$,

$$\log [R_n(z)/z] - \log [\pi(z)/z] = \log [R_n(z)/\pi(z)] = \log [(1 + z/\gamma_1)/(1 + z/\gamma_{1,n})]$$
$$- \sum_{m=1}^{\infty} \{z^m/m\} \left\{ \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^{n} (1/\delta_{2k+1,n}^m) - \sum_{k=1}^{n} (2/\gamma_{2k,n}^m) \right.$$
$$\left. - \sum_{k=1}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right\} .$$

Therefore, for $n_0 > 2$, as $n \to \infty$,

$$0 \leqq \lim \sup \left| \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^{n} (1/\delta_{2k+1,n}^m) \right.$$

$$- \sum_{k=1}^{n} (2/\gamma_{2k,n}^m) - \sum_{k=1}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \Big|$$

$$\leqq \lim\ \sup \Big| \sum_{k=n_0}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=n_0}^{n} (1/\delta_{2k+1,n}^m) - \sum_{k=n_0}^{n} (2/\gamma_{2k,n}^m)$$

$$- \sum_{k=n_0}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \Big|$$

$$\leqq \lim\ \sup \, | (1/\delta_{2n_0,n}^m) + (1/\delta_{2n_0+1,n}^m) + (1/\delta_{2n_0}^m) + (1/\delta_{2n_0+1}^m) |$$

$$= (2/\delta_{2n_0}^m) + (2/\delta_{2n_0+1}^m) .$$

Since $\delta_{2n_0} \to \infty$ and $\delta_{2n_0+1} \to \infty$ as $n_0 \to \infty$, it follows that the limit as $n \to \infty$ of each coefficient of the preceding expansion of $\log [R_n(z)/\pi(z)]$ is zero. Furthermore, because as $n \to \infty$, $\{\log [R_n(z)/\pi(z)]\}_{n=1}^{\infty}$ converges uniformly on $\{| z | < r\}$, then $\log [R_n(z)/\pi(z)] \to 0$ as $n \to \infty$. Thus $\pi(z) = \lim_{n\to\infty} R_n(z) = f(z)$.

LEMMA 7. $\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$, $\sum_{k=1}^{\infty} 1/\beta_k < \infty$, $\sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty$, *and*

$$\sum_{k=2}^{\infty} 1/\delta_k < \infty .$$

Proof. Again by Lemma 4, there exists $r > 0$ such that $f'(z) \neq 0$ and $R_n'(z) \neq 0$ for $| z | < r$. Since $R_n(z) \to f(z)$, it follows that $R_n'(z) \to f'(z)$ and thus $\log R_n'(z) \to \log f'(z)$ uniformly in $\{| z | < r\}$ as $n \to \infty$. Thus for $| z | < r$, $\log R_n'(z)$

$$= \sum_{m=1}^{\infty} [z^m/m] \Big[- \sum_{k=0}^{n} 1/\alpha_{2k+1,n}^m$$

$$+ \sum_{k=1}^{2n+1} (-1)^{m+1}/\beta_{k,n}^m + 2(-1)^m/\gamma_{1,n}^m + \sum_{k=1}^{n} 3/\gamma_{2k,n}^m \Big] .$$

Hence, for $m = 1$,

$$\lim_{n\to\infty} \Big| - \sum_{k=0}^{n} 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n} + \sum_{k=1}^{n} 3/\gamma_{2k,n} \Big| < \infty .$$

Because $0 < \gamma_{1,n} < \beta_{1,n}$ and $0 < \gamma_{2k,n} < \alpha_{2k+1,n}$, then

$$0 < \sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^{n} 2/\gamma_{2k,n}$$

$$< - \sum_{k=0}^{n} 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n}$$

$$+ \sum_{k=1}^{n} 3/\gamma_{2k,n} + 1/\alpha_{1,n} + 2/\gamma_{1,n} .$$

Therefore, as $n \to \infty$,

$$0 \leqq \lim\ \sup \Big[\sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^{n} 2/\gamma_{2k,n} \Big] < \infty ,$$

$$\limsup \sum_{k=1}^{2n+1} 1/\beta_{k,n} < \infty, \text{ and } \limsup \sum_{k=1}^{n} 1/\gamma_{2k,n} < \infty .$$

Furthermore, because for

$$k \geqq 1,\ \gamma_{2k,n} < \delta_{2k+1,n} < \alpha_{2k+1,n} < \delta_{2k+2,n} ,$$
$$\limsup_{n\to\infty} \sum_{k=3}^{2n+2} 1/\delta_{k,n} \leqq \limsup_{n\to\infty} \sum_{k=1}^{n} 2/\gamma_{2k,n} < \infty$$

and

$$\limsup_{n\to\infty} \sum_{k=1}^{n} 1/\alpha_{2k+1,n} \leqq \limsup_{n\to\infty} \sum_{k=1}^{n} 1/\gamma_{2k,n} < \infty .$$

Hence

$$\limsup_{n\to\infty} \sum_{k=0}^{n} 1/\alpha_{2k+1,n} < \infty \quad \text{and} \quad \limsup_{n\to\infty} \sum_{k=2}^{2n+2} 1/\delta_{k,n} < \infty .$$

For all $N > 0$, as $n \to \infty$,

$$\sum_{k=0}^{N} 1/\alpha_{2k+1} = \sum_{k=0}^{N} \lim 1/\alpha_{2k+1,n} \leqq \limsup \sum_{k=0}^{n} 1/\alpha_{2k+1,n} < \infty ,$$

and thus $\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$. The convergence of the other series is established in a similar manner.

LEMMA 8. *Each of the three infinite products in*

$$P(z) = [1/(1 + z/\gamma_1)^2]\left[\prod_{k=0}^{\infty} \alpha^*_{2k+1} \prod_{k=1}^{\infty} \beta^*_k / \prod_{k=1}^{\infty} (\gamma^*_{2k})^3\right]$$

converges uniformly on any compact subset of the z-plane.

Proof. This is a consequence of Lemma 7.

LEMMA 9. $f'(z) = [\exp(\delta z)][P(z)]$ *where* δ *is real.*

Proof. By Lemma 4, there exists $r > 0$ such that for $|z| < r$, $R'_n(z) \neq 0$ and $f'(z) \neq 0$. For $m \geqq 1$, consider the coefficient of z^m/m in the Taylor expansion of $\log [R'_n(z)/P(z)]$ about $z = 0$ for $|z| < r$. Because of Lemma 7, there exists $M > 0$ such that for all $n \geqq 1$,

$$\sum_{k=1}^{n} 1/\gamma_{2k,n} < M \quad \text{and} \quad \sum_{k=1}^{\infty} 1/\gamma_{2k} < M .$$

Then because of the ordering of the $\gamma_{k,n}$ and γ_k, for each $k < n$, $k/\gamma_{2k,n} < M$ and $k/\gamma_{2k} < M$. Thus for each $N > 1$, as $n \to \infty$,

$$\limsup \left| \sum_{k=1}^{n} 1/\gamma_{2k,n}^{m} - \sum_{k=1}^{\infty} 1/\gamma_{2k}^{m} \right|$$
$$\leqq \limsup \left| \sum_{k=N}^{n} 1/\gamma_{2k,n}^{m} - \sum_{k=N}^{\infty} 1/\gamma_{2k}^{m} \right| \leqq 2M^{m} \sum_{k=N}^{\infty} 1/k^{m},$$

which implies for $m \geqq 2$, as $n \to \infty$

$$\lim \left[\sum_{k=1}^{n} 1/\gamma_{2k,n}^{m} - \sum_{k=1}^{\infty} 1/\gamma_{2k}^{m} \right] = 0 .$$

Similarly, the other terms in the coefficient of z^m/m have a limit of zero for $m \geqq 2$, and the coefficient of z is real. Then as $n \to \infty$, $\log [R_n'(z)/P(z)] \to \log [f'(z)/P(z)] = \delta z$, and thus $f'(z) = [\exp(\delta z)][P(z)]$.

LEMMA 10. $\delta = 0$.

Proof. Because the factors of $P(z)$ are canonical products of genus zero with real zeros, for $\varepsilon > 0$ and $0 < \rho \leqq |\arg z| \leqq \pi - \rho$, $P(z) = 0[\exp(\varepsilon |z|)]$ and $1/P(z) = 0[\exp(\varepsilon |z|)]$. Then if $\arg z$ satisfies the preceding conditions and $|z|$ is sufficiently large, then

$$\exp [\delta \mathscr{R}(z) - \varepsilon |z|] \leqq |f'(z)| \leqq \exp [\delta \mathscr{R}(z) + \varepsilon |z|] .$$

Let $A_1 = \{z \mid \pi/4 \leqq \arg z \leqq \pi/3\}$ and $A_2 = \{z \mid 2\pi/3 \leqq \arg z \leqq 3\pi/4\}$. If $\delta > 0$, then there exists $\varphi_1 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geqq \exp(\varphi_1 |z|)$ when $z \in A_1$ and $|f'(z)| \leqq \exp(-\varphi_1 |z|)$ when $z \in A_2$. Thus as $z \to \infty$ in A_2, $f'(z) \to 0$, and because $f(z) \geqq b_{2n} > 0$ for z on the curve C_{2n}, $f(z) \to k \geqq 0$ as $z \to \infty$ in A_2. Thus for n sufficiently large, $b_{2n} < k + 1$. Since $f'(z)dz > 0$ in the positive sense on the part of the curve C_{2n+1} in A_1, $b_{2n+1} - a_{2n+1} \to \infty$ as $n \to \infty$, where $a_{2n+1} > 0$ and thus $b_{2n+1} \to \infty$ as $n \to \infty$. Because $b_{2n+1} < b_{2n}$, a contradiction has been reached and $\delta \not> 0$. If $\delta < 0$, then there exists $\varphi_2 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geqq \exp(\varphi_2 |z|)$ when $z \in A_2$ and $|f'(z)| \leqq \exp(-\varphi_2 |z|)$ when $z \in A_1$. Similarly, $\delta \not< 0$.

THEOREM. *A Riemann surface of the class defined is parabolic and its mapping function f is given by*

$$f(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^{*} \ \delta_{2k+1}^{*}/(\gamma_{2k}^{*})^2]$$

where

$$f'(z) = [1/(1 + z/\gamma_1)^2] \left[\prod_{k=0}^{\infty} \alpha_{2k+1}^{*} \prod_{k=1}^{\infty} \beta_k^{*} / \prod_{k=1}^{\infty} (\gamma_{2k}^{*})^3 \right] .$$

Furthermore,

$$\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty, \ \sum_{k=1}^{\infty} 1/\beta_k < \infty, \ \sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty, \ \textit{and} \ \sum_{k=2}^{\infty} 1/\delta_k < \infty .$$

REMARKS. Lemmas 5 and 6 establish the representation of $f(z)$ as the product of quotients, while Lemmas 8 and 9 show a representation of $f'(z)$ as a quotient of products. However, Lemma 7 can be used to show that the representation of $f(z)$ can also be considered as the quotient of products.

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FREE COMPLETE EXTENSIONS OF BOOLEAN ALGEBRAS

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From considering questions about the existence of free α-complete Boolean algebras and free complete Boolean algebras, one is led naturally to the following problem: Given a Boolean algebra B, is it possible to embed B as a subalgebra in a complete Boolean algebra B^* in such a way that homomorphisms of B into complete Boolean algebras can be extended to complete homomorphisms on B^*? In general, the answer is "no"; this paper establishes that B can be so embedded if and only if every homomorphic image of B is atomic. Several other equivalent conditions on B are also developed.

To express these ideas more precisely, we say that the complete Boolean algebra B^* is a *free complete extension* of the Boolean B provided that there exist an isomorphism i of B into B^* such that

(i) if h is a homomorphism of B into a complete Boolean algebra C, then there is a complete homomorphism h^* of B^* into C such that $h^* \circ i = h$;

(ii) B^* has no regular complete proper subalgebra which contains $i[B]$ — that is, $i[B]$ completely generates B^*. A Boolean algebra B is said to be *superatomic* if every homomorphic image of B is atomic (or, equivalently, if every subalgebra of B is atomic). Our principal result, then, is that a Boolean algebra B has a free complete extension if and only if B is superatomic.

The problem of determining which Boolean algebras have free complete extensions arose from a conjecture made by L. Rieger in [7]. This conjecture is, in effect, that no infinite free Boolean algebra has a free complete extension; this has been verified by H. Gaifman in [3] and, independently and by different methods, by A. Hales, in [4].

Rieger proved, on the other hand, that for any cardinal numbers α and β there exists a unique free α-complete Boolean algebra on β generators. This result can be expressed by the statement that the free Boolean algebra on β generators has a unique free α-extension (a *free α-extension* of a Boolean algebra is defined in the same manner as a free complete extension, with the concepts of completeness and complete homomorphism replaced by those of α-completeness and

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α-homomorphism). F. M. Yaqub generalized this results in showing ([10], § 2) that every Boolean algebra has a unique free α-extension.

In addition, Yaqub found that the free α-extension of a Boolean algebra is α-representable if and only if it is isomorphic to an α-field of sets. He also proved that if $\alpha \geqq 2^{\aleph_0}$ and the free α-extension of a Boolean algebra B is α-representable, then B is superatomic. In § 4, we prove that the converse of the second of these results follows from the first, when the first is supplemented with our own results.

In § 2, we derive several characterizations of the superatomic Boolean algebras. In our third section it is shown that every superatomic Boolean algebra has a free complete extension. Then, using the result of Gaifman and Hales mentioned above, we show that every Boolean algebra having a free complete extension is superatomic.

Boolean concepts which are used without definition in this paper are defined in [2] and [8]. We denote the join and meet of two elements a and b of a Boolean algebra by $a + b$ and ab, respectively; the complement of b will be denoted by $\bar{b}$; the join and meet of a set S of elements of a Boolean algebra will be denoted by ΣS and ΠS, when they exist. To simplify certain arguments, we have assumed that if $S = \phi$, then ΣS and ΠS are the zero and unit, respectively, of the Boolean algebra in which the operations are performed.

2. Superatomic Boolean algebras. As defined by Yaqub in [10], a Boolean algebra B is superatomic if every subalgebra and every homomorphic image of B is atomic. This concept had been formulated previously in a somewhat more general context by Mostowski and Tarski, who, in [6], defined a hereditarily atomic Boolean ring as a Boolean ring each of whose homomorphic images is atomic; they observed that a Boolean ring R has this property if and only if every subring of R is atomic. In addition, Mayer and Pierce, in § 3 of [5], discussed Boolean algebras with scattered ordered bases (a totally ordered set, or chain, is *scattered* if it contains no subset which is order-isomorphic to the chain of rational numbers); it follows from their work that every such Boolean algebra is superatomic. However, there do exist superatomic Boolean algebras which are not of this form — a simple example is the Boolean algebra of finite and co-finite subsets of an uncountably infinite set, ordered by set inclusion.

THEOREM. *If B is a Boolean algebra, then the following are equivalent:*

(i) *every homomorphic image of B is atomic;*

(ii) *every subalgebra of B is atomic;*

(iii) *the Stone space of B is clairsemé (has no nonempty dense-*

in-itself set);

(vi) *every chain of elements of B is scattered;*

(v) *there is a sublattice S of B, whose elements generate B, such that every chain of elements of S is scattered;*

(vi) *no subalgebra of B is an infinite free Boolean algebra.*

Proof. The equivalence of (i) and (ii) is proven in [6], Theorem 3.12. We could, in fact, include in our collection of equivalent statements,

(i′) no homomorphic image of B is atomless; since it is easily shown that this is equivalent to (i).

It is well-known that in the natural duality between Boolean algebras and their Stone spaces, the homomorphic images and atoms of a Boolean algebra correspond to the closed subspaces and isolated points, respectively, of its Stone space. Thus, (iii) is equivalent to (i) (this was observed in [5]).

A Boolean algebra generated by a chain of elements which is order-isomorphic to the rationals is necessarily atomless, and, in fact, is isomorphic to the free Boolean algebra on $\aleph_0$ generators (as is shown on page 54 of [2]). Thus, (ii) implies (iv), and (vi) implies (iv). Moreover, if B is a nonatomic Boolean algebra, then B has an atomless element b; if a is any element of B such that $a < b$, then for some element c of B, $a < c < b$. Every nonatomic Boolean algebra can thus be shown to have a chain of elements which is order isomorphic to the rationals; hence, (iv) implies both (ii) and (vi).

It is clear that (iv) implies (v). We shall conclude our proof by showing that (v) implies (i′). Let us assume that B is a Boolean algebra generated by a sublattice S. Suppose that h is a homomorphism of B onto the Boolean algebra B'. If a and b are elements of S such that $a < b$ and $h(\bar{a}b)$ is neither an atom nor the zero element of B', then there are elements s and t of S such that $s < t$, $h(\bar{s}t)$ is not the zero of B', and $h(\bar{s}t) < h(\bar{a}b)$. This implies that either

$$h(a) < h(a + bs) < h(b)$$

or $h(a) < h(a + bt) < h(b)$. In either case, we have found an element c of S such that $a < c < b$ and neither $h(\bar{a}c)$ nor $h(\bar{c}b)$ is the zero of B'. Thus we may conclude that either B' has an atom or S has a chain of elements which is order-isomorphic to the rationals. Hence, if B is generated by a sublattice S such that every chain of elements of S is scattered, then no homomorphic image of B is atomless.

REMARK. The equivalence of conditions (i) and (iii) of the theorem indicates that every clairsemé compact Hausdorff space whose topology has a base of open-and-closed sets is the Stone space of a superatomic Boolean algebra. We can, in fact, make a stronger statement. If a compact Hausdorff space is clairsemé, then it is clearly totally disconnected; that is, it has only one-point maximal connected components. It is shown in Chapter I of [1] that every locally compact totally disconnected space has a base of open-and-closed sets. Thus, every clairsemé compact Hausdorff space is the Stone space of some superatomic Boolean algebra. (The author is indebted to Prof. M. Henriksen of Purde University for this argument.)

3. Free complete extensions of Boolean algebras. In this section, we show that a Boolean algebra has a free complete extension if and only if it is superatomic. Let us observe first that the definition of the free complete extension of a Boolean algebra implies that if a Boolean algebra has a free complete extension, then it is unique in the following sense: if B_1^* and B_2^* are free complete extensions of the Boolean algebra B, and i_1 and i_2 are the isomorphisms of the definition which carry B into B_1^* and B_2^* respectively, then there is an isomorphism i^* of B_1^* onto B_2^* such that $i^* \circ i_1 = i_2$.

THEOREM. *If B is a superatomic Boolean algebra, then the field of all subsets of the Stone space of B is a free complete extension of B, with respect to the natural isomorphism i of B onto the field of open-and-closed subsets of its Stone space.*

Proof. Suppose that B is superatomic and that X is the Stone space of B. Let B^* denote the field of all subsets of X. It is clear that $i[B]$ completely generates B^*, since X is Hausdorff and its open-and-closed sets form a base for its topology. It remains to be proven that if h is a homomorphism of B into a complete Boolean algebra C, then there is a complete homomorphism h^* of B^* into C such that $h^* \circ i = h$.

Recall that for ordinal β, we define $X^{(\beta)}$, the β-th derivative of X, as follows: $X^{(0)} = X$; $X^{(\beta+1)}$ is the set of all limit points of $X^{(\beta)}$; and if β is a limit ordinal, then $X^{(\beta)} = \cap \{X^{(\alpha)}: \alpha < \beta\}$. Since X is clairsemé, every point of X is an isolated point of some derivative of X; moreover, X has empty derivatives.

If $p \in X$ and p is an isolated point of $X^{(\beta)}$, let b_p be an element of B such that $i(b_p) \cap X^{(\beta)} = \{p\}$.

Now suppose that h is a homomorphism of B into a complete Boolean algebra C. We define a mapping h^* of B^* into C inductively (for convenience, we will write $h^*(p)$ instead of $h^*(\{p\})$). If p is an isolated point of X, let $h^*(p) = h(b_p)$. If p is an isolated point of

$X^{(\beta)}$ and $h^*(q)$ is defined for each $q \in X \sim X^{(\beta)}$, let

$$h^*(p) = h(b_p)\Pi\{\overline{h^*(q)} : q \in i(b_p) \sim X^{(\beta)}\} .$$

If $S \subset X$, let $h^*(S) = \Sigma\{h^*(p) : p \in S\}$.

Next, we prove that for every ordinal β,

S_β : (i) if $b \in B$ and $i(b) \cap X^{(\beta)} = \phi$, then $h^*(i(b)) = h(b)$;

(ii) if $b \in B$ and $i(b) \cap X^{(\beta)} = \{p\}$, then

$$h(b)\Pi\{\overline{h^*(q)} : q \in i(b) \sim X^{(\beta)}\} = h^*(p) .$$

Since X is Hausdorff, the truth of (ii) implies that if $p, q \in X$ and $p \neq q$, then $h^*(p)$ is disjoint from $h^*(q)$. Thus, if S_1 and S_2 are subsets of X and $S_1 \cap S_2 = \phi$, then $h^*(S_1)$ and $h^*(S_2)$ are disjoint. Since h^* is completely additive by definition, (i) then implies that h^* is a complete homomorphism of B^* into C, and $h^* \circ i = h$.

S_0 is clearly true, since $X^{(0)} = X$. Now suppose that β is an ordinal such that S_α is true for every ordinal α such that $\alpha < \beta$, and that b is an element of B such that $i(b) \cap X^{(\beta)} = \phi$. If β is a limit ordinal, then since $i(b)$ is compact, there is an ordinal $\alpha, \alpha < B$, such that $i(b) \cap X^{(\alpha)} = \phi$; hence, $h^*(i(b)) = h(b)$. If β is not a limit ordinal, and $i(b) \cap X^{(\beta-1)} \neq \phi$, then $i(b) \cap X^{(\beta-1)}$is finite, since it is compact and discrete. We can, without loss of generality, assume that $i(b) \cap X^{(\beta-1)} = \{p\}$. Then, by $S_{\beta-1}(ii)$, $h^*(p) = h(b)\Pi\{\overline{h^*(q)}: q \in i(b) \sim X^{(\beta-1)}\}$; it follows that

$$h(b) \leqq h^*(p) + \Sigma\{h^*(q) : q \in i(b) \sim X^{(\beta-1)}\} ,$$

and thus, $h^*(p) \leqq h(b) \leqq h^*(i(b))$. Moreover, if $q \in i(b) \sim X^{(\beta-1)}$, there is an ordinal $\alpha, \alpha < \beta - 1$, and a $c \in B$, $c \leqq b$, such that $h^*(q) \leqq h(c)$; thus, $h(b) = h^*(i(b))$. Hence, if S_α is true for every ordinal α such that $\alpha < \beta$, then $S_\beta(i)$ is true.

Next, suppose that b is an element of B such that $i(b) \cap X^{(\beta)} = \{p\}$. Using $S_\beta(i)$, we have

$$\begin{aligned} &h(b)\Pi\{\overline{h^*(q)} : q \in i(b) \sim X^{(\beta)}\} \\ &\quad = [h(bb_p) + h^*(i(b\overline{b_p}))]\Pi\{\overline{h^*(q)} : q \in i(b) \sim X^{(\beta)}\} \\ &\quad = h(bb_p)\Pi\{\overline{h^*(q)} : q \in i(bb_p) \sim X^{(\beta)}\}\Pi\{\overline{h^*(q)} : q \in i(b\overline{b_p})\} \\ &\quad = h(bb_p)\Pi\{\overline{h^*(q)} : q \in i(bb_p) \sim X^{(\beta)}\} . \end{aligned}$$

In the same way, we can show that

$$h(b_p)\Pi\{\overline{h^*(q)} : q \in i(b_p) \sim X^{(\beta)}\} = h(bb_p)\Pi\{\overline{h^*(q)} : q \in i(bb_p) \sim X^{(\beta)}\} .$$

This completes our proof.

REMARK. It is easily seen that there is only one complete homomorphism of B^* into C which is an extension of h, since such an extension must have the property that if $p \in X$, then

$$h^*(p) = \Pi\{h(b) : p \in i(b)\} ,$$

and if $S \subset X$, then $h^*(S) = \Sigma\{h^*(p) : p \in S\}$.

It remains only to be shown that a non-superatomic Boolean algebra has no free complete extension.

LEMMA. *If the Boolean algebra B has a free complete extension, then so does every subalgebra of B.*

Proof. Suppose that B has a free complete extension B^*, that i is the associated isomorphism of B into B^*, and that B' is a subalgebra of B. Let B^{**} be the complete subalgebra of B^{**} which is completely generated by $i[B']$. If h' is a homomorphism of B' into the complete Boolean algebra C, then, according to a result of Sikorski ([8], p. 112), h' can be extended to a homomorphism h of B into C. Since B^* is the free complete extension of B, there is a complete homomorphism h^* of B^* into C such that $h^* \circ i = h$. The restriction of h^* to B^{**} is a complete homomorphism of B^{**} into C, and for every $b \in B'$, $h^*(i(b)) = h'(b)$.

THEOREM. *A Boolean algebra has a free complete extension if and only if it is superatomic.*

Proof. It was shown above that every superatomic Boolean algebra has a free complete extension. Now suppose that B is a nonsuperatomic Boolean algebra. According to the theorem of § 2, B must have a subalgebra which is an infinite free Boolean algebra. It follows immediately from the result of Gaifman ([3]) and Hales ([4]) mentioned in our introduction, and the above lemma, that B has no free complete extension.

4. The free α-extensions of superatomic Boolean algebras. F. M. Yaqub showed in [10] that if a Boolean algebra B has a free α-extension B_α which is α-representable (that is, if B_α is the image of an α-field of sets under an α-complete homomorphism), then B_α is isomorphic to an α-field of sets, and if $\alpha \geqq 2^{\aleph_0}$, then B is superatomic. The results of § 3 enable us to prove the following converse of this statement.

THEOREM. *If B is a superatomic Boolean algebra, then the free α-extension of B is isomorphic to an α-field of sets.*

Proof. It was shown in § 3 that a superatomic Boolean algebra B has a free complete extension B^* which is a complete field of sets. For each cardinal α, let β_α denote the α-complete subfield of B^* which is α-generated by B. Suppose that h is a homomorphism of B into an α-complete Boolean algebra C. Let C' denote the normal completion of C, and let h^* be the complete homomorphism of B^* into C' which is an extension of h. h^* carries B_α into C since B α-generates B_α, $h^*[B] \subset C$, and C is an α-complete, α-regular subalgebra of C'. The restriction of h^* to B is thus an α-complete homomorphism of B_α into C, and is an extension of h. Hence B_α is the free α-extension of B.

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THE BOREL SPACE OF VON NEUMANN ALGEBRAS ON A SEPARABLE HILBERT SPACE

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Let $(S, \mathscr{S})$ be a Borel space (see G.W. Mackey, *Borel structures in groups and their duals,* Trans. Amer. Math. Soc. 85, (1957) 134–165), $\mathscr{H}$ a separable Hilbert space, $\mathfrak{L}$ the bounded linear operators on $\mathscr{H}$ with the Borel structure generated by the weak topology, and $\mathscr{A}$ the collection of von Neumann algebras on $\mathscr{H}$. A *field of $\mathscr{H}$ von Neumann algebras on S* is a map $s \rightarrow \mathfrak{A}(s)$ of S into $\mathscr{A}$. We prove that there is a unique standard Borel structures on $\mathscr{A}$ with the property that $s \rightarrow \mathfrak{A}(s)$ is Borel if and only if there exist countably many Borel functions $s \rightarrow A_i(s)$ of S into $\mathfrak{L}$ such that for each s, the operators $A_i(s)$ generate $\mathfrak{A}(s)$. This is a consequence of the more general result that when it is provided with a suitable Borel structure, the space of weakly* closed subspaces of the dual of a separable Banach space has sufficiently many Borel choice functions.

We show that the commutant, join, and intersection operations on $\mathscr{A}$ are Borel. It follows that the Borel space of factors is standard. The relevance of $\mathscr{A}$ to the theory of group representations is also investigated.

Essentially following von Neumann [9], we say that a field $s \rightarrow \mathfrak{A}(s)$ is *Borel* if there exist countably many Borel functions $s \rightarrow A_i(s)$ of S into $\mathfrak{L}$ such that for each s the operators $A_i(s)$ generate $\mathfrak{A}(s)$. This definition may be regarded as somewhat artificial. Rather than state which maps of S into $\mathscr{A}$ are Borel, one would conjecture that there is a standard Borel structure on $\mathscr{A}$ for which this characterization of the Borel maps of S into $\mathscr{A}$ is then valid. In § 2 and § 3 we shall show that this is the case. The demonstration depends on two results: a theorem in [4] showing that a certain Borel structure on the closed subsets of a polonais space is standard, and Theorem 2 of this paper. In the latter we prove the existence of Borel choice functions for the weakly* closed subspaces of the dual of a separable Banach space.

The Borel space $\mathscr{A}$ is of importance in representation theory. If G is a second countable locally compact group, and $G^c(\mathscr{H})$ are the weakly continuous unitary representions of G on $\mathscr{H}$ with the weak Borel structure (see [8]), the map $L \rightarrow L(G)'$ (prime indicates commutant) of $G^c(\mathscr{H})$ into $\mathscr{A}$ is Borel. By proving in § 3 that the factors $\mathscr{F}$ are

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a Borel subset of $\mathscr{A}$, we obtain new proof in § 4 of Dixmier's result that the factor representations $G^f(\mathscr{H})$ form a Borel subset of $G^c(\mathscr{H})$. We are also able to show that the quasi-equivalence relation is a Borel subset of $G^c(\mathscr{H}) \times G^c(\mathscr{H})$.

It is interesting to speculate about the isomorphism relation on $\mathscr{F}$. Conceivably, one might find an argument similar to those in [3] to prove that the quotient space was not smooth, and thus in particular, that there are uncountably many essentially distinct factors on $\mathscr{H}$.

We remark that an analogous problem of a "nonintrinsic" definition of structure, solved for $\mathscr{A}$ below, exists in Spanier's definition of a quasi-topology [12]. As is shown in [12], one must look for structures more general than topologies.

We are indebted to E. Alfsen and E. Størmer, who enabled us to simplify the proofs of Theorem 2 (by the convexity argument for the continuity of L) and Theorem 5, respectively.

2. Separable Banach spaces. Let $\mathfrak{X}$ be a separable real or complex Banach space, $\mathfrak{X}^*$ the dual of $\mathfrak{X}$, $\mathscr{N}(\mathfrak{X})$ the norm closed subspaces of $\mathfrak{X}$, and $\mathscr{W}(\mathfrak{X}^*)$ the weakly* closed subspaces of $\mathfrak{X}^*$. We wish to define a Borel structure on $\mathscr{W}(\mathfrak{X}^*)$. As $\mathfrak{Y} \to \mathfrak{Y}^\perp$ (the annihilator of $\mathfrak{Y}$) is a one-to-one correspondence between $\mathscr{N}(\mathfrak{X})$ and $\mathscr{W}(\mathfrak{X}^*)$, it suffices to find a Borel structure on $\mathscr{N}(\mathfrak{X})$ and then to transfer it to $\mathscr{W}(\mathfrak{X}^*)$.

$\mathscr{N}(\mathfrak{X})$ is a subset of $\mathscr{C}_0(\mathfrak{X})$, the collection of nonempty closed subsets of the polonais space $\mathfrak{X}$. In [4] we showed that convergence of subsets in $\mathscr{C}_0(\mathfrak{X})$ defines a standard Borel structure on $\mathscr{C}_0(\mathfrak{X})$. Recalling the procedure, if F_α is a net in $\mathscr{C}_0(\mathfrak{X})$ let $\underline{\lim}\, F_\alpha$ be those x in $\mathfrak{X}$ for which there is a net $x_\alpha \in F_\alpha$ with $x_\alpha \to x$. Let $\overline{\lim}\, F_\alpha$ be those x in $\mathfrak{X}$ for which there is a subnet F_{α_β} and $x_{\alpha_\beta} \in F_{\alpha_\beta}$ with $x_{\alpha_\beta} \to x$. If $F \in \mathscr{C}_0(\mathfrak{X})$, we say that F_α converges to the limit F, $F_\alpha \to F$, if $F = \underline{\lim}\, F_\alpha = \overline{\lim}\, F_\alpha$. If $\Sigma \subseteq \mathscr{C}_0(\mathfrak{X})$, we let $\bar{\Sigma}$ be the limits of nets in Σ, and we say that Σ is convergence closed if $\bar{\Sigma} = \Sigma$. The convergence closed sets form a topology, and generate a standard Borel structure on $\mathscr{C}_0(\mathfrak{X})$. We let $\mathscr{N}(\mathfrak{X})$ have the relative Borel structure. It is easily verified that $\mathscr{N}(\mathfrak{X})$ is convergence closed in $\mathscr{C}_0(\mathfrak{X})$, hence $\mathscr{N}(\mathfrak{X})$ and $\mathscr{W}(\mathfrak{X}^*)$ have standard Borel structures.

If d is any metric on $\mathfrak{X}$ compatible with the topology of $\mathfrak{X}$, $x \in \mathfrak{X}$, and $F \in \mathscr{C}_0(\mathfrak{X})$, define $d(x, F) = \text{glb}\,\{d(x, y) : y \in F\}$. For any positive c,

$$\{F \in \mathscr{C}_0(\mathfrak{X}) : d(x, F) \geqq c\} \tag{1}$$

is convergence closed. It follows that $F \to d(x, F)$ is a Borel function on $\mathscr{C}_0(\mathfrak{X})$. As in the proof of the first theorem in [4], sets of the form (1) separate points in $\mathscr{C}_0(\mathfrak{X})$, and thus as $\mathscr{C}_0(\mathfrak{X})$ is standard, generate the Borel structure. It follows that the Borel structure on

$\mathscr{C}_0(\mathfrak{X})$ is the weakest for which the functions $F \to d(x, F)$ are Borel (actually it would suffice to restrict to the x in a countable dense subset).

Let d be the norm metric on $\mathfrak{X}$. Then for $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$, $d(x, \mathfrak{Y}^{\perp}) = \| x + \mathfrak{Y}^{\perp} \|$, the latter being the quotient norm in $\mathfrak{X}/\mathfrak{Y}^{\perp}$. As $\mathfrak{Y}$ is weakly* closed, $\mathfrak{Y}^{\perp}$ $\mathfrak{Y}$, and we have a natural isometry $(\mathfrak{X}/\mathfrak{Y}^{\perp})^* \cong \mathfrak{Y}$. The corresponding isometry of $\mathfrak{X}/\mathfrak{Y}^{\perp}$ into $\mathfrak{Y}^*$ is defined by $x + \mathfrak{Y}^{\perp} \to x \,|\, \mathfrak{Y}$, where $x \,|\, \mathfrak{Y}$ in the restriction of x, regarded as an element of $\mathfrak{X}^{**}$, to $\mathfrak{Y}$. We conclude:

THEOREM 1. *Let $\mathfrak{X}$ be a separable Banach space, $\mathscr{W}(\mathfrak{X}^*)$ the weakly* closed subspaces of $\mathfrak{X}^*$. The Borel structure on $\mathscr{W}(\mathfrak{X}^*)$ is standard, and may be described as the smallest structure for which the functions*

$$\mathfrak{Y} \to \| x + \mathfrak{Y}^{\perp} \| = \| x \,|\, \mathfrak{Y} \| , \qquad x \in \mathfrak{X}$$

are Borel.

If $\mathfrak{X}$ is a real or complex separable Banach space, the *weak* Borel structure* on $\mathfrak{X}^*$ is that generated by the weak* topology. In other words, it is the smallest structure for which the functions $f \to f(x)$, $x \in \mathfrak{X}$ are Borel. Although we shall not use this fact, we remark that this structure is standard (see the proof of [8, Th. 8.1]).

Theorem 2 may be regarded as an elaborate form of the Hahn-Banach Theorem. Recalling the usual argument, suppose that $\mathfrak{X}$ is a real Banach space, and that we wish to construct a function in the closed unit ball $\mathfrak{X}_1^*$ of $\mathfrak{X}^*$. Suppose that f has been defined on a linear subspace $\mathfrak{B}$ of $\mathfrak{X}$, and is in $\mathfrak{B}_1^*$. If we extend f to the space generated by $\mathfrak{B}$ and a vector x, we must insist that

$$|f(x + w)| \leqq \| x + w \| \tag{2}$$

for all $w \in \mathfrak{B}$, i.e.,

$$- \| x + u \| - f(u) \leqq f(x) \leqq \| x + v \| - f(v)$$

for all $u, v \in \mathfrak{B}$. Let

$$\begin{aligned} L(f) &= \text{lub}\,\{- \| x + u \| - f(u) \colon u \in \mathfrak{B}\} , \\ M(f) &= \text{glb}\,\{\| x + v \| - f(v) \colon v \in \mathfrak{B}\} . \end{aligned} \tag{3}$$

These exist as for any $u, v \in \mathfrak{B}$,

$$f(v - u) \leqq \| v - u \| \leqq \| x + v \| + \| x + u \| ,$$

(4) i.e., $$- \| x + u \| - f(u) \leqq \| x + v \| - f(v) .$$

Thus we may rewrite (2):

(5) $$L(f) \leqq f(x) \leqq M(f) .$$

We shall assume below that $\mathfrak{V}$ is finite dimensional, and let $\mathfrak{V}^*$ have the norm topology. The functions $f \rightarrow L(f)$ and $f \rightarrow M(f)$ are defined on the closed unit ball $\mathfrak{V}_1^*$. As it is the least upper bound of convex functions, $f \rightarrow L(f)$ is convex, and thus continuous on the interior of of $\mathfrak{V}_1^*$ (see [1, p. 92]). From

(6) $$M(f) = - L(- f) ,$$

$f \rightarrow M(f)$ is also continuous on the interior of $\mathfrak{V}_1^*$.

THEOREM 2. *Let $\mathfrak{X}$ be a separable Banach space, $\mathscr{W}(\mathfrak{X}^*)$ the weakly* closed subspaces of $\mathfrak{X}^*$. There exist countably many Borel choice functions f_n: $\mathscr{W}(\mathfrak{X}^*) \rightarrow \mathfrak{X}^*$ such that for each $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$, the vectors $f_n(\mathfrak{Y})$ are weakly* dense in the closed unit ball $\mathfrak{Y}_1$ of $\mathfrak{Y}$.*

Proof. Suppose that $\mathfrak{X}$ is real. If $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$, we may identify $\mathfrak{Y}$ with $(\mathfrak{X}/\mathfrak{Y}^\perp)^*$, the norms and the weak* topologies will coincide.

For each sequence of real numbers $t = (t_1, t_2, \cdots)$ with $0 \leqq t_i \leqq 1$, we shall construct a function $f_t^{\mathfrak{Y}} \in (\mathfrak{X}/\mathfrak{Y}^\perp)_1^*$. Let $x_1, x_2, \cdots$ be norm dense in $\mathfrak{X}$, with $x_1 = 0$. Let $x_n(\mathfrak{Y}) = x_n + \mathfrak{Y}^\perp$, and $\mathfrak{V}_n(\mathfrak{Y})$ be the linear space spanned by $x_1(\mathfrak{Y}), \cdots, x_n(\mathfrak{Y})$ in $\mathfrak{X}/\mathfrak{Y}^\perp$. Define $f_{t_1}^{\mathfrak{Y}}(0) = 0$. Suppose that we have defined $f_{t_1,\cdots,t_n}^{\mathfrak{Y}}$ to be an element of $\mathfrak{V}_n(\mathfrak{Y})_1^*$. Letting $\mathfrak{V}_n(\mathfrak{Y}) = \mathfrak{V}$, $f_{t_1,\cdots,t_n}^{\mathfrak{Y}} = f$, and $x_{n+1}(\mathfrak{Y}) = x$ in our previous discussion, define

(7) $$f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) = t_{n+1}L(f) + (1 - t_{n+1})M(f) .$$

If $x \in \mathfrak{V}$, letting $u = v = - x$, we have from (3), (5), and (7)

$$- f(u) \leqq L(f) \leqq f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) \leqq M(f) \leqq - f(v),$$

i.e.,

$$f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) = f(x) .$$

Thus defining $f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}$ on $\mathfrak{V}_{n+1}(\mathfrak{Y})$ by

$$f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(cx + w) = cf_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) + f(w) ,$$

we obtain an extension of $f_{t_1,\cdots,t_n}^{\mathfrak{Y}}$ to an element of $\mathfrak{V}_{n+1}(\mathfrak{Y})^*$. As $f = f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}$ satisfies (5), it readily follows that $f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}$ is in $\mathfrak{V}_{n+1}(\mathfrak{Y})_1^*$. Define $f_t^{\mathfrak{Y}}$ on the space spanned by the $x_n(\mathfrak{Y})$ to be the union of the functions $f_{t_1,\cdots,t_n}^{\mathfrak{Y}}$. This extends by continuity to an element of $(\mathfrak{X}/\mathfrak{Y}^\perp)_1^*$.

It is clear that any function in $(\mathfrak{X}/\mathfrak{Y}^\perp)_1^*$ must have the form $f_t^{\mathfrak{Y}}$

for some sequence $t = (t_1, t_2, \cdots)$. We claim that the countable family of functions $f_r^{\mathfrak{Y}}$, $r = (r_1, r_2, \cdots)$ with the r_i rational, and all but a finite number equal to 0, are weakly* dense in $(\mathfrak{X}/\mathfrak{Y}^\perp)_1^*$. It suffices to prove that for all n, the functions $f_{r_1,\ldots,r_n}^{\mathfrak{Y}}$ are weakly*, or equivalently, norm dense in the interior of $(\mathfrak{B}_n(\mathfrak{Y}))_1^*$. This is trivial if $n = 1$. Suppose that it is true for n. If $g \in \mathfrak{B}_{n+1}(\mathfrak{Y})^*$ and $\|g\| \leqq 1$, let f be the restriction of g to $\mathfrak{B}_n(\mathfrak{Y})$. From our hypothesis and the earlier discussion, we may select rationals $r_1, \cdots, r_n$ with $f_{r_1,\ldots,r_n}^{\mathfrak{Y}}$ close to f in the norm topology, and $L(f_{r_1,\ldots,r_n}^{\mathfrak{Y}})$ and $M(f_{r_1,\ldots,r_n}^{\mathfrak{Y}})$ close to $L(f)$ and $M(f)$, respectively. Thus by a suitable choice of r_{n+1}, we obtain

$$f_{r_1,\ldots,r_{n+1}}^{\mathfrak{Y}}(x_{n+1}(\mathfrak{Y}))$$

close to $g(x_{n+1}(\mathfrak{Y}))$.

For any sequence $(t_1, t_2, \cdots)$ we have that $\mathfrak{Y} \rightarrow f_t^{\mathfrak{Y}}(x_n)$ is Borel (regarding $f_t^{\mathfrak{Y}}$ as an element of $\mathfrak{Y}$). This is trivial if $n = 1$. Suppose that it is true for $k \leqq n$. Then

$$\begin{aligned} f_t^{\mathfrak{Y}}(x_{n+1}) &= f_{t_1,\ldots,t_{n+1}}^{\mathfrak{Y}}(x_{n+1}(\mathfrak{Y})) \\ &= t_{n+1}L(f_{t_1,\ldots,t}^{\mathfrak{Y}}) + (1 - t_{n+1})M(f_{t,\ldots,t}^{\mathfrak{Y}}) \end{aligned} \tag{8}$$

If $\mathfrak{B}_n$ is the linear span of $x_1, \cdots, x_n$,

$$L(f_{t_1,\ldots,t_n}^{\mathfrak{Y}}) = \text{lub}\{-\|x_{n+1} + u + \mathfrak{Y}^\perp\| - f_t^{\mathfrak{Y}}(u) \colon u \in \mathfrak{B}_n\} .$$

From Theorem 1 and the induction hypothesis,

$$\mathfrak{Y} \rightarrow -\|x_{n+1} + u + \mathfrak{Y}^\perp\| - f_t^{\mathfrak{Y}}(u)$$

is Borel for any $u \in \mathfrak{B}_n$. Restricting to u that are rational linear combinations of the x_k for $k \leqq n$, $\mathfrak{Y} \rightarrow L(f_{t_1,\ldots,t_n}^{\mathfrak{Y}})$ is the least upper bound of a countable number of Borel functions, and is thus Borel. From (6) and (8), $\mathfrak{Y} \rightarrow f_t^{\mathfrak{Y}}(x_{n+1})$ is Borel. For any $x \in \mathfrak{X}$, $\mathfrak{Y} \rightarrow f_t^{\mathfrak{Y}}(x)$ is a limit of functions of the form $\mathfrak{Y} \rightarrow f_t^{\mathfrak{Y}}(x_n)$, and hence is Borel. Thus $\mathfrak{Y} \rightarrow f_t^{\mathfrak{Y}}$ is Borel.

Finally, suppose that $\mathfrak{X}$ is a complex Banach space. Letting $\mathfrak{X}_R$ be the corresponding real Banach space, $\mathscr{N}(\mathfrak{X})$ is a convergence closed subset of $\mathscr{N}(\mathfrak{X}_R)$. Define a map of $\mathscr{W}(\mathfrak{X}^*)$ into $\mathscr{W}((\mathfrak{X}_R)^*)$ by $\mathfrak{Y} \rightarrow \text{Re}\,\mathfrak{Y}$, where the latter consists of all real functions $\text{Re}\,f$ with $f \in \mathfrak{Y}$ (the customary argument shows that $f \rightarrow \text{Re}\,f$ is an isometry of $\mathfrak{X}^*$ onto $(\mathfrak{X}_R)^*$). For $\mathfrak{Z} \in \mathscr{N}(\mathfrak{X})$, $\text{Re}\,(\mathfrak{Z}^\perp) = \mathfrak{Z}^\perp$, where annihilators are taken in $\mathfrak{X}^*$ and $(\mathfrak{X}_R)^*$, respectively. It follows that $\mathfrak{Y} \rightarrow \text{Re}\,\mathfrak{Y}$ defines a Borel isomorphism of $\mathscr{W}(\mathfrak{X}^*)$ onto a Borel subset of $\mathscr{W}((\mathfrak{X}_R)^*)$. Choose real choice functions f_n: $\mathscr{W}((\mathfrak{X}_R)^*) \rightarrow (\mathfrak{X}_R)^*$ with $f_n(\mathfrak{Y})$ weakly* dense in $\mathfrak{Y}_1$ for each $\mathfrak{Y} \in \mathscr{W}((\mathfrak{X}_R)^*)$. Let g_n: $\mathscr{W}(\mathfrak{X}^*) \rightarrow \mathfrak{X}^*$ be the corresponding complex functions, i.e., for $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$ and $x \in \mathfrak{X}$, let

$$g_n(\mathfrak{Y})(x) = f_n(\operatorname{Re}\mathfrak{Y})(x) - if_n(\operatorname{Re}\mathfrak{Y})(ix) .$$

Then $\operatorname{Re} g_n(\mathfrak{Y}) = f_n(\operatorname{Re}\mathfrak{Y}) \in (\operatorname{Re}\mathfrak{Y})_1$, implies $g_n(\mathfrak{Y}) \in \mathfrak{Y}_1$. Given an arbitrary $g \in \mathfrak{Y}_1, x_1, \cdots, x_k \in \mathfrak{X}$, and $\varepsilon > 0$, choose an f_n with

$$|f_n(\operatorname{Re}\mathfrak{Y})(x_j) - \operatorname{Re} g(x_j)| < \varepsilon$$
$$|f_n(\operatorname{Re}\mathfrak{Y})(ix_j) - \operatorname{Re} g(ix_j)| < \varepsilon ,$$

for $j = 1, \cdots, k$. Then as

$$g(x) = \operatorname{Re} g(x) - i \operatorname{Re} g(ix) ,$$

we have

$$| g_n(\mathfrak{Y})(x_j) - g(x_j) | < 2\varepsilon$$

for $j = 1, \cdots, k$. Thus the $g_n(\mathfrak{Y})$ are weakly* dense in $\mathfrak{Y}_1$. Clearly the g_n are Borel.

COROLLARY. *If $(S, \mathscr{S})$ is a Borel space, then a map $s \to \mathfrak{Y}(s)$ of S into $\mathscr{W}(\mathfrak{X}^*)$ is Borel if and only if there exist countably many Borel functions $s \to f_n^s$ of S into $\mathfrak{X}^*$, such that for each s, the vectors f_n^s are weakly dense in $\mathfrak{Y}(s)_1$.*

Proof. If $s \to \mathfrak{Y}(s)$ is Borel, the functions f_n^s are obtained by composing this map with the choice functions of Theorem 2. Conversely, if such functions exist, we have from the isometry

$$\mathfrak{Y}(s) \cong (\mathfrak{X}/\mathfrak{Y}(s)^\perp)^* ,$$

$$|| x + \mathfrak{Y}(s)^\perp || = \sup \{| f_i^s(x) | : i = 1, 2, \cdots\}$$

for each $x \in \mathfrak{X}$. Thus $s \to || x + \mathfrak{Y}(s)^\perp ||$ is Borel for each $x \in \mathfrak{X}$, and by Theorem 1, $s \to \mathfrak{Y}(s)$ is Borel.

3. Von Neumann algebras. Let $\mathscr{H}$, $\mathfrak{L}$, $\mathscr{A}$, and $\mathscr{F}$ be as in § 1. We have that $\mathfrak{L} = (\mathfrak{L}_*)^*$, where $\mathfrak{L}_*$ is the separable Banach space of ultra-weakly continuous functions on $\mathfrak{L}$ (or by a natural identification, the trace class operators with a suitable norm-see [10]). The ultra-weak and weak* topologies coincide on $\mathfrak{L}$. Thus letting $\mathscr{W}(\mathfrak{L})$ be the ultra-weakly closed subspaces of $\mathfrak{L}$, we may give it the Borel structure described in § 2.

If $\mathfrak{Y} \in \mathscr{W}(\mathfrak{L})$, write $\mathfrak{Y}^*$ and $\mathfrak{Y}'$ for the adjoints of elements in $\mathfrak{Y}$, and the commutant of $\mathfrak{Y}$, respectively. The proof of the following theorem is largely patterned after that of [6, Th. 2.8].

THEOREM 3. *$\mathfrak{Y} \to \mathfrak{Y}^*$ and $\mathfrak{Y} \to \mathfrak{Y}'$ define Borel transformations of*

$\mathscr{W}(\mathfrak{L})$.

Proof. For $f \in \mathfrak{L}_*$, define $f^* \in \mathfrak{L}_*$ by $f^*(A) = \overline{f(A^*)}$, the bar indicating complex conjugate. This is an isometry of $\mathfrak{L}_*$, hence the transformation $\mathfrak{B} \to \mathfrak{B}^*$ on $\mathscr{N}(\mathfrak{L})$ is a homeomorphism (in the sense of convergence), and a Borel isomorphism. For $\mathfrak{Y} \in \mathscr{W}(\mathfrak{L})$, $(\mathfrak{Y}^\perp)^* = (\mathfrak{Y}^*)^\perp$, i.e., the adjoint operation on $\mathscr{N}(\mathfrak{L}^*)$ is carried into that on $\mathscr{W}(\mathfrak{L})$, and thus is a Borel isomorphism on the latter.

From Theorem 2, we may let $\mathfrak{Y} \to A_n^{\mathfrak{Y}}$ be Borel choice functions on $\mathscr{W}(\mathfrak{L})$ with $A_n^{\mathfrak{Y}}$ ultra-weakly dense in $\mathfrak{Y}_1$. We have

$$\mathfrak{Y}' = \{B \in \mathfrak{L}: BA_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}}B = 0 \text{ for } n = 1, 2, \cdots\}.$$

Let $\mathfrak{M}$ and $\mathfrak{M}_*$ be the sequences (A_n) and (f_n) of elements in $\mathfrak{L}$ and $\mathfrak{L}_*$, respectively, with $\sup\{\|A_n\|: n = 1, 2, \cdots\} < \infty$ and $\sum_{n=1}^{\infty} \|f_n\| < \infty$. With the norms $\|(A_n)\| = \sup\{\|A_n\|: n = 1, 2, \cdots\}$ and $\|(f_n)\| = \sum_{n=1}^{\infty} \|f_n\|$, $\mathfrak{M}$ and $\mathfrak{M}_*$ are Banach spaces, and defining $(f_n)((A_n)) = \sum_{n=1}^{\infty} f_n(A_n)$, $\mathfrak{M}$ may be identified with the dual of $\mathfrak{M}_*$. We have

$$\mathfrak{Y}' = \text{kernel } T^{\mathfrak{Y}} ,$$

where $T^{\mathfrak{Y}}: \mathfrak{L} \to \mathfrak{M}$ is defined by

$$T^{\mathfrak{Y}}(B) = (BA_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}}B) .$$

we claim that $T^{\mathfrak{Y}}$ is continuous in the weak* topologies. If $(f_n) \in \mathfrak{M}_*$,

$$(f_n)T^{\mathfrak{Y}}(B) = \sum_{n=1}^{\infty} g_n(B) ,$$

where $g_n(B) = f_n(BA_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}}B)$. The partial sums $\sum_{n=1}^{N} g_n$ are weakly* continuous, and converge uniformly on the unit ball $\mathfrak{L}_1$ of $\mathfrak{L}$, as if $B \in \mathfrak{L}_1$,

$$\left| \sum_{n=N+1}^{\infty} g_n(B) \right| \leqq 2 \sum_{n=N+1}^{\infty} \|f_n\| .$$

It follows that $B \to (f_n)T^{\mathfrak{Y}}(B)$ is continuous on $\mathfrak{L}_1$, and thus on $\mathfrak{L}$ (see [2, p. 41]). Define $T_*^{\mathfrak{Y}}: \mathfrak{M}_* \to \mathfrak{L}_*$ by

$$T_*^{\mathfrak{Y}}((f_n))(B) = (f_n)(T^{\mathfrak{Y}}(B)) .$$

We have that (kernel $T^{\mathfrak{Y}})^\perp$ is the closure of the range of $T_*^{\mathfrak{Y}}$. Thus letting B_i be ultra-weakly dense in $\mathfrak{L}_1$ and $g_j = (f_n^j)$ be norm dense in $\mathfrak{M}_*$, we have for any $f \in \mathfrak{L}_*$,

$$\|f + (\mathfrak{Y}')^\perp\| = \text{glb}\{\|f + T_*^{\mathfrak{Y}}(g_j)\| , \ j = 1, 2, \cdots\}$$

where

$$\begin{aligned}\| f + T_*^{\mathfrak{Y}}(g_j) \| &= \text{lub}\,\{| f(B_i) + T_*^{\mathfrak{Y}}(g_j)(B_i) |: i = 1, 2, \cdots\} \\ &= \text{lub}\,\{| f(B_i) + \sum_{n=1}^{\infty} f_n^j(B_i A_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}} B_i) |: i = 1, 2, \cdots\}.\end{aligned}$$

As $\mathfrak{Y} \to A_n^{\mathfrak{Y}}$ is ultra-weakly Borel, $\mathfrak{Y} \to \| f + (\mathfrak{Y}')^{\perp} \|$ is Borel, and as f is arbitrary, we have from Theorem 1 that $\mathfrak{Y} \to \mathfrak{Y}'$ is Borel.

COROLLARY 1. *$\mathscr{A}$ is a Borel subset of $\mathscr{W}(\mathfrak{Y})$, and thus is standard under the relative Borel structure.*

Proof. $\mathscr{A}$ consists of the $\mathfrak{Y} \in \mathscr{W}(\mathfrak{L})$ invariant under the Borel transformations $\mathfrak{Y} \to \mathfrak{Y}^*$ and $\mathfrak{Y} \to \mathfrak{Y}''$. In general say that θ is a Borel transformation of Borel space $(S, \mathscr{S})$. If Δ is the diagonal of $S \times S$, and $\theta \times \iota: S \to S \times S$ is defined by $\theta \times \iota(s) = (\theta(s), s)$, we have

$$\{s \in S: \theta(s) = s\} = (\theta \times \iota)^{-1}(\Delta) .$$

Thus if $(S, \mathscr{S})$ is standard, Δ is a Borel subset of $S \times S$, and the set of fixed points of θ is Borel.

Given von Neumann algebras $\mathfrak{A}$ and $\mathfrak{B}$, we let $\mathfrak{A} \vee \mathfrak{B}$ denote the von Neumann algebra generated by $\mathfrak{A}$ and $\mathfrak{B}$. Providing $\mathscr{A} \times \mathscr{A}$ with the product structure,

COROLLARY 2. *The maps of $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$ defined by $(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{A} \cap \mathfrak{B}$ and $(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{A} \vee \mathfrak{B}$ are Borel.*

Proof. As $\mathfrak{A} \cap \mathfrak{B} = (\mathfrak{A}' \vee \mathfrak{B}')'$, it suffices to prove the second assertion. From Theorem 2, there exist Borel choice functions $A_i: \mathscr{A} \to \mathfrak{L}$ with $A_i(\mathfrak{A})$ ultra-weakly dense in $\mathfrak{A}_1$, for each $\mathfrak{A} \in \mathscr{A}$. For each pair $(\mathfrak{A}, \mathfrak{B}) \in \mathscr{A} \times \mathscr{A}$, let $\mathscr{C}(\mathfrak{A}, \mathfrak{B})$ be the self-adjoint linear algebra generated by the elements $A_i(\mathfrak{A})$ and $A_j(\mathfrak{B})$. Let $B_k(\mathfrak{A}, \mathfrak{B})$ be an enumeration of the finite complex rational combinations of finite products of the elements $A_i(\mathfrak{A})$, $A_j(\mathfrak{B})$ and their adjoints. The $B_k(\mathfrak{A}, \mathfrak{B})$ are norm dense in $\mathscr{C}(\mathfrak{A}, \mathfrak{B})$, hence defining $B_k'(\mathfrak{A}, \mathfrak{B}) = B_k(\mathfrak{A}, \mathfrak{B})$ if $\| B_k(\mathfrak{A}, \mathfrak{B}) \| \leqq 1$, and $B_k'(\mathfrak{A}, \mathfrak{B}) = 0$ otherwise, the $B_k'(\mathfrak{A}, \mathfrak{B})$ are norm dense in $\mathscr{C}(\mathfrak{A}, \mathfrak{B})_1$. From the Kaplansky Density Theorem, the latter is ultra-weakly dense in $(\mathfrak{A} \vee \mathfrak{B})_1$. As $(\mathfrak{A}, \mathfrak{B}) \to B_k'(\mathfrak{A}, \mathfrak{B})$ are Borel, our assertion follows from the corollary to Theorem 2.

COROLLARY 3. *$\mathscr{F}$ is a Borel subset of $\mathscr{A}$, and thus is standard in the relative Borel structure.*

Proof. Let $\mathfrak{J}$ be the von Neumann algebra on $\mathscr{H}$ consisting of complex multiples of the identity operator. Then $\mathscr{F}$ is the inverse image of the element $\mathfrak{J}$ under the Borel map of $\mathscr{A}$ into $\mathscr{A}$ defined by $\mathfrak{A} \rightarrow \mathfrak{A} \cap \mathfrak{A}'$.

The argument used in the proof of Corollary 2 shows that a map $s \rightarrow \mathfrak{A}(s)$ of a Borel space $(S, \mathscr{S})$ into $\mathscr{A}$ is Borel if and only if there exist Borel functions $s \rightarrow A_i(s)$ of S into $\mathfrak{L}$ such that the $A_i(s)$ *generate* $\mathfrak{A}(s)$. Thus we have recaptured the original definition of § 1.

In direct integral theory, it is of some importance to know that various other subsets of $\mathscr{A}$ are measurable (see [9, 11]). We suspect that constructive procedures similar to that used in Theorem 2, would enable one to show that many of these sets are Borel.

4. **Representation spaces.** Let $\mathscr{H}$, $\mathfrak{L}$, $\mathscr{A}$, and $\mathscr{F}$ be as above, and G be a second countable locally compact group (an analogous theory exists for separable C^*-algebras). Let $G^c(\mathscr{H})$ be the weakly continuous unitary representations of G on $\mathscr{H}$, with the standard Borel structure defined by Mackey (see [8]). Let $G^f(\mathscr{H})$ be the subset of factor representations, i.e. those representations $L \in G^c(\mathscr{H})$ with $L(G)'$ a factor von Neumann algebra.

If $L, M \in G^c(\mathscr{H})$, let $\mathfrak{R}(L, M)$ be the ring of intertwining operators for L and M, i.e., those $B \in \mathfrak{L}$ with $BL(t) = M(t)B$ for all $t \in G$. In particular, $\mathfrak{R}(L, L) = L(G)'$. As was the case for Theorem 3, the following is simply a refinement of [6, Th. 2.8].

THEOREM 4. *The map* $G^c(\mathscr{H}) \times G^c(\mathscr{H}) \rightarrow G^c(\mathscr{H})$ *defined by* $(L, M) \rightarrow \mathfrak{R}(L, M)$ *is Borel.*

Proof. Let t_n be dense in G, and define $\mathfrak{M}$ and $\mathfrak{M}_*$ as in the proof of Theorem 3. Defining $S^{(L,M)}: \mathfrak{L} \rightarrow \mathfrak{M}$ by

$$S^{(L,M)}(B) = (BL(t_n) - M(t_n)B) ,$$

we have that

$$\mathfrak{R}(L, M) = \text{kernel } S^{(L,M)} ,$$

and that $S^{(L,M)}$ is continuous in the weak* topologies. $S^{(L,M)}$ is the adjoint of a map $S_*^{(L,M)}: \mathfrak{M}_* \rightarrow \mathfrak{L}_*$, and choosing B_i ultra-weakly dense in $\mathfrak{L}_1$, and $g_j = (f_j^n)$ norm dense in $\mathfrak{M}_*$, we have for any $f \in L_*$,

$$\| f + \mathfrak{R}(L, M)^{\perp} \| = \text{glb}\, \{\| f + S_*^{(L,M)}(g_j) \|: j = 1, 2, \cdots\} ,$$

where

$$\| f + S_*^{(L,M)}(g_j) \| = \text{lub}\, \{| f(B_i) + \sum_{n=1}^{\infty} f_j^n(B_i L(t_n) - M(t_n)B_i) |: i = 1, 2, \cdots\} .$$

$(L, M) \to f_j^n(B_i L(t_n) - M(t_n)B_i)$ is Borel when $G^c(\mathscr{H}) \times G^c(\mathscr{H})$ is given the product of the Mackey Borel structures, as any ultra-weakly continuous function is a norm limit of weakly continuous functions. It follows that $(L, M) \to \| f + \mathfrak{R}(L, M)^\perp \|$ is Borel, and from Theorem 1, $(L, M) \to \mathfrak{R}(L, M)$ is Borel.

COROLLARY 1. *The map* $G^c(\mathscr{H}) \to \mathscr{A}$ *defined by* $L \to L(G)'$ *is Borel*

COROLLARY 2. (*This was first proved by J. Dixmier—see* [5, *Theorem* 1].) *The set* $G^f(\mathscr{H})$ *of factor representation of* G *forms a Borel subset of* $G^c(\mathscr{H})$, *and thus is standard under the relative Borel structure.*

Following Mackey (see [7]), if $L, M \in G^c(\mathscr{H})$, we say that L is *covered* by $M, L \prec M$, if very subrepresentation of L contains a subrepresentation that is unitarily equivalent to a subrepresentation of M. L is *quasi-equivalent* to $M, L \sim M$, if $L \prec M$ and $M \prec L$.

If E is a projection in $L(G)'$, and $E \neq 0$, let L^E denote the corresponding subrepresentation of G on the range of E. If there exists a projection $E \in L(G)'$ with $E \neq 0$ and $L^E \prec M$, let $C(L, M)$ be the least upper bound of all such projections. Otherwise, let $C(L, M) = 0$. $C(L, M)$ is an element of $L(G)' \cap L(G)''$.

THEOREM 5. *The map* $G^c(\mathscr{H}) \times G^c(\mathscr{H}) \to \mathfrak{L}$ *defined by* $(L, M) \to C(L, M)$ *is Borel.*

Proof. If $A \in \mathfrak{L}$, let E_A and F_A be the projections on the closure of the range, and the orthogonal complement of the kernel of A. If $A \in \mathfrak{R}(L, M)$, then $F_A \in L(G)'$ and $E_A \in M(G)'$. If $A \neq 0$, and U is the partial isometry in the polar decomposition of A with $U^*U = F_A$, then U determines a unitary equivalence of L^{F_A} and M^{E_A}, and $F_A \leqq C(L, M)$. From Theorems 4 and 2, there exist Borel functions $A_i(L, M)$ that are ultra-weakly dense in the unit ball of $\mathfrak{R}(L, M)$ for each L and M. We claim that

$$C(L, M) = \bigvee_{i=1}^{\infty} F_{A_i(L, M)} , \tag{9}$$

where on the right we have taken the least upper bound in the complete projection lattice of $L(G)'$.

Suppose that there exist L and M with

$$F = C(L, M) - \bigvee_{i=1}^{\infty} F_{A_i(L, M)} \neq 0 .$$

As $L^F \prec M$, there exists a projection $F_0 \leqq F$ with $F_0 \neq 0$ and $F_0 = U^*U$ where $U \in \Re(L, M)$. Choosing i_k for which $A_{i_k}(L, M) \to U$ ultra-weakly,

$$0 = A_{i_k}(L, M)F_0 \to UF_0 = F_0 ,$$

a contradiction.

The map of $\mathfrak{L}$ into itself defined by $A \to F_A$ is Borel. To see this, note that $A \to A^*A$ is weakly Borel, as if $x, y \in \mathscr{H}$, letting x_i be an orthonormal basis we have

$$A^*Ax \cdot y = \sum_{i=1}^{\infty} (Ax \cdot x_i)(Ay \cdot x_i)^- .$$

A similar expansion shows that for positive integers n, $A \to A^n$ is Borel, hence for any polynominal p, $A \to p(A)$ is Borel. Suppose that f is a bounded real Borel function on the reals, and that there is a sequence of real polynomials p_n converging to f point-wise, uniformly bounded on compact sets. If A is a self-adjoint element in $\mathfrak{L}$, we have from spectral theory that $p_n(A) \to f(A)$ weakly. Thus $A \to f(A)$ is Borel. Letting g be the characteristic function of the open set $(0, \infty)$, $A \to F_A = g((A^*A)^{1/2})$ is Borel.

For all i, $(L, M) \to F_{A_i(L,M)}$ is Borel. If $F_1, \cdots, F_n$ are propections, then

$$F_1 \vee \cdots \vee F_n = F_{(F_1 + \cdots + F_n)} ,$$

hence

$$(L, M) \to \bigvee_{i=1}^{n} F_{A_i(L,M)}$$

is Borel. As the projections $\bigvee_{i=1}^{n} F_{A_i(L,M)}$ converge weakly to $\bigvee_{i=1}^{\infty} F_{A_i(L,M)}$, we conclude from (9) that $(L, M) \to C(L, M)$ is Borel.

Ernest remarked in the proof of [5, Prop. 2] that the quasi-equivalence relation on $G^f(\mathscr{H})$ is a Borel subset of $G^f(\mathscr{H}) \times G^f(\mathscr{H})$. The above theorem implies:

COROLLARY 1. *The covering and quasi-equivalence relations are Borel subsets of* $G^c(\mathscr{H}) \times G^c(\mathscr{H})$.

COROLLARY 2. *The quasi-equivalence class* $[L]$ *of a representation* L *in* $G^c(\mathscr{H})$ *is a Borel subset of* $G^c(\mathscr{H})$.

Proof. Let $\pi_i: G^c(\mathscr{H}) \times G^c(\mathscr{H}) \to G^c(\mathscr{H})$, $i = 1, 2$, *be* the projections on the first and second co-ordinates. Then $[L] = \pi_2(\pi_1^{-1}(L) \cap \sim)$, and as π_2 is one-to-one on $\pi_1^{-1}(L) \cap \sim$, and the latter is standard, $[L]$ is Borel.

It would seem likely that the unitary equivalence relation is also a Borel subset of $G^c(\mathscr{H}) \times G^c(\mathscr{H})$. Presumably one must prove the existence of a Borel choice function on spaces of the form $\mathfrak{R}(L, M)$, that selects a unitary operator when such exists. If unitary equivalence were a Borel set, it would follow that the representations $L \in G^c(\mathscr{H})$ with $L(G)'$ finite was also Borel. It should be noted that the unitary analogue of Corollary 2 above is true (see [3, Lemma 2.4]).

If G is the free group on countably many generators, the map described in Corollary 1 of Theorem 4 is onto. As the given structure and the corresponding quotient structure on $\mathscr{A}$ must coincide, a subset of $\mathscr{A}$ will be Borel if and only if the inverse image in $G^c(\mathscr{H})$ is Borel.

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A SET OF NONNORMAL NUMBERS

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Let P be the set of real polynomials and let $E(P)$ be the the set of real numbers whose nth binary digit from a certain point on is 0 or 1 according as $[\varphi(n)]$ is even or odd for some $\varphi \in P$. We prove that no number in $E(P)$ is normal in the binary system and that $E(P)$ has Hausdorff dimension 0.

Some notations and definitions. It is well known that every real number x of the unit interval which is not a binary fraction can be expanded in the binary system

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{2^n}$$

where $(\varepsilon_n(x))_{n \in \mathrm{N}}$ is a uniquely determined sequence of functions taking values 0 or 1. The functions $r_n(x) = 1 - 2\,\varepsilon_n(x)$ are known as the Rademacher functions.

We shall say that x is a normal number (in the binary system) if for every positive integer s and every sequence of positive, strictly increasing integers $k_1, k_2, \cdots, k_s$ one has:

$$(1) \qquad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} r_{n+k_1}(x) \cdots r_{n+k_s}(x) = 0 .$$

One can prove that this definition is equivalent to the other usual ones [3], [4], [6].

If t is a real number, $[t]$ will denote the greatest integer not greater than t and $\{t\} = t - [t]$ the fractional part of t.

Let P be the set of real polynomials and let $E(P)$ be the set of points x such that for some $\varphi \in P$ and for some $n_0 \geqq 0$, $r_n(x) = \exp i\pi[\varphi(n)]$ for all integers $n > n_0$.

We wish to prove first the following theorem:

THEOREM 1. *$E(P)$ contains only nonnormal numbers.*

This result shows that the measure of $E(P)$ is null, since almost all numbers are normal. Now the question arises if $E(P)$ contains "almost all" (in a sense soon to be made precise) nonnormal numbers or not. We answer this question by stating the known result:

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The Hausdorff dimension of the set of nonnormal numbers is 1, (see for example [1]),

and by proving our second theorem:

THEOREM 2. *The Hausdorff dimension of $E(P)$ is 0.*

2. **Proof of Theorem 1.** Let x be an element of $E(P)$. We show that for a certain sequence of increasing positive integers $k_1, k_2, \cdots, k_s$ the equation (1) does not hold.

Let φ be a polynomial such that $r_n(x) = \exp i\pi[\varphi(n)]$ for all sufficiently large integers n. Without loss of generality we may suppose that this relation holds for all positive integers, for normality or nonnormality are asymptotic properties. Let the expansion of φ be

$$(2) \qquad \varphi(n) = \alpha_\mu n^\nu + \alpha_{\nu-1} n^{\nu-1} + \cdots + \alpha_1 n + \alpha_0 , \qquad \nu \geqq 1 .$$

If all the numbers $\alpha_j (1 \leqq j \leqq \nu)$ are rational, then x is clearly rational, hence nonnormal. If one of the numbers $\alpha_j (1 \leqq j \leqq \nu)$ is irrational, we can without loss of generality suppose that the leading coefficient α_ν is irrational. Indeed, suppose that $\alpha_\mu (1 \leqq \mu < \nu)$ is irrational and that $\alpha_{\mu+1}, \cdots, \alpha_\nu$ are rational. Let q be the least common denominator of the $\nu - \mu$ fractions $\alpha_{\mu+1} \cdots, \alpha_\nu$. If x is normal, then so is the number y defined by $r_n(y) = \exp i\pi[\varphi(2qn)]$ for all integers n. But clearly $[\varphi(2qn)] \equiv [\psi(n)] \pmod 2$ where $\psi(n) = \alpha_\mu (2q)^\mu n^\mu + \cdots + \alpha_0$. This shows that we can now deal with ψ, the leading coefficient of which is irrational.

From now on in this section, φ is defined by equation (2) where α_ν is an irrational number.

We need the known identity for polynomials of degree ν:

$$(3) \qquad \varphi(n + \nu) \equiv \binom{\nu}{1}\varphi(n + \nu - 1) - \binom{\nu}{2}\varphi(n + \nu - 2) + \cdots + (-1)^{\nu-1}\binom{\nu}{\nu}\varphi(n) + \nu\,!\,\alpha_\nu$$

and the lemma:

LEMMA 1. *If $F(x_1, x_2, \cdots, x_\nu)$ is a Riemann integrable function which is of period 1 in each variable and if φ is a real polynomial of degree ν, the leading coefficient of which is irrational, then the following equality holds:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(\varphi(n),\ \varphi(n + 1), \cdots, \varphi(n + \nu - 1)) = \int_{T^\nu} F(x_1, x_2, \cdots, x_\nu) dx_1, \cdots, dx_\nu .$$

This is a very well known corollary of Weyl's theorems on uniform distribution (see for example [2]).

Combining equality (3) and Lemma 1, one can write:

$$L = \lim \frac{1}{N} \sum_{n=1}^{N} \exp i\pi\Big([\varphi(n+\nu)]$$

$$- \binom{\nu}{1}[\varphi(n+\nu-1)] + \cdots + (-1)^{\nu}\binom{\nu}{\nu}[\varphi(n)]\Big)$$

$$= \lim \frac{1}{N} \sum_{n=1}^{N} \exp i\pi\Big(\Big[\binom{\nu}{1}\varphi(n+\nu-1)$$

$$- \binom{\nu}{2}\varphi(n+\nu-2) + \cdots + \nu\,!\,\alpha_{\nu}\Big]$$

$$- \binom{\nu}{1}[\varphi(n+\nu-1)] + \cdots + (-1)^{\nu}[\varphi(n)]\Big)$$

$$= \int_{T^{\nu}} \exp i\pi\Big(\Big[\binom{\nu}{1}2x_{\nu} - \binom{\nu}{2}2x_{\nu-1} + \cdots + \nu\,!\,\alpha_{\nu}\Big]$$

$$- \binom{\nu}{1}[2x_{\nu}] + \cdots + (-1)^{\nu}[2x_1]\Big)dx_1 \cdots dx_{\nu}\,.$$

By putting $2x_j = y_j$, $j = 1, 2, \cdots, \nu$, the integral becomes

$$L = \frac{1}{2^{\nu}} \int_{(0,2)^{\nu}} \exp i\pi\left(\Big[\binom{\nu}{1}y_{\nu} - \binom{\nu}{2}y_{\nu-1} + \cdots + \nu\,!\,\alpha_{\nu}\Big]\right.$$

$$\left.- \binom{\nu}{1}[y_{\nu}] + \cdots + (-1)^{\nu}[y_1]\right)dy_1 \cdots dy_{\nu}\,.$$

Now the identity $[x + \varepsilon y] = [x] + \varepsilon[y] + [\{x\} + \varepsilon\{y\}]$, $\varepsilon = \pm 1$ shows that one has:

$$\Big[\binom{\nu}{1}y_{\nu} - \binom{\nu}{2}y_{\nu-1} + \cdots + \nu\,!\,\alpha_{\nu}\Big] = \binom{\nu}{1}[y_{\nu}] - \binom{\nu}{2}[y_{\nu-1}] + \cdots + [\nu\,!\,\alpha_{\nu}]$$
$$+ \Big[\binom{\nu}{1}\{y_{\nu}\} - \cdots + \{\nu\,!\,\alpha_{\nu}\}\Big]$$

so that:

$$L = \frac{\pm 1}{2^{\nu}} \int_{(0,2)^{\nu}} \exp i\pi\Big[\binom{\nu}{1}\{y_{\nu}\} - \binom{\nu}{2}\{y_{\nu-1}\} + \cdots + \{\nu\,!\,\alpha_{\nu}\}\Big]dy_1 \cdots dy_{\nu}$$

$$= \pm \int_{T^{\nu}} \exp i\pi\Big[\binom{\nu}{1}y_{\nu} - \binom{\nu}{2}y_{\nu-1} + \cdots + \{\nu\,!\,\alpha_{\nu}\}\Big]dy_1 \cdots dy_{\nu}\,.$$

Consider the hyperplane $\binom{\nu}{1}y_{\nu} - \binom{\nu}{2}y_{\nu-1} + \cdots + (-1)^{\nu-1}y_1 = -\{\nu\,!\,\alpha_{\nu}\}$ in the euclidean space R^{ν}. It has rational coefficients except for the constant term, which is irrational. Hence it cannot split the unit cube $(0, 1)^{\nu}$ into two regions of equal volume. Therefore the integral L

cannot be 0. Finally we notice that L may be written

$$L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} r_{n+\nu}(x)(r_{n+\nu-1}(x))^{\binom{\nu}{1}} \cdots r_n(x) ;$$

this completes the demonstration.

3. **Proof of Theorem 2.** Let P_ν denote the set of real polynomials of degree ν, the coefficients of which are all in the interval $[0, 2[$. It is easily seen that to prove Theorem 2, it is sufficient to prove the lemma:

LEMMA 2. *Let E^ν be the set of numbers x such that for some $\varphi \in P_\nu$, $r_n(x) = \exp i\pi[\varphi(n)]$ for all integers n. Then the Hausdorff dimension of E^ν is* 0.

Let $\qquad \varphi(n) = \alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0, \quad \alpha_j \in [0, 2]$

and let $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_\nu)$ be a point in the space $(0, 2)^{\nu+1}$. We are going to estimate the number $N_\nu(p)$ of regions in $(0, 2)^{\nu+1}$ which have the following property: when α ranges over one of these regions, the sequence $[\varphi(1)]$, $[\varphi(2)], \cdots, [\varphi(p)]$ stays invariant. First let us show:

LEMMA 3. *The h-dimensional measure $(0 \leqq h \leqq 1)$ of the set $E_p^\nu = \{x \mid r_n(x) = \exp i\pi[\varphi(n)]$; $n = 1, 2, \cdots, p$; $\varphi \in P_\nu\}$ satisfies the inquality*

$$h\text{-meas } (E_p^\nu) \leqq \frac{N_\nu(p)}{2^{ph}} .$$

Indeed, when φ runs through P_ν, α ranges over $(0, 2)^{\nu+1}$. The set E_p^ν is composed of at most $N_\nu(p)$ intervals, each of which has h-length $\left(\frac{1}{2^p}\right)^h$.

Now, if one notices that $E^\nu = \bigcap_{p=1}^\infty E_p^\nu$, one gets the result that the Hausdorff dimension of E^ν cannot be greater than

$$\delta = \liminf_{p\to\infty} \frac{\log N_\nu(p)}{p \log 2} .$$

We wish to show that $\delta = 0$ and we shall do so by proving our last lemma:

LEMMA 4. *When p goes to infinity, one has*

$$N_\nu(p) = 0(p^{(\nu+1)^2}) .$$

Proof. Let q be an integer such that

$$0 \leqq q \leqq 2\,(n^\nu + n^{\nu-1} + \cdots + n + 1) - 1$$

Consider the set $R_{n,q}$ of the points $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_\nu)$ defined by

$$q \leqq \alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0 < q + 1.$$

Clearly, when α runs through the region $R_{n,q}$, the quantity $[\varphi(n)] = [\alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0]$ stays equal to q. Then let $q_1, q_2, \cdots, q_p$ be any sequence of integers such that $0 \leqq q_j < 2\ (j^\nu + \cdots + j + 1)$, $j = 1, 2, \cdots, p$. When α ranges over the set $\bigcap_{n=1}^{p} R_{n,q_n}$, the sequence $[\varphi(1)], [\varphi(2)], \cdots, [\varphi(p)]$ does not change. But the number of these regions is at most the number of different regions one can obtain by dissecting the space $\mathrm{R}^{\nu+1}$ by hyperplanes $\alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0 = q$. These hyperplanes are at most $M = M_\nu(p) = \sum_{j=1}^{p} 2(j^\nu + \cdots + j + 1) = 0(p^{\nu+1})$. Now, one can show that the space $\mathrm{R}^{\nu+1}$ is dissected into $0(M^{\nu+1})$ regions by M hyperplanes [5] and therefore:

$$N_\nu(p) = 0(p^{(\nu+1)^2}) .$$

REMARK 1. It is easy to generalize Theorem 2 and obtain the following result. Let $(f_n)_{n\in \mathrm{N}}$ be a countable set of real functions such that

$$\lim_{p\to\infty} \frac{\log^+ |f_n(p)|}{p} = 0 , \qquad \forall n \in \mathrm{N} .$$

($\log^+$ denotes the maximum of 0 and log). Let Q be the set of all real finite linear combinations of the family (f_n). Then the Hausdorff dimension of the set $E(Q)$ is 0.

REMARK 2. The proof of Theorem 2 shows that the set E^ν is not dense on the unit interval (0, 1). On the other hand, E^ν is invariant under the mapping $x \to \{2x\}$. From these two remarks, one sees that E^ν is a Rajchman H-set and that $E(p)$ is therefore a set of uniqueness for trigonometric series. This result is to be compared with the following corollary of Pyatetski-Shapiro's theorem:

The set of nonnormal numbers is not a set of uniqueness.

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POLYNOMIALS ORTHOGONAL OVER A DENUMERABLE SET

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This paper concerns itself with characterizing the orthogonality domain and the distribution function for polynomials which satisfy

$$\phi_{n+1}(x) = (x - a_n)\phi_n(x) - b_n\phi_{n-1}(x) \qquad (n \geqq 0) \tag{1.1}$$

with

$$\phi_{-1}(x) = 0 \text{ and } \phi_0(x) = 1 \tag{1.2}$$

under the restriction $a_n = 0$, $b_n > 0$ $(n \geqq 0)$ **and** $\lim b_n = 0$.

This extends the results of Dickinson, Pollak and Wannier [6] by replacing their restriction $\Sigma b_n < \infty$ with the weaker assumption $\lim b_n = 0$, by correcting an apparent oversight, and by characterizing the distinction between the cases $\Sigma b_n < \infty$ and $\lim b_n = 0$. In the course of this study we prove some theorems with occur rather naturally and seem of interest in their own right. Our approach owes its origins to ideas expressed in [4] and [6] and our techniques to the product and the series representations for a certain subclass of analytic functions studied by Richards [9] and related to functions characterized by a certain Stieltjes transform and continued fraction expansion.

More specifically; from a theorem of Favard [7] and Shohat [11], equations (1.1) and (1.2) and the assumptions a_n real and $b_n > 0$ $(n \geqq 0)$ are sufficient to imply that $\{\phi_n(x)\}$ is a real orthogonal set. Under the additional restrictions $a_n = 0$ $(n \geqq 0)$ and $\Sigma b_n < \infty$, Dickinson, Pollak and Wannier [6] have shown:

(i) The domain of orthogonality is a bounded denumerable set S, symmetric with respect to $x = 0$, with $x = 0$ the only non-isolated point.

(ii) The distribution function (unique, after normalization, because of the boundeness of S) with respect to which the polynomials $\{\phi_n(x)\}$ are orthogonal, is bounded, nondecreasing and with spectrum (the points of increase) the point set S of (i). The points S are the poles of a certain function, meromorphic in $1/x$, whose residues are the values of the jumps of the distribution. (This statement appears to require modification because of the possibility of nonzero mass at $x = 0$, a

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point of S not a pole. This oversight is examined in some detal below, see Theorem 3 and § 5.)

(iii) The sequence $\{x^n\phi_n(x)\}$ converges to an entire function which is shown to be the denominator of the meromorphic function referred to in (ii).

Later, Carlitz [2], studing polynomials, remotely related to Laguerre polynomials, showed that properties (i) and (ii) hold for the polynomial sets (under a different normalization)

$$g_{n+1}(x) = xg_n(x) - \frac{n}{(n+\gamma)(n+\gamma-1)}\, g_{n-1}(x) \qquad (n \geqq 1) \tag{1.4}$$

where $\gamma > 0$, $g_0(x) = 1$ and $g_1(x) = x$. For these polynomials $\Sigma\, b_n$ diverges and hence (Corollary 4, below) (iii) is false. Subsequently, Chihara [3, pg. 15] noted, and offered an independent proof of, a theorem implicit in the works of Stieltjes [13], equivalent to the proposition that $\lim b_n = 0$ is necessary as well as sufficient to insure a denumerable spectrum (with $x = 0$ the only limit point) for the distribution relative to $\{\phi_n(x)\}$, $a_n = 0$, $n = 0, 1, \cdots$.

In § 2 we sketch the fundamental theorems of continued fractions and the theory of moments that are pertainent to our work. In § 3 we prove the corrected generalization of the Dickinson, Pollak and Wannier theorem (Theorem 3) and set forth necessary and sufficient conditions that $\lim x^n\phi_n(1/x)$ be entire. Section 4 provides a representation theorem for the class of meromorphic functions relevent to our study and provides us with a means for investigating in § 6, conditions under which mass at $x = 0$ is not present. Finally we offer an example, due essentially to Wall [15] which explicitly contradicts (ii) (without the modification supplied in Theorem 3) but in which (iii) holds. The example is of interest independently of our use.

2. Preliminary theorems and notational conventions. We use this section to set forth those parts of the theory of continued fractions, theory of moments and theory of orthogonal polynomials which bear on the problems with which we wish to concern ourselves. None of these theorems are novel. They are stated in a form suitable for our purposes with their proofs outlined only in such detail that a specific reference may be quoted for their completion.

Consider the class of polynomial sets defined recursively by

$$\phi^{(s)}_{n+1}(x) = x\phi^{(s)}_n(x) - b_{n+s}\phi^{(s)}_{n-1}(x) \qquad (n \geqq 0) \tag{2.1}$$

with

$$\phi^{(s)}_{-1}(x) = 0\,, \qquad \phi^{(s)}_0(x) = 1 \tag{2.2}$$

and

$$b_n > 0 \qquad (n \geqq 0). \tag{2.3}$$

We write $\phi_n^{(0)}(x)$ as $\phi_n(x)$ and agree in general to omit all zero superscripts. We reserve the use of superscripts in parenthesis for nonnegative integers fixed in advance of any argument and never for use as a derivative.

It may be easily seen that $\{\phi_n^{(s)}(x)\}$ are the successive denominators and $\{\phi_{n-1}^{(s+1)}(x)\}$ the successive numerators (here our convention assures that s is fixed and the sequences are indexed by n) of the convergents of

$$\frac{1|}{|1} - \frac{b_{1+s}|}{|\ x} - \frac{b_{2+s}|}{|\ x} - \cdots - \frac{b_{n+s}|}{|\ x} - \cdots. \tag{2.4}$$

Such polynomial sequences have been studied by Dickison [4], [5], Dickinson, Pollak and Wannier [6], and perhaps most completely by Sherman [10], in which more references may be found. We pause to mention the important recursion relationship

$$\phi_n^{(s)}(x) = x\phi_{n-1}^{(s+1)}(x) - b_{1+s}\phi_{n-2}^{(s+2)}(x) \qquad (n \geqq 1), \tag{2.5}$$

which follows from (2.4) but which may be proved independently by induction. If (2.5) is established first, one may observe directly that

$$\frac{\phi_{n-1}^{(s+1)}(x)}{\phi_n^{(s)}(x)} = \frac{1|}{|x} - \frac{b_{1+s}|}{|\ x} - \cdots - \frac{b_{n+s-1}\phi_0^{(n+s)}(x)}{|\qquad x}, \tag{2.6}$$

and hence the equivalence of (2.4) and $\lim_n \phi_{n-1}^{(s+1)}(x)/\phi_n^{(s)}(x)$. The definitions (2.1) and (2.2), the theorem of Favard-Shohat and the standard properties of real orthogonal polynomials lead to the observations which we state as

LEMMA 1. *All the zeros of the monic polynomials $\phi_n^{(s)}(x)$ are real and simple. The degree of $\phi_n^{(s)}(x)$ is precisely n and $\phi_n^{(s)}(x)$ is an $\binom{even}{odd}$ function if n is $\binom{even}{odd}$. The zeros of $\phi_n^{(s)}(x)$ and $\phi_{n-1}^{(s+1)}(x)$ alternate; there is a zero of one polynomial separating two consecutive zeros of the other.*

Next consider the sequence $\{\phi_{n-1}^{(s+1)}(x)/\phi_n^{(s)}(x)\}$. Set $z = 1/x$ in (2.3) and define $G^{(s)}(z)$ by

$$zG^{(s)}(z) = \frac{1|}{|1/z} - \frac{b_{1+s}|}{|1/z} - \frac{b_{2+s}|}{|1/z} - \cdots - \frac{b_{n+s}|}{|1/z} - \cdots.$$

so that after and equivalence transformation

$$G^{(s)}(z) = \frac{1|}{|1} - \frac{b_{1+s}z^2|}{|1} - \frac{b_{2+s}z^2|}{|1} - \cdots - \frac{b_{n+s}z^2|}{|1} - \cdots . \tag{2.7}$$

But a theorem of Stieltjes states: *A necessary and sufficient condition that* (2.7) *be a nonrational meromorphic function is* $b_n > 0$ *and* $\lim b_n = 0$ (see Wall [14; Theorems 54.1 and 54.2]). Hence,

THEOREM 1. *If* $\{\phi_n^{(s)}(x)\}$ *is defined by* (2.1), (2.2) *and* (2.3) *with* $\lim b_n = 0$, *then for each nonnegative integer s.*

$$\lim_{n\to\infty} \frac{\phi_{n-1}^{(s+1)}(1/z)}{z\phi_n^{(s)}(1/z)} = G^{(s)}(z) . \tag{2.8}$$

$G^{(s)}(z)$ *is a transcendental meromorphic function and the convergence in* (2.8) *is uniform in compact sets which exclude the poles of* $G^{(s)}(z)$.

As a corollary of this theorem we prove the properties of the orthogonality domain listed in (i) of the introduction. (See Chihara [3, pg. 15] for an alternate proof along different lines). Consider an interval $[c, d]$ free of poles of $G^{(s)}(1/x)$. Lemma 1 assures that the zeros of $\phi_n^{(s)}(x)$ and $\phi_{n-1}^{(s+1)}(x)$ are not common. Hence, Theorem 1 and and Hurwitz's theorem imply that $[c, d]$ is ultimately free of zeros of $\phi_n^{(s)}(x)$. Thus any distribution function for $\{\phi_n^{(s)}(x)\}$ is constant in $[c, d]$, Szegö [12, Theorem 6.1.1]. But the poles of $G^{(s)}(1/x)$ are a bounded set, symmetrically distributed with respect to $x = 0$. Hence, we have proved:

COROLLARY 1. *The orthogonality domain for the* $\{\phi_n^{(s)}(x)\}$ *of Theorem* 1 *is the bounded, denumerably infinite set of singularities of* $G^{(s)}(1/x)$. *This set is isolated except at* $x = 0$, *and symmetric with respect to the origin.*

Suppose, by way of a converse, that S is any bounded, denumerably infinite point set with $x = 0$ the only limit point. Suppose the distribution function $\beta(x)$ (bounded and nondecreasing) has S as its spectrum. We normalize $\beta(x)$ and all distribution function considered herein by specifying:

$$\begin{aligned} &\text{(i)} \quad \int_{-\infty}^{\infty} d\beta = 1 \\ &\text{(ii)} \quad \beta(x) = \frac{1}{2}[\beta(x+0) + \beta(x-0)] , \qquad x \in S. \end{aligned} \tag{2.9}$$

Suppose further that $\beta(-x) = -\beta(x)$ (so that S is symmetric) and that $\{p_n(x)\}$ is the unique set of monic polynomials orthogonal over S with respect to $d\beta$. Then the symmetry of $d\beta$ leads directly to

$$p_{n+1}(x) = xp_n(x) - B_n p_{n-1}(x) \qquad (n \geqq 0) \tag{2.10}$$

with $p_{-1}(x) = 0$ and $p_0(x) = 1$. This in turn defines the function (Szegö [12, Theorem 3.5.4])

$$\begin{aligned} F(z) &= \int_{-\infty}^{\infty} \frac{d\beta(t)}{1 - tz} \\ &= \frac{1|}{|1} - \frac{B_1 z^2|}{|1} - \frac{B_2 z^2|}{|1} - \cdots . \end{aligned} \tag{2.11}$$

It is now a consequence of the hypotheses on $\beta(x)$ that $F(z)$ is transcendental and meromorphic. Hence the theorem of Stieltjes mentioned prior to the statement of Theorem 1 yields $B_n > 0$, $n = 1, 2, \cdots$ and $\lim B_n = 0$. Thus,

COROLLARY 2. *A nondecreasing, symmetric distribution function, normalized by* (2.9) *having discrete, bounded spectrum with* $x = 0$ *the only limit point, determines a transcendental meromorphic function with an expansion of the form* (2.11), *where* $\{B_n\}$ *is a positive null-sequence.*

It is useful to have these two corollaries and Theorem 1 restated in somewhat different form.

COROLLARY 3. *The denominators of the successive convergents of any continued fraction of the form* (2.11) *with* $B_n > 0$ $(n > 0)$ *and* $\lim B_n = 0$ *form a sequence of real orthogonal polynomials with the discrete domain of orthogonality described in Corollary* 1.

Our proof of Theorem 3 (below) requires a Mittag-Leffler expansion of $G^{(s)}(z)$. To this end we call attension to the following theorem of Montel (Obrechkoff [8; Theorem XXI]):

If a sequence of rational functions converges uniformly to a meromorphic function and if the zeros and poles of each rational function are simple, real, and alternate, then the meromorphic function has the expansion

$$B - Az + \sum_{n=1}^{\infty} A_n \left(\frac{1}{z - \alpha_n} + \frac{1}{\alpha_n} \right)$$

where $\sum A_n/\alpha_n^2$ *converges and* $A, B, A_1, A_2, \cdots, A_n, \cdots$ *have the same sign and the* α_n *are real and distinct.*

The hypotheses of this theorem are satisfied by the rational functions $\phi_{n-1}^{(s+1)}(1/z)/\phi_n^{(s)}(1/z)$ because of Lemma 1 and Theorem 1. Furthermore, $G^{(s)}(z)$ is even. Hence, after simplification,

$$G^{(s)}(z) = -A^{(s)} + \sum_{n=1}^{\infty} \frac{2A_n^{(s)}}{z^2 - (\alpha_n^{(s)})^2} . \tag{2.13}$$

Finally, $G^{(s)}(0) = 1$ so that $A^{(s)} \leqq 0$ and $A_n^{(s)} < 0$ $(n > 0)$. In this representation, and in all similar expansions, we agree to order the the poles, $0 < \alpha_1^{(s)} < \alpha_2^{(s)} < \cdots, \alpha_n \rightarrow \infty$.

THEOREM 2. *The transcendental meromorphic function of Theorem 1 has the expansion*

$$G^{(s)}(z) = -A^{(s)} + \sum_{n=1}^{\infty} \frac{2A_n^{(s)}}{z^2 - (\alpha_n^{(s)})^2} ,$$

where $-\Sigma A_n^{(s)}(\alpha_n^{(s)})^{-2} < \infty$, $A^{(s)} \leqq 0$ *and* $A_n^{(s)} < 0$, $n = 1, 2, \cdots$. *The convergence is uniform in compact sets which exclude the poles of* $G^{(s)}(z)$.

3. The construction of the distribution function. With the preliminaries now settled, we can proceed with a consideration of the first of our goals; namely, the construction which explicitly exhibts the relationship between $G^{(s)}(1/x)$ and $\{\phi_n^{(s)}(x)\}$. We state this result as Theorem 3, a generalization and correction of the corresponding theorem in [6, Theorem 5]. The approach in this section owes its inspiration to the ideas expressed in [6].

Set $z = 1/x$ in (2.1), (2.2) and (2.5). Define $F_n^{(s)}(z) = z^n\phi_n^{(s)}((1/z)$. Then,

$$F_{n+1}^{(s)}(z) = F_n^{(s)}(z) - b_{n+s}z^2F_{n-1}^{(s)}(z) \qquad (n \geqq 1) \tag{3.1}$$

with

$$F_0^{(s)}(z) = 1, \qquad F_1^{(s)}(z) = 1 \tag{3.2}$$

and

$$F_n^{(s)}(z) = F_{n-1}^{(s+1)}(z) - b_{1+s}z^2F_{n-2}^{(s+2)}(z) \qquad (n \geqq 2). \tag{3.3}$$

Furthermore,

$$\frac{\phi_{n-1}^{(s+1)}(1/z)}{z\phi_n^{(s)}(1/z)} = \frac{F_{n-1}^{(s+1)}(z)}{F_n^{(s)}(z)} \qquad (n \geqq 1).$$

Now divide (3.3) by $F_{n-1}^{(s+1)}(z)$ and let $n \rightarrow \infty$. This yields

$$\frac{1}{G^{(s)}(z)} = 1 - b_{1+s}z^2G^{(s+1)}(z) , \tag{3.4}$$

which may be interpreted as an alternate expression for (2.7). We combine (3.3) and (3.4) to obtain

$$F_n^{(s)}G^{(s)} - F_{n-1}^{(s+1)} = b_{1+s}z^2G^{(s)}(F_{n+1}^{s+1}G^{(s+1)} - F_{n-2}^{(s+2)})$$

for $n \geqq 1$. (Here and in the next equation $F_n^{(s)} = F_n^{(s)}(z)$, $G^{(s)} = G^{(s)}(z)$, etc.). Such an expression suggests iteration. With the aid of (3.4) written as

$$\begin{aligned} F_1^{(s+n-1)}G^{(s+n-1)} - F_0^{(s+n)} &= G^{(n+s-1)} - 1 \\ &= b_{n+s}z^2G_0^{(s+n)}G^{(n+s-1)} , \end{aligned}$$

and after some simplication, including multiplying through by z^{-n-p-1}, we compute

$$\begin{aligned} & z^{-n-p-1}G^{(s)}(z)F_n^{(s)}(z) - z^{-n-p-1}F_{n-1}^{(s+1)}(z) \\ &= z^{n-p-1}\prod_{k=1}^{n} b_{k+s}G^{(k+s)}(z)G^{(s)}(z) \end{aligned} \tag{3.5}$$

for any p and all $n \geqq 1$. Now choose C' a circle small enough to exclude all the singularities of $G^{(s)}(z), G^{(s+1)}(z), \cdots, G^{(s+n)}(z)$. Such a circle exists because $G^{(m)}(0) = 1$ for all m. The degree of $F_k^{(s)}(z)$ is $2[k/2]$. Hence the residues at the origin of each term in (3.5) is readily computed and we have

$$\frac{1}{2\pi i}\int_{C'} z^{-n-p-1}F_n^{(s)}(z)G^{(s)}(z)dz = \delta_{np}\prod_{k=1}^{n} b_{k+s} \tag{3.6}$$

for $0 \leqq p \leqq n$ and $n = 1, 2, 3, \cdots$. If we define the empty product as unity, then (3.6) holds for $n = 0$ also. The change of variables, $z = 1/x$, casts (3.6) into

$$\frac{1}{2\pi i}\int_C x^p\phi_n^{(s)}(x)\{x^{-1}G^{(s)}(1/x)\}dx = \delta_{np}\prod_{k=1}^{n} b_{k+s} \tag{3.7}$$

$0 \leqq p \leqq n$ and $n \geqq 0$. Here C is the circle reciprocal to C'. C surrounds all of the singularities of $G^{(k+s)}(1/x), k = 0, 1, \cdots, n$. The integration is taken in the positive direction. We may convert (3.7) into a real orthogonality relationship by substituting the representation of $G^{(s)}(z)$, given in Theorem 2, into (3.7) and interchanging the order of integration and summation. With the observation that the residue of $x^{-1}G^{(s)}(1/x)$ at $x = \pm 1/\alpha_n^{(s)}$ is $-A_n^{(s)}(\alpha_n^{(s)})^{-2} > 0$, we have

$$-A^{(s)}\delta_{p0}\varphi_n^{(s)}(0) + \Sigma x^p\phi_n^{(s)}(x)\,\mathrm{Res}\,\{x^{-1}G^{(s)}(1/x)\} = \delta_{np}\prod_{k=1}^{n} b_{k+s} \tag{3.8}$$

for $0 \leqq p \leqq n, n = 0, 1, 2, \cdots$. The summation is extended over all the poles of $x^{-1}G^{(s)}(1/x)$. We express this and the results of section two as

THEOREM 3. *Let $\{b_n\}$ be an arbitrary sequence of positive constants with $\lim b_n = 0$. Suppose $\{\phi_n^{(s)}\}$ are the sets of orthogonal polynomials determined by* (2.1) *and* (2.2). *Suppose $G^{(s)}(z)$ is defined by* (2.7) *and $\beta^{(s)}$ is the unique normalized distribution function associated with $\{\phi_n^{(s)}(x)\}$. Then for each nonnegative integer s,*

(i) *the spectrum of $\beta^{(s)}$ is the closure of the set of poles of $G^{(s)}(1/z)$; namely, $x = 0$ and $x = \pm 1/\alpha_n^{(s)}$, $n = 1, 2, \cdots$.*

(ii) *the jump of $\beta^{(s)}$ at these poles is equal to the residue of $x^{-1}G^{(s)}(1/x)$ there. That is,*

$$\beta^{(s)}(x+0) - \beta^{(s)}(x-0) = -A_n^{(s)}(\alpha_n^{(s)})^{-2}$$

for $x = \pm 1/\alpha_n^{(s)}$.

(iii) $\beta^{(s)}(+0) - \beta^{(s)}(-0) = -A^{(s)}$.

(iv) *For each $p, 0 \leqq p \leqq n$ and all $n = 0, 1 2, \cdots$,*

$$\int_{-a}^{a} x^p \phi_n^{(s)}(x) d\beta^{(s)} = \int_{-a}^{a} \phi_p^{(s)}(x)\phi_n^{(s)}(x) d\beta^{(s)} = \delta_{np} \prod_{k=1}^{n} b_{k+s} \, ,$$

where $[-a, a]$ is an interval large enough to include the bounded set, $\{\pm 1/\alpha_n^{(s)}\}$.

The criterion $\Sigma b_n < \infty$ is both necessary and sufficient to imply the existence of $\lim_n x^n \phi_n^{(s)}(1/x)$.

COROLLARY 4. *If $\Sigma b_n < \infty$ then (uniformly)*

$$\lim_{n\to\infty} \frac{\phi_{n+1}^{(s+1)}(1/x)}{x\phi_n^{(s)}(1/x)} = \frac{\lim\limits_{n\to\infty} x^{n-1}\phi_{n-1}^{(s+1)}(1/x)}{\lim\limits_{n\to\infty} x^n \phi_n^{(s)}(1/x)} = \frac{E^{(s+1)}(x)}{E^{(s)}(x)} = G^{(s)}(x) \, .$$

Here, $E^{(s+1)}(x)$ and $E^{(s)}(x)$ are entire functions. Conversely, if $\lim_n x^n \phi_n^{(s)}(1/x)$ converges uniformly in a bounded closed domain about $x = 0$ then $\Sigma b_n < \infty$.

Proof. The sufficiency, with the modification at $x = 0$ previously mentioned, is the main theorem of Dickinson, Pollak and Wannier. Their proof depends only upon $\Sigma b_n < \infty$ and hence is applicable here. The necessity is proved by an appeal to a theorem of Polya (Obrechkoff, [8, Theorem IV] which states in effect that the limit of a uniformly converging sequence of polynomials with real, symmetric zeros is an entire function. Therefore, the coefficient of x^2 in $x^n\phi_n^{(s)}(1/x)$ converges to the coefficient of x^2 in the series expansion of the limit function. But, we see that $x^n\phi_n^{(s)}(1/x) = 1 - (b_{1+s} + b_{2+s} + \cdots + b_{n+s})x^2 + 0(x^4)$ for $n = 1, 2, \cdots$. This proves the necessity. More interestingly;

THEOREM 4. *A necessary and sufficient condition that* $\lim x^n\phi_n^{(s)}(1/x)$ *converges uniformly in some bounded, closed domain containing* $x=0$ *(and hence converges to an entire function) is that* $\Sigma\alpha_n^{-2} < \infty$.

Proof. Assume that $\lim_n x^n\phi_n^{(s)}(1/x)$ converges uniformly. But the aforementioned theorem of Polya we know the limit function is entire. Denote its zeros by $\pm\alpha_n^{(s)}$, $0 < \alpha_1^{(s)} < \alpha_2^{(s)} < \cdots$. Let $\pm\alpha_{k,n}$, $k = 1, 2, \cdots [n/2]$ be the $2[n/2]$ zeros of $x^n\phi_n^{(s)}(1/x)$ ordered

$$0 < \alpha_{1,n} < \alpha_{2,n} \cdots < \alpha_{[n/2],n} .$$

Now a theorem of Hurwitz asserts that $\lim_n \alpha_{k,n} = \alpha_k^{(s)}$, $k = 1, 2, \cdots$. Referring once again to Polya's Theorem we conclude that $\Sigma(\alpha_k^{(s)})^{-2}$ converges. Of course, the zeros $\pm\alpha_k^{(s)}$ are the poles of $G^{(s)}(1/x)$. From (3.4) they are also the zeros of $G^{(s-1)}(1/x)$. But the zeros and poles of $G^{(s-1)}(1/x)$ (for any s) alternate on the real axis. Hence $\Sigma(\alpha_k^{(s-1)})^{-2}$ also converges. Successive applications of this reasoning yields the convergence of $\Sigma(\alpha_k^{(0)})^{-2} = \Sigma\alpha_k^{-2}$ after s steps. We prove the sufficiency by showing that the convergence of $\Sigma\alpha_k^{-2}$ implies the convergence of Σb_n. Towards this end we note that the zeros of $\phi_n^{(s)}(x)$ and $\phi_{n-1}^{(s+1)}(x)$ are interlaced (Lemma 1). In our notation, the reciprocals of these zeros are ordered as follows;

$$\cdots < \alpha_{1,n+1} < \alpha_{1,n} < \alpha_{2,n+1} < \alpha_{2,n} < \cdots ,$$

for $n = 1, 2, \cdots$. Hence,

$$(3.9) \qquad \begin{array}{ccccccc} \dfrac{1}{\alpha_{1,n}^2} & < & \dfrac{1}{\alpha_{1,n+1}^2} & < & \dfrac{1}{\alpha_{1,n+2}^2} & < & \cdots \\ \dfrac{1}{\alpha_{2,n}^2} & < & \dfrac{1}{\alpha_{2,n+1}^2} & < & \dfrac{1}{\alpha_{2,n+2}^2} & < & \cdots \\ \vdots & & \vdots & & \vdots & & \\ \dfrac{1}{\alpha_{r,n}^2} & < & \dfrac{1}{\alpha_{r,n+1}^2} & < & \dfrac{1}{\alpha_{r,n+2}^2} & < & \cdots \end{array}$$

for $r < [n/2]$. Thus $\alpha_{r,n}^{-2} < \alpha_r^{-2}$. By hypothesis $\Sigma\alpha_n^{-2} < \infty$. Hence by Tannery's theorem (Browwich [1, pg. 136]).

$$\lim_{n\to\infty} \sum_{k=1}^{[n/2]} \alpha_{k,n}^{-2} = \sum_{k=1}^{\infty} \alpha_k^{-2} ,$$

But $b_1 + b_2 + \cdots + b_n = \sum_{k=1}^{[n/2]} \alpha_{k,n}^{-2}$, since the coefficient of x^2 in $x^n \phi_n^{(s)}(1/x)$ is the sum of the squares of the zeros of $\phi_n^{(s)}(x)$. Corollary 4 completes the proof.

For a special case, Dickinson [4] has computed the moments, $\{m_n^{(s)}\}$, of $\beta^{(s)}$ in terms of the parameters of $G^{(s)}(z)$. We know in advance that the odd moments are zero (Shohat [11, Theorem II]) and that the moments are the coefficients in the Taylor series expansion of $G^{(s)}(z)$ about $z = 0$. Specifically,

COROLLARY 5. *Under the hypothesis of Theorem* 3;

$$m_n^{(s)} = 0 \,, \qquad n \text{ odd} \,,$$

$$m_n^{(s)} = -A^{(s)} - 2 \sum_{k=1}^{\infty} A_k^{(s)} (\alpha_k^{(s)})^{-2} = 1 \,,$$

$$m_n^{(s)} = -2 \sum_{k=0}^{\infty} A_k^{(s)} (\alpha_k^{(s)})^{-n-2} \,, \qquad n > 0 \text{ and even} \,.$$

Proof. We have from Theorem 2 that

$$\begin{aligned} G^{(s)}(z) &= -A^{(s)} + \sum_{n=1}^{\infty} \frac{2A_n^{(s)}}{z^2 - (\alpha_n^{(s)})^2} \\ &= -A^{(s)} - 2 \sum_{n-1}^{\infty} A_n^{(s)} (\alpha_n^{(s)})^{-2} \{1 - z^2/(\alpha_n^{(s)})^2\}^{-1} \\ &= -A^{(s)} - \sum_{k=0}^{\infty} z^{2k} \sum_{n=1}^{\infty} 2A_n^{(s)} (\alpha_n^{(s)})^{-2k-2} \end{aligned}$$

for $|z| < |\alpha_1^{(s)}|$. Thus

$$x^{-1} G^{(s)}(1/x) = -A^{(s)}/x - \sum_{k=0}^{\infty} x^{-2k-1} \sum_{n=1}^{\infty} 2A_n^{(s)} (\alpha_n^{(s)})^{-2k-2} \,. \tag{3.10}$$

But then

$$\frac{1}{2\pi i} \int_C x^n \{x^{-1} G^{(s)}(1/x)\} dx = \int_{-a}^{a} x^n d\beta^{(s)}$$

and the Corollary is proved.

4. A representation theorem for the meromorphic function $G^{(s)}(z)$. In previous sections we have concentrated on the determination of the spectrum of $\beta^{(s)}$ from a knowledge of $\{b_n\}$. In this and the succeeding section we direct our attention to the class of meromorphic functions which determine real orthogonal polynomials with distributions having spectra of the type described in Corollary 1. We denote the class of such meromorphic functions by PIF; a notation motivated by the notation for a related class of functions. We express our main theorem by the following:

THEOREM 5. *The following four statements are equivalent**:

(1) $$zF(z) \in PIF$$

(2) $$F(z) = \frac{1|}{|1} - \frac{B_1z^2|}{|1} - \frac{B_2z^2|}{|1} - \cdots,$$

with $B_n > 0$ $n = 1, 2, \cdots$, *and* $\lim B_n = 0$.

(3) $$F(z) = -A + \sum_1^\infty \frac{2A_n}{z^2 - \alpha_n^2},$$

where $A \leqq 0, A_n < 0, n = 1, 2, 3, \cdots, -A - \Sigma 2A_n\alpha_n^{-2} = 1, 0 < \alpha_1 < \alpha_2 < \cdots,$ $\alpha_n \to \infty$.

(4) $$F(z) = \prod_{n=1}^\infty \frac{1 - z^2/\gamma_n^2}{1 - z^2/\alpha_n^2}$$

where $0 < \alpha_1 < \gamma_1 < \alpha_2 < \gamma_2 < \cdots, \alpha_n \to \infty$ *and* $\prod_{n=1}^\infty (1 + \gamma_n^{-2})(1 + \alpha_n^{-2})^{-1}$ *converges (and is therefore* $\neq 0$).

Proof. (1)$\Leftrightarrow$(2) is established in § 2 along with (2)$\Rightarrow$3 (Theorem 2). We shall prove (3)$\Rightarrow$(2) and then (3)$\Leftrightarrow$(4) to complete the proof.

(a) We prove (3)$\Rightarrow$(2). Suppose F is defined by (3). Then $-A - \Sigma 2A_n\alpha_n^{-2} = 1$ establishes the uniform convergence of the right-hand side of (3). Hence F is transcendentally meromorphic and analytic at $z = 0$. The Taylor series (in z^2) for F at $z = 0$ has only positive coefficients. From the theory of continued fractions (Shohat [11, pg. 455]), we deduce a representation for F in the form (2) with the stated conditions on $\{B_n\}$.

(b) To prove the equivalence of (3) and (4) we set $z = it$ and define $f(t) = tF(it)$. Then $f(t)$ is a meromorphic function which maps the right half-plane into itself, the imaginary axis into itself and the reals into the reals. These properties of $f(t)$ follow, if we assume F is given by (3), by taking real and imaginary parts of (3). Richards [9] made a detailed study of such functions which he named *iPRF* (*PR* for positive real part; *PI* for positive imaginary part in our case). The transformation, $t = -iz$ and the definition $F(z) = if(-iz)/z$, therefore, converts Richards' theorems into results for $F(z)$. In particular then, (3)$\Rightarrow$(4) as a consequence of [9; Theorem 12] and because $f'(0) = F'(0) = 1$. Conversely, if F is given by (4) then $f(t) \in iPRF$ by [9, Corollary 12.1] and (4)$\Rightarrow$(3) by [9; Corollary 10.1]. This completes the proof.

5. **The constants $A^{(s)}$.** The mass assigned by the weight function

* We suspend our convention on superscripts for this section.

to each point of the orthogonality domain may be determined, as we have seen, by examination of the residue at each pole of a function meromorphic in $1/x$ with the single exception of the mass at zero, $-A^{(s)}$. If the mass at zero is zero for each s, as it is in the examples considered by Dickinson, Pollak and Wannier [6] and Carlitz [2], the exception is vacuous. It is of some interest then, to consider the problem of characterizing in function-theoretic terms those *PIF* functions with $A^{(s)} = 0$, $s = 0, 1, 2, \cdots$. In the course of this section we derive some theorems, parts of which yield conditions assuring nonzero mass at $x = 0$. We begin by proving a Lemma fundamental to this part of our study.

LEMMA 2. *In the Stolz domain,*

$$0 < \theta_1 \leqq \arg z \leqq \theta_2 < \pi, \qquad \textit{and for each } s \geqq 0\,,$$

$$\lim_{|z|\to\infty} G^{(s)}(z) = -A^{(s)}\,. \tag{5.1}$$

Proof. This is a well-known theorem in a different guise. For, [9, Theorem 5 and Corollary 10.1] shows that $\lim_{|z|\to\infty} f(t)/t$ exists (t in the domain $|\arg t| \leqq \theta < \pi/2$) and is nonnegative. Now $G^{(s)}(z)$ is *PIF* from Theorem 5 (2), so that $f(t) = tG^{(s)}(it) \in IPR$ and the Lemma follows from [9; Corollary 10.1]. A second Lemma follows from (3.4) and Lemma 2.

LEMMA 3. *Either the terms of* $\{A^{(s)}\}_{s=1}^{\infty}$ *are all zero or they are alternately zero and nonzero.*

Proof. Set $z = iy$, (y real) in (3.4) and write

$$\frac{1}{G^{(s)}(iy)} = 1 + b_{1+s}y^2G^{(s+1)}(iy)\,. \tag{5.2}$$

Then one interation of (5.2) yields

$$\frac{1}{G^{(s)}(iy)} = 1 + b_{1+s}(1/y^2 + b_{2+s}G^{(s+2}(iy))^{-1}\,. \tag{5.3}$$

Now let $y \to \infty$ in (5.2) and (5.3) and evoke Lemma 2. Equation (5.2) shows that $A^{(s)}A^{(s+1)} \neq 0$ is impossible and (5.3) shows that either $A^{(s)} = A^{(s+2)} = 0$ or $A^{(s)}A^{(s+2)} \neq 0$ for every s. But this is just an alternate way of expressing the content of Lemma 3.

THEOREM 6. *For all* $s = 0, 1, 2, \cdots$

$$-A^{(s)} = \lim_{n\to\infty} \prod_{1}^{n} (\alpha_k^{(s)}/\alpha_k^{(s+L)})^2 < \infty\,. \tag{5.4}$$

Proof. Set $z = (iy)^{-1}$ in Theorem 5 and note that

$$\lim_{|z|\to\infty} G^{(s)}(z) = -A^{(s)} = \lim_{y\to 0} \prod_1^\infty \left[\frac{y^2 + (\alpha_n^{(s+1)})^{-2}}{y^2 + (\alpha_n^{(s)})^{-2}}\right]. \tag{5.5}$$

Since

$$\frac{(\alpha_n^{(s)})^{-2} - (\alpha_n^{(s+1)})^{-2}}{y^2 + (\alpha_n^{(s)})^{-2}} < 1 - \left(\frac{\alpha_n^{(s)}}{\alpha_n^{(s+1)}}\right)^2$$

for all y and every n, the hypothesis that $\prod (\alpha_n^{(s)}/\alpha_n^{(s+1)})^2$ converges implies the uniform convergence of the rightmost factor in (5.5) in every set $y^2 \leqq R^2$ and thus the continuity of $G^{(s)}(1/iy)$ at $y = 0$. This proves (5.4) when the product converges. Now $|\alpha_n^{(s)}| < |\alpha_n^{(s+1)}|$ so that divergence of $\prod (\alpha_n^{(s)}/\alpha_n^{(s+1)})^2$ is divergence to zero. Hence, given any $\varepsilon > 0$ there exists an N such that for all $n > N$

$$\prod_1^N (\alpha_n^{(s)}/\alpha_n^{(s+1)})^2 < \varepsilon .$$

But

$$\prod_1^\infty \left[\frac{y^2 + (\alpha_n^{(s+1)})^{-2}}{y^2 + (\alpha_n^{(s)})^{-2}}\right] \leqq \prod_1^N \left[\frac{y^2 + (\alpha_n^{(s+1)})^{-2}}{y^2 + (\alpha_n^{(s)})^{-2}}\right]$$

$$\leqq \prod_1^N \left[\frac{y^2 + (\alpha_n^{(s+1)})^{-2}}{(\alpha_n^{(s)})^{-2}}\right] \leqq \left(1 + \frac{1}{N}\right)^N \prod_1^N (\alpha_n^{(s)}/\alpha_n^{(s+1)})^2 .$$

The leftmost inequality holds for every N because each term in the product is less than one for every y. The last inequality holds for all y satisfying $y \leqq (\sqrt{N}\,\alpha_n^{(s+1)})^{-1}$ $n = 0, 1, \cdots, N$. Because of the ordering of the poles of $G^{(s)}(z)$, this can be accomplished by the restriction $y \leqq (\sqrt{N}\,\alpha_N^{(s+1)})^{-1}$. Hence the limit on the right side of (5.5) must be zero, proving Theorem 6 if the product diverges. A less complete result follows from (2.13).

THEOREM 7. *If $A^{(s+1)} = 0$ then a necessary and sufficient condition for $A^{(s)} \neq 0$ is that $\Sigma A_n^{(s+1)}$ converge. In either case*

$$A^{(s)} = -(1 - b_{1+s}\Sigma 2A_n^{(s+1)})^{-1}$$

(suitably interpreted if $\Sigma A_n^{(s)}$ diverges).

Proof. Set $z = (iy)^{-1}$ (y real) in the representation for $G^{(s+1)}$ given by (2.13). Since $A^{(s+1)} = 0$ and $0 < \alpha_1^{(s+1)} < \alpha_2^{(s+1)} < \cdots$, we have for all $y \leqq (\alpha_1^{(s+1)})^2$,

$$(-1/2)\sum_1^N A_n^{(s+1)} \leqq \frac{-A_n^{(s+1)}}{1 + (\alpha_1^{(s+1)})^{-2}y^2} \leqq \sum_1^N \frac{-A_n^{(s+1)}}{1 + (\alpha_n^{(s+1)})^{-2}y^2}$$

$$\leqq \sum_1^\infty \frac{-A_n^{(s+1)}}{1+(\alpha_n^{(s+1)})^{-2}y^2} = y^{-2}G^{(s+1)}(-i/y) .$$

But from (3.4), with $z=(iy)^{-1}$,

$$y^{-2}G^{(s+1)}(-i/y) = b_{1+s}^{-1}\left(\frac{1}{G^{(s)}(-i/y)} - 1\right) . \tag{5.6}$$

Let $y \to 0$ and assume $\Sigma A_n^{(s+1)}$ diverges. Then

$$\lim_{y\to 0} G^{(s)}(-i/y) = -A^{(s)} = 0 .$$

Now suppose $\Sigma A_n^{(s+1)}$ converges. Since

$$y^{-2}G^{(s+1)}(-i/y) = \sum_1^\infty \frac{-A_n^{(s+1)}}{1+(\alpha_n^{(s+1)})^{-2}y^2} \leqq \sum_1^\infty -A_n^{(s+1)} , \tag{5.7}$$

we conclude that $\Sigma\, A_n^{(s+1)}[1+(\alpha_n^{(s+1)})^2 y^2]^{-1}$ converges uniformly for all y in, say $y^2 \leqq R$, and hence represents a continuous function at $y=0$. Therefore, from (5.6) and (5.7).

$$b_{1+s}^{-1}(-1/A^{(s)} - 1) = -\sum_1^\infty A_n^{(s+1)} .$$

This completes the proof.

Hence if $\{A^{(s)}\}$ is a sequence of zeros, $\sum A_n^{(s)}$ must diverge for each s. Conversely, if all such series diverge $\{A^{(s)}\}$ is a sequence of zeros. Is it possible for $\Sigma A_n^{(s)}$ to converge and $A^{(s)} \neq 0$? $A^{(s)}=0$? In other words, if $\{A^{(s)}\}$ is alternately zero and nonzero, what can be said about the convergence of ΣA_n, $\Sigma A_n^{(1)}$, $\Sigma A_n^{(2)}$, $\cdots$, besides the statement that they all cannot diverge? We leave this question open.

Finally,

THEOREM 8. *The constants* $\{b_{n+s}\}$ *and* $A^{(s)}$ *are related by*

$$\frac{-1}{A^{(s)}} = 1 + \frac{b_{1+s}}{b_{2+s}} + \frac{b_{1+s}b_{3+s}}{b_{2+s}b_{4+s}} + \frac{b_{1+s}b_{3+s}b_{5+s}}{b_{2+s}b_{4+s}b_{6+s}} + \cdots , \tag{5.8}$$

so that $A^{(s)}=0$ *if and only if the series* (5.8) *diverges.*

This theorem is known (see [3; Theorem 2] and the references therein). We include the statement here for completeness.

6. **An example.** Wall [15] studied a certain continued fraction which arose in a number-theoretic context. By suitable changes of variable this example may be used to illustrate the theorems of the

previous sections. There seem to be relatively few cases of interesting special functions for which the sequences $\{b_n\}$ and $\{A_n^{(s)}\}$ are known explicitly and $\{A^{(s)}\}$ is not all zeros. The author was able to find only this one example. Choose $0 < r < 1$ and $0 < q < 1$ and define $b_2 = r$,

$$b_{2k+2} = (1 - rq^k)q^{k+1},\ b_{2k+3} = rq^{k+1}(1 - q^{k+1}) \tag{6.1}$$

$k = 0, 1, 2, \cdots$;

$$M_1 = r \prod_{k=1}^{\infty} (1 - rq^k)\ ; \tag{6.2}$$

$$M_k = \frac{r^{k-1}M_1}{(1-q)(1-q^2)\cdots(1-q^{k-1})}\ , \qquad k \geqq 2\ . \tag{6.3}$$

Then Wall [15] has shown, in our notation,

$$\frac{1}{G^{(0)}(z)} = 1 + z^2 \sum_{k=1}^{\infty} \frac{M_k q^{-k}}{z^2 - q^{-k}}\ . \tag{6.4}$$

From (3.4) and (6.4) we deduce that

$$G^{(1)}(z) = \sum_{k=1}^{\infty} \frac{-M_k(rq^k)^{-1}}{z^2 - q^{-k}} \tag{6.5}$$

so that $A^{(1)} = 0,\ -2A_k^{(1)} = M_k(rq^k)^{-1}$ and $(\alpha_k^{(1)})^2 = q^{-k}$. We use Theorem 7 with $s = 0$ to deduce that $A^{(0)} \neq 0$ if $\Sigma M_k(rq^k)^{-1}$ converges. The ratio test yields the convergence of this series if $r < q$ and its divergence if $q < r$. Thus the terms in $\{A^{(s)}\}$ are all zero if $q < r$ and are alternately zero and nonzero, $A^{(0)} \neq 0$, if $r < q$. Wall has also shown that

$$G^{(1)}(z) = \sum_{n=0}^{\infty} \left[q^n \prod_{k=0}^{n-1} (1 - rq^k) \right] z^{2n}\ . \tag{6.6}$$

Hence, the moments $m_{2n}^{(1)}$ can be read off immediately (see Corollary 5)

$$m_{2n}^{(1)} = q^n \prod_{k=0}^{n-1} (1 - rq^k) \qquad (n \geqq 0) \tag{6.7}$$

where the empty product is defined as unity. Incidentally, $\Sigma b_k < \infty$ for all choices of r and q so that this example is one that is included in the Dickinson, Pollak and Wannier theory.

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NORM DECREASING HOMOMORPHISMS OF GROUP ALGEBRAS

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The homomorphisms φ of the group algebra $L^1(F)$ into the algebra $M(G)$ of measures, where F and G are locally compact groups, has been completely determined when both groups are abelian by P. J. Cohen, and when G is compact and the homomorphism is norm decreasing and order-preserving by Glicksberg. In this paper the structure of norm decreasing homomorphisms φ is determined for arbitrary locally compact F and G. As an application the special structure of all norm decreasing monomorphisms is determined, along with the rather elegant structure of all norm decreasing homomorphisms mapping $L^1(F)$ onto $L^1(G)$.

The analysis is effected by finding all multiplicative subgroups of the unit ball of measures on a locally compact group for, as we show, each φ extends to a norm decreasing homomorphism $\varphi: M(F) \rightarrow M(G)$, and is determined by the image under φ of the group of point masses on G, a multiplicative subgroup of the unit ball in $M(G)$.

This paper completes a study of norm decreasing homomorphisms on group algebras initiated by Glicksberg in [4] and [5]. If G is a locally compact group we will denote its group algebra by $L^1(G)$ and its convolution algebra of bounded regular Borel measures by $M(G)$. We present a complete structural analysis of the subgroups of the unit ball in $M(G)$, and a structure theory classifying all norm decreasing homomorphisms $\varphi: L^1(F) \rightarrow M(G)$ where F and G are locally compact groups. As an application we determine the special structure of all monomorphisms φ mapping $L^1(F)$ into $M(G)$ and all norm decreasing homomorphisms which map $L^1(F)$ onto $L^1(G)$.

Let $C_0(G)$ be the sup norm algebra of all continuous complex valued functions on G which vanish at infinity, and recall that $C_0(G)^* = M(G)$. If $\mu \in M(G)$ its support $s(\mu)$ is defined so that $x \in s(\mu) \Leftrightarrow$ for each neighborhood U of x there is some $\psi \in C_0(G)$, vanishing outside of U, with $\langle \mu, \psi \rangle \neq 0$. Then $s(\mu)$ is a Borel set. If Γ is a subset in $M(G)$ we define $\operatorname{supp}(\Gamma) = \cup\{s(\mu): \mu \in \Gamma\}$. The convolution of $\mu, \lambda \in M(G)$ is given as an element of $C_0(G)^*$ by defining

$$\langle \mu * \lambda, \psi \rangle = \int_G \left[\int_G \psi(st) d\mu(s) \right] d\lambda(t)$$

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for all $\psi \in C_0(G)$. If $M(G)$ is given the total variation norm it becomes a Banach algebra under this multiplication.

We first show that if $\mu, \lambda \in M(G)$ then

(1) $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\| \Rightarrow s(\mu*\lambda) = (s(\mu)s(\lambda))^-$

(2) $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\| \Rightarrow |\mu*\lambda| = |\mu|*|\lambda|$.

These facts were first pointed out, in somewhat less general form, by Wendel [11] and Glicksberg [4]. These results suffice for the analysis of the subgroups of the unit ball in $M(G)$.

In order to determine the norm decreasing homomorphisms $\varphi: L^1(F)\to M(G)$ we use an important observation that such a map always extends to a norm decreasing homomorphism $\bar{\varphi}: M(F) \to M(G)$ which is continuous on norm bounded sets as a mapping of $(M(F), (so))$ into $(M(G), (\sigma))$. Here (σ) is the usual weak $*$ topology on $M(G)$, and (so) is the strong operator topology on $M(F)$ gotten by letting $M(F)$ act by left convolution on the ideal $L^1(F) \subset M(F)$.

The author is greatly indebtted to the earlier work of Glicksberg presented in [4], [5]. He is also pleased to acknowledge Professor Glicksberg's helpful commentary in private correspondence. It will be clear to the reader familiar with [4] that the proof of the fundamental relation $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\| \Rightarrow |\mu*\lambda| = |\mu|*|\lambda|$ is a simple adaptation of a Glicksberg theorem dealing with compact groups. The simpler proof given here was suggested by Glicksberg.

1. Preliminaries. Throughout this paper we will find it convenient to write convergence of a net $\{x_j\}$ to a point x in a topological space (X, τ) as $x_j \xrightarrow{(\tau)} x$ or $x \xleftarrow{(\tau)} x_j$, interchangeably. To avoid confusion in discussing homomorphisms we will use the terms homomorphism (epimorphism, monomorphism) for into (onto, $1:1$) homomorphisms; we reserve the term isomorphism for $1:1$ onto homomorphisms.

Most measure theoretic notions are taken from Halmos [3], including definition of Baire and Borel sets. In the following discussion let $\boldsymbol{B} = \boldsymbol{B}(G)$ ($\boldsymbol{B}_0 = \boldsymbol{B}_0(G)$) be the collection of Borel (Baire) sets in G. If a function f is defined on G and if $\boldsymbol{H}$ is a σ-ring of sets in G, we say that f is $\boldsymbol{H}$ measurable on all $\boldsymbol{H}$ sets of G if $\chi_E f$ is $\boldsymbol{H}$ measurable for each set $E \in \boldsymbol{H}$ (χ_E = characteristic function of E). It is clear that $\boldsymbol{B}_0$ measurability on $\boldsymbol{B}_0$ sets implies $\boldsymbol{B}$ measurability on $\boldsymbol{B}$ sets in G.

If $\mu \in M(G)$ define its Baire contraction μ' by restricting its domain of definition to be $\boldsymbol{B}_0(G)$. A regular Borel measure is uniquely determined by its Baire contraction (see [3], 54. D). If $E \in \boldsymbol{B}(G)$ it must be σ-bounded, and hence there is a Baire set $A \supset E$; this applies in particular to $s(\mu)$ where $\mu \in M(G)$. If $E \in \boldsymbol{B}$ and f is $\boldsymbol{B}$ measurable on G, we let $\int_E f d\mu$ denote the integral $\int_G \chi_E f d\mu$.

In applying the Fubini theorem, Borel functions have rather pathological properties when compared to those of Baire functions. These difficulties arise from the fact that the product σ-ring $\boldsymbol{B}_0 \times \boldsymbol{B}_0 = \boldsymbol{B}_0(G \times G)$, while we only know $\boldsymbol{B}_0 \times \boldsymbol{B}_0 \subset \boldsymbol{B} \times \boldsymbol{B} \subset \boldsymbol{B}(G \times G)$ for the corresponding Borel sets. If f is $\boldsymbol{B}_0$ measurable on G and if $A, B \in \boldsymbol{B}_0$, then the function $\chi_{A\times B}(s, t)f(st)$ is $\boldsymbol{B}_0 \times \boldsymbol{B}_0$ measurable on $G \times G$ (hence $\boldsymbol{B} \times \boldsymbol{B}$ measurable) and we can apply Fubini to the convolution-like integral

$$(*) \qquad \int_{G\times G} \chi_{A\times B}(s, t)f(st)d\mu \times \lambda(s, t) .$$

If f is $\boldsymbol{B}$ measurable, the best we can say is that $\chi_{A\times B}(s, t)f(st)$ is $\boldsymbol{B}(G \times G)$ measurable, but this does not give the $\boldsymbol{B} \times \boldsymbol{B}$ measurability required to make $(*)$ well defined. To avoid these difficulties we will rely on the following well known observations.

(R1) If f is bounded and $\boldsymbol{B}_0$ measurable, and if $E, F \in \boldsymbol{B}_0$, then $\chi_{E\times F}(s, t)f(st)$ is $\boldsymbol{B}_0 \times \boldsymbol{B}_0$ measurable on $G \times G$.

(R2) If f is bounded and $\boldsymbol{B}_0$ measurable, and if $\mu, \lambda \in M(G)$, let us choose any sets $E, F \in \boldsymbol{B}_0$ such that $E \supset s(\mu)$, $F \supset s(\lambda)$. Then

$$\int_G f(x)d\mu * \lambda(x) = \int_{G\times G} \chi_{E\times F}(s, t)f(st)d\mu \times \lambda(s, t) .$$

(R3) If $\mu \in M(G)$ there exists a unimodular function f_μ which is $\boldsymbol{B}_0$ measurable on $\boldsymbol{B}_0$ sets in G, such that $\mu = f_\mu |\mu|$. Thus if $E \in \boldsymbol{B}$,

$$\mu(E) = \int_G \chi_E(s)d\mu(s) = \int_G \chi_E(s)f_\mu(s)d|\mu|(s) .$$

Notice that $\Psi(t) = \int \psi(st)d\mu(s)$ is in $C_0(G)$ if $\psi \in C_0(G)$, and all $C_0(G)$ functions are $\boldsymbol{B}_0$ measurable on G, so the definition of convolution is meaningful for $\mu, \lambda \in M(G)$. Convolution is actually independent of the order of iteration of the integrals used to define it, in fact the above remarks show that

$$\int_G \left[\int_G \psi(st)d\mu(s)\right]d\lambda(t) = \int_{G\times G} \chi_{E\times F}(s, t)\psi(st)d\mu \times \lambda(s, t)$$
$$= \int_G \left[\int_G \psi(st)\,d\lambda(t)\right]d\mu(s) ,$$

if $E, F \in \boldsymbol{B}_0$ are such that $E \supset s(\mu)$, $F \supset s(\lambda)$.

If G is a locally compact group and if $Q \subset G$ we let $\mathscr{E}_Q = \{\delta_x : x \in Q\}$ where δ_x is the point mass at x for $x \in G$. Let $co[X]$ be the convex span of a set $X \subset M(G)$ and if γ is a vector space topology on $M(G)$, denote the (γ)-closed convex span of X as $co[X : \gamma]$. We will need the following lemmas about the (σ) and (so) topologies on $M(G)$ (see introduction).

LEMMA 1.1.1. *If* $\mu_j \xrightarrow{(so)} \mu$ *with* $\|\mu_j\| \leqq M < \infty$ *in* $M(F)$, *and if* ψ *is right uniformly continuous and bounded on* F, *then we have* $\langle \mu_j, \psi \rangle \rightarrow \langle \mu, \psi \rangle$.

Proof. Since ψ is uniformly continuous, there exists $f \in L^1(F)$ corresponding to $\varepsilon > 0$ such that $\|f\| = 1$ and $\left| \int_F \psi(st)f(t)dt - \psi(s) \right| < \varepsilon/M$ for all $s \in F$. Then we have $|\langle \mu_j * f, \psi \rangle - \langle \mu_j, \psi \rangle| < \varepsilon$ for $j \in J$ and likewise for μ, so that $|\langle \mu_j, \psi \rangle - \langle \mu, \psi \rangle| < 3\varepsilon$ for $j \geqq j_0$.

LEMMA 1.1.2. *If* Q *is a compact set in locally compact group* G, *and if* S *is the circle group, then* $co[S\mathscr{E}_Q : so] = co[S\mathscr{E}_Q : \sigma] = \{\mu \in M(G) : \|\mu\| \leqq 1,\ s(\mu) \subset Q\}$, *and on these sets the* (σ) *and* (so) *topologies coincide.*

Proof. Clearly $S\mathscr{E}_Q$ is both (σ) and (so) compact, thus from [2, p. 511] we see that $co[S\mathscr{E}_Q : so]$ is compact in the (so) topology, as is $co[S\mathscr{E}_Q : \sigma]$ in the (σ) topology. On the unit ball the identity $j : (M(G), so) \rightarrow (M(G), \sigma)$ is continuous by 1.1.1, so $co[S\mathscr{E}_Q : so]$ is (σ) compact and hence must contain $co[S\mathscr{E}_Q : \sigma]$. Since Q is compact it is known that $co[S\mathscr{E}_Q : \sigma] = \{\mu \in M(G) : \|\mu\| \leqq 1,\ s(\mu) \subset Q\}$. But $\mu \in co[S\mathscr{E}_Q : so] \Rightarrow s(\mu) \subset Q$ and $\|\mu\| \leqq 1$, which gives the reverse containment. It is obvious that the topologies are the same on these compact sets, once they are known to coincide.

LEMMA 1.1.3. *If* G *is a locally compact group,* $co[S\mathscr{E}_G : so]$ *is the unit ball in* $M(G)$ *if* S *is the circle group.*

Proof. Let $\mu \in M(G)$, $\|\mu\| = 1$, and let K_n be compacta such that $K_{n+1} \supset K_n$ and $\bigcup_{n=1}^{\infty} K_n \supset s(\mu)$. Then $\mu_n = \mu \mid K_n \in M(G)$ is such that $\|\mu_n\| \leqq 1$, $\mu_n \in co[S\mathscr{E}_{K_n} : so]$, and $\mu_n \xrightarrow{\text{norm}} \mu$. Thus μ is in the norm closure of $\bigcup_{n=1}^{\infty} co[S\mathscr{E}_{K_n} : so]$, which lies within $co[S\mathscr{E}_G : so]$.

LEMMA 1.1.4. *On the unit ball in* $M(G)$, *convolution is a jointly strong operator continuous operation.*

Proof. Let $\mu_j \xrightarrow{(so)} \mu$ and $\lambda_k \xrightarrow{(so)} \lambda$ in the unit ball. If $f \in L^1(G)$, then because $\|\mu_j\| \leqq 1$ for all $j \in J$ we have $\|\mu_j * \lambda_k * f - \mu * \lambda * f\| \leqq \|\mu_j * (\lambda_k * f) - \mu_j * (\lambda * f)\| + \|\mu_j * (\lambda * f) - \mu * \lambda * f\| \leqq \|\lambda_k * f - \lambda * f\| + \|\mu_j * (\lambda * f) - \mu * (\lambda * f)\| \rightarrow 0$.

LEMMA 1.1.5. *If* G *is a locally compact group, the unit ball in*

$L^1(G) \subset M(G)$ is (so) dense in the unit ball in $M(G)$; in particular, $L^1(G)$ is (so) dense in $M(G)$.

Proof. Clearly there exists a left approximate identity $\{e_j\}$ of norm one in $L^1(G)$. If $\mu \in M(G)$ then $\mu * e_j \in L^1(G)$ and $\| \mu * e_j \| \leqq \| \mu \|$; furthermore, if $f \in L^1(G)$ we have

$$\| \mu * f - (\mu * e_j) * f \| = \| \mu * f - \mu * (e_j * f) \| \longrightarrow 0 .$$

2. Idempotent measures of norm one. If G is a locally compact group and $K \subset G$ is a compact subgroup, define $m_K \in M(G)$ to be the normalized Haar measure on K, so that

$$\langle m_K, \psi \rangle = \int_K \psi(s) dm_K(s)$$

for $\psi \in C_0(G)$. Let $K^\wedge$ be the set of all continuous unimodular multiplicative complex valued functions on K, and if $\beta \in K^\wedge$ let βm_K denote the Haar measure on K weighted with the function β. Then βm_K is an idempotent of norm one in $M(G)$; it is our purpose to show that these are the only idempotent measures of norm one in $M(G)$.

THEOREM 2.1.1. *Let G be a locally compact group. Then if $\mu, \lambda \in M(G)$ are such that $\| \mu * \lambda \| = \| \mu \| \cdot \| \lambda \|$ it follows that $s(\mu * \lambda) = (s(\mu)s(\lambda))^-$, the closure in G of $s(\mu)s(\lambda)$.*

Proof. It is sufficient to consider the case $\| \mu \| = \| \lambda \| = 1$. Clearly $(s(\mu)s(\lambda))^- \supset s(\mu * \lambda)$. If this inclusion is proper we can find a compact Baire set U which is such that $(\text{int } U) \cap (s(\mu)s(\lambda))^- \neq \varnothing$, while at the same time $U \cap s(\mu * \lambda) = \varnothing$. Let $E, F \in \boldsymbol{B}_0(G)$ be such that $E \supset s(\mu)$, $F \supset s(\lambda)$, and define $V = \{(s, s^{-1}u) : u \in U, s \in E\} \subset G \times G$; notice that $V \in \boldsymbol{B}_0 \times \boldsymbol{B}_0$ and is such that $\chi_V(s, t) = \chi_U(st)$ for $s \in E$, $t \in G$, thus

$$\begin{aligned} &\int_{(E \times F) \cap V} \psi(st) d\mu \times \lambda(s, t) \\ &\qquad = \int_{E \times F} \chi_V(s, t) \psi(st) d\mu \times \lambda(s, t) \\ &\qquad = \int_{E \times F} \chi_U(st) \psi(st) d\mu \times \lambda(s, t) \\ &\qquad = \int_G \chi_U(x) \psi(x) d\mu * \lambda(x) . \end{aligned}$$

Given $\varepsilon > 0$, there is a function $\psi \in C_0(G)$ such that $\| \psi \|_\infty = 1$ and $| \langle \mu * \lambda, \psi \rangle | > 1 - \varepsilon$. If V is any Baire set in $G \times G$, then

$$
(*)\qquad
\begin{aligned}
1 - \varepsilon < |\langle \mu * \lambda, \psi \rangle| &= \left| \int_{G\times G} \chi_{E\times F}(s, t)\psi(st) d\mu \times \lambda(s, t) \right| \\
&= \left| \int_{E\times F} \psi(st) d\mu \times \lambda(s, t) \right| \\
&= \left| \int_{(E\times F)\setminus V} \psi(st) d\mu \times \lambda(s, t) \right. \\
&\quad \left. + \int_{(E\times F)\cap V} \psi(st) d\mu \times \lambda(s, t) \right| .
\end{aligned}
$$

For V as above, the right hand side of $(*)$ consists of the single term

$$
\begin{aligned}
\left| \int_{(E\times F)\setminus V} \psi(st) d\mu \times \lambda(s, t) \right| &\leqq \int_{(E\times F)\setminus V} |\psi(st)|\, d|\mu| \times |\lambda|(s, t) \\
&\leqq \int_{E\times F} d|\mu| \times |\lambda| - \int_{(E\times F)\cap V} d|\mu| \times |\lambda| \\
&= 1 - \int_{(E\times F)\cap V} d|\mu| \times |\lambda| .
\end{aligned}
$$

But from our definition of U it is clear that $\int_{(E\times F)\cap V} d|\mu| \times |\lambda| = \delta > 0$, and thus for all $\varepsilon > 0$ we get $1 - \varepsilon \leqq 1 - \delta$, a contradiction.

THEOREM 2.1.2. *If G is a locally compact group and if $\mu, \lambda \in M(G)$ are measures such that $\|\mu * \lambda\| = \|\mu\| \cdot \|\lambda\|$, then $|\mu * \lambda| = |\mu| * |\lambda|$.*

Proof. Again it suffices to consider the case $\|\mu\| = \|\lambda\| = 1$. If $F \supset s(\mu)$ is a Baire set, it is σ-bounded and from the Radon-Nikodym theorem we know that there is a Baire measurable function f_μ on F such that $\mu(E) = \int \chi_E(x) f_\mu(x) d|\mu|(x)$ for all $E \in \boldsymbol{B}(G)$ such that $E \subset F$. clearly $|f_\mu(x)| = 1$ $|\mu|$-a.e. on F; we define a new function

$$
\rho_\mu(x) = \begin{cases} f_\mu(x) \text{ if } x \in F \text{ and } |f_\mu(x)| = 1 \\ 1 \text{ for all other } x \in G . \end{cases}
$$

Then ρ_μ is a unimodular function on G Baire measurable on Baire sets in G.

We will show $|\mu * \lambda| \leqq |\mu| * |\lambda|$. Since these are positive measures, both of norm one (since $\|\mu * \lambda\| = \|\mu\| \cdot \|\lambda\|$ for any positive measures $\mu, \lambda \in M(G)$), our result must follow. If $\psi \in C_0(G)$ and $\psi \geqq 0$, then

$$
\begin{aligned}
\langle |\mu * \lambda|, \psi \rangle &= \int_G \frac{\psi(x)}{\rho_{\mu*\lambda}(x)} d\mu * \lambda(x) \\
&= \int_G \left[\int_G \frac{\psi(st)}{\rho_{\mu*\lambda}(st)} d\mu(s) \right] d\lambda(t) \\
&= \int_G \left[\int_G \psi(st) \frac{\rho_\mu(s)\rho_\lambda(t)}{\rho_{\mu*\lambda}(st)} d|\mu|(s) \right] d|\lambda|(t) .
\end{aligned}
$$

Now the last integral is positive and the integrand is a unimodular multiple of $\psi(st)$, so it must be less than or equal to

$$\int_G \left[\int_G \psi(st)d\,|\,\mu\,|\,(s)\right]d\,|\,\lambda\,|\,(t) = \langle |\,\mu\,|*|\,\lambda\,|, \psi\rangle .$$

The following lemma is given in Loynes [6] and Pym [8], and is also a simple consequence of 2.1.1 and 2.1.2.

PROPOSITION 2.1.3. If G is a locally compact group and if $\mu \in M(G)$ is a positive idempotent of norm one, then there is a compact subgroup $K \subset G$ such that $\mu = m_K$.

We can now prove the main assertion of this section.

THEOREM 2.1.4. *If G is a locally compact group and $\mu \in M(G)$ is an idempotent of norm one, then there is a compact subgroup $K \subset G$ and a function $\rho \in K^\wedge$ such that $\mu = \rho m_K$.*

Proof. Write $\mu = \rho\,|\,\mu\,|$ where ρ is a unimodular function on G, Baire measurable on Baire sets in G. From 2.1.2 we see that $|\,\mu\,|$ is a positive idempotent of norm one, so that $|\,\mu\,| = m_K$ for some compact subgroup $K \subset G$ from 2.1.3.

Now ρ is a bounded Borel measurable function on K since $\boldsymbol{B}(K) = \{E \cap K : E \in \boldsymbol{B}(G)\}$, so the function $\rho \star \rho(t) = \int_K \rho(s^{-1}t)\rho(s)dm_K(s)$ is continuous on K (we are taking $\star$ as the convolution of two functions on K here). If $\psi \in C_0(G)$ and if $F \in \boldsymbol{B}_0(G)$ is such that $F \supset s(\mu) = K$, we have

$$\begin{aligned}\int_K \psi(x)\rho(x)\, dm_K(x) &= \langle \mu, \psi\rangle = \langle \mu*\mu, \psi\rangle \\ &= \int_G \left[\int_G \psi(st)\,\rho(t)\,dm_K(t)\right]\rho(s)dm_K(s) \\ &= \int_K \left[\int_K \psi(t)\rho(s^{-1}t)dm_K(t)\right]\rho(s)dm_K(s) \\ &= \int_{G\times G} \chi_{F\times F}(s, t)\rho(s^{-1}t)\cdot\psi(t)\rho(s)\chi_{K\times K}(s, t)dm_K \times m_K(s, t) .\end{aligned}$$

Clearly $\psi(t)\rho(s)\chi_{K\times K}(s, t)$ is $\boldsymbol{B} \times \boldsymbol{B}$ measurable and a slight modification of $R1$ gives the $\boldsymbol{B}_0 \times \boldsymbol{B}_0$ measurability of $\rho(s^{-1}t)\chi_{F\times F}(s, t)$ on $G \times G$. Thus Fubini applies and we get

$$\begin{aligned}&= \int \chi_{K\times K}(s, t)\psi(t)\rho(s)\rho(s^{-1}t)dm_K \times m_K(s, t) \\ &= \int_K \left[\int_K \rho(s)\rho(s^{-1}t)dm_K(s)\right]\psi(t)dm_K(t) \\ &= \int_K \psi(t)\rho \star \rho(t)dm_K(t) ,\end{aligned}$$

so ρ is $|\mu|$-a.e. identical to a continuous function on K. Taking ρ to be continuous on K, it is clear that $(s, t) \to \rho(st)$ is continuous on $K \times K$. But we can apply the argument of 2.1.2:

$$1 = \|\mu * \mu\| = \int_K \left[\int_K \frac{\rho_\mu(s)\rho_\mu(t)}{\rho_{\mu*\mu}(st)}\, d|\mu|(s)\right] d|\mu|(t)$$
$$= \int_K \left[\int_K \frac{\rho(s)\rho(t)}{\rho(st)}\, dm_K(s)\right] dm_K(t)\,,$$

which means that ρ is a multiplicative function on K.

3. Subgroups of the unit ball in a measure algebra. In this section we consider a locally compact group G and let Γ be a subgroup in the unit ball of $M(G)$. We will denote this unit ball by $\Sigma_{M(G)}$ and refer to the weak $*$ topology on $M(G)$ as the (σ) topology. Given Γ we denote $H_0 = \operatorname{supp}(\Gamma) = \cup\{s(\mu) : \mu \in \Gamma\}$.

LEMMA 3.1.1. *Both H_0 and its closure in G are subgroups of G, and if the unit of Γ is denoted i ($i = \rho m_K$ for some compact subgroup $K \subset G$ and some $\rho \in K^\wedge$), then K is a normal subgroup of both H_0 and its closure. Furthermore, if $\mu \in \Gamma$ then $s(\mu)$ is a single coset of the group K in H_0.*

Proof. If $\mu \in \Gamma$ then $s(\mu)$ is a union of right (or of left) cosets of K because $i * \mu = \mu * i = \mu \Rightarrow (s(\mu)s(i))^- = s(\mu)\cdot K = K\cdot s(\mu) = s(\mu)$ from 2.1.1. If $\mu \in \Gamma$ then $s(\mu^{-1}) = s(\mu)^{-1}$. In fact, if $x \in s(\mu)$, $y \in s(\mu^{-1})$ then $xy = k \in (s(\mu)s(\mu^{-1}))^- = s(\mu * \mu^{-1}) = K$, so that $x^{-1} = yk^{-1} \in s(\mu^{-1})K = s(\mu^{-1})$, and vice versa.

If $g_1 \in s(\mu)$, $g_2 \in s(\mu^{-1})$ we have the relations

$$(*) \qquad K = s(i) = (s(\mu)s(\mu^{-1}))^- \supset s(\mu)s(\mu^{-1}) \supset g_1K^2g_2 \supset g_1Kg_2$$

$$(**) \qquad K = s(i) = (s(\mu)s(\mu^{-1}))^- \supset s(\mu)s(\mu^{-1}) \supset Kg_1g_2K \supset \{g_1g_2\}\,.$$

Thus $s(\mu)$ is a single coset of K; otherwise we could find $g_1, g_2 \in s(\mu)$ with $g_1 \notin Kg_2$, and this would $\Rightarrow g_1g_2^{-1} \notin K$. But $g_2^{-1} \in s(\mu)^{-1} = s(\mu^{-1})$ and $(**) \Rightarrow g_1g_2^{-1} \in K$, a contradiction. We see now that all supports are compact and hence $s(\mu)s(\lambda) = s(\mu * \lambda)$ for all $\mu, \lambda \in \Gamma$.

Clearly H_0 is a subgroup of G since $s(\mu * \lambda) = s(\mu)s(\lambda)$ and $s(\mu)^{-1} = s(\mu^{-1})$; hence its closure is also a subgroup in G. We get normality of K by considering $g \in H_0$ and taking any $\mu \in \Gamma$ such that $g \in s(\mu)$. Then if we take $g_1 = g$, $g_2 = g^{-1} \in s(\mu^{-1})$ in $(*)$, we get $K \supset gKg^{-1}$.

The following theorem gives the structure of a subgroup Γ; it gives only a necessary condition on the structure of a collection of measures Γ in the unit ball of $M(G)$ in order that Γ be a subgroup. Necessary and sufficient conditions will be given later.

PROPOSITION 3.1.2. If Γ is a subgroup of $\Sigma_{M(G)}$ for locally compact group G, let $i = \rho m_K$ be its unit and let $H_0 = \operatorname{supp}(\Gamma)$. Then there exists a subgroup $\Omega \subset S \times G$, with the property

$$H_0 = \{g \in G : (\alpha, g) \in \Omega \text{ for some } |\alpha| = 1\} ,$$

such that $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$.

REMARK. Here S is the circle group and $S \times G$ is the usual product group. In 2.1.4 we have already shown that the unit is $i = \rho m_K$ where K is normal in H_0 and $\rho \in K^\wedge$.

Proof. Let ρ_μ be a unimodular function on G, Baire measurable on Baire sets in G, such that $\mu = \rho_\mu |\mu|$ for $\mu \in \Gamma$. If $g \in s(\mu)$ we have shown that $s(\mu) = gK$ and we know that ρ_μ is determined $|\mu|$-a.e. on $s(\mu)$. But if $s(\mu) = gK$, then $|\mu| = \delta_g * m_K$; in fact, we have $\mu * i = \mu$, which $\Rightarrow |\mu| * |i| = |\mu| * m_K = |\mu|$, and this gives $|\mu| = \delta_g * m_K$ since $\|\mu\| = 1$. We first show that ρ_μ is $|\mu|$-a.e. identical to a continuous function on $s(\mu)$, or equivalently that $\rho_\mu^g(x) = \rho_\mu(gx)$ is m_K-a.e. equal to a continuous function on K. We have

$$\int_G \psi d\mu = \int_K \psi(gt)\rho_\mu^g(t) dm_K(t)$$

for $\psi \in C_0(G)$, while $\mu = i * \mu \Rightarrow$

$$\begin{aligned}\int_G \psi d\mu &= \int_{gK}\left[\int_K \psi(st)\rho(s)\, dm_K(s)\right] d\mu(t)\\ &= \int_K\left[\int_K \psi(sgt)\rho(s)\rho_\mu(gt)\, dm_K(s)\right] dm_K(t) .\end{aligned}$$

For $g \in H_0$, the map $\pi_g : s \longrightarrow gsg^{-1}$ is an automorphism of K such that $m_K(\pi_g E) = m_K(E)$ for Borel sets $E \subset K$; thus if we define $\pi_g^*\beta(s) = \beta(gsg^{-1})$ for $s \in K$, $g \in H_0$, and $\beta \in K^\wedge$, then $\pi_g^*\beta \in K^\wedge$ and the last expression above is

$$\begin{aligned}&= \int_K\left[\int_K \psi(gst)\pi_g^*\rho(s)\rho_\mu(gt)\, dm_K(s)\right] dm_K(t)\\ &= \int_K \psi(gt)\left[\int_K \pi_g^*\rho(s)\rho_\mu^g(s^{-1}t)\, dm_K(s)\right] dm_K(t)\\ &= \int_K \psi(gt)[\pi_g^*\rho \star \rho_\mu^g](t)\, dm_K(t)\end{aligned}$$

where $\star$ gives the convolution of two functions on K rather than functions on G. Since $\pi_g{}^*\rho$ and $\rho_\mu{}^g$ are bounded and $\boldsymbol{B}(K)$ measurable functions on K, their convolution on K is a continuous function, and the above equalities $\Rightarrow \rho_\mu{}^g = \pi_g{}^*\rho \star \rho_\mu{}^g$ m_K-a.e. on K.

Take each function ρ_μ to be continuous on $s(\mu)$ for $\mu \in \Gamma$. Then we have $\rho_\mu(x)\rho_\lambda(y) = \rho_{\mu*\lambda}(xy)$ for all $(x, y) \in s(\mu) \times s(\lambda)$ in $G \times G$, because

$$1 = \int_G d\,|\,\mu*\lambda\,| = \int_G \frac{1}{\rho_{\mu*\lambda}(z)} d\mu*\lambda(z)$$
$$= \int_{s(\mu)} \left[\int_{s(\lambda)} \frac{\rho_\mu(s)\rho_\lambda(t)}{\rho_{\mu*\lambda}(st)} d\,|\,\mu\,|\,(s)\right] d\,|\,\mu\,|\,(t)\ ,$$

and since the last integrand is continuous and unimodular. If $s \in K$ and $g \in s(\mu)$ for $\mu \in \Gamma$, then we have

$$\rho_\mu{}^g(s) = \rho_\mu(gs) = \rho_{\mu*i}(gs) = \rho_\mu(g)\rho_i(s)$$
$$= \rho_\mu(g)\rho(s) = c_g\cdot\rho(s)$$

which means that $\rho_\mu{}^g = c_g\cdot\rho$ on K where $c_g = \rho_\mu(g)$ is a scalar of modulus one. Clearly $\mu = \rho_\mu(g)\cdot(\delta_g*\rho m_K)$ if $g \in s(\mu)$; i.e. if $g \in s(\mu)$, then for some scalar α with $|\,\alpha\,| = 1$ we have $\mu = \alpha\delta_g*\rho m_K$.

Let $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g*\rho m_K \in \Gamma\}$. We have shown that for each $g \in H_0$ we can find a scalar $|\,\alpha\,| = 1$ such that $(\alpha, g) \in \Omega$, so we only have to show that

$$(\alpha_1\delta_{g_1}*\rho m_K)*(\alpha_2\delta_{g_2}*\rho m_K) = \alpha_1\alpha_2\delta_{g_1g_2}*\rho m_K\ .$$

Since the left side is in Γ we get $(\alpha_1\alpha_2, g_1g_2) \in \Omega$, and this will give the group property. But $\alpha_2\delta_{g_2}*\rho m_K \in \Gamma$ and $i = \rho m_K$ is the unit of Γ; hence

$$\alpha_1\delta_{g_1}*\rho m_K*(\alpha_2\delta_{g_2}*\rho m_K) = \alpha_1\delta_{g_1}*\alpha_2\delta_{g_2}*\rho m_K = \alpha_1\alpha_2\delta_{g_1g_2}*\rho m_K$$

as required. Clearly $\Gamma = \{\alpha\delta_g*\rho m_K : (\alpha, g) \in \Omega\}$.

COROLLARY 3.1.3. *If $\mu, \lambda \in \Gamma$ we have $s(\mu) = s(\lambda) \Leftrightarrow \mu = \alpha\lambda$ for some scalar α with $|\,\alpha\,| = 1$.*

Proof. If $s(\mu) = s(\lambda) = gK$ then there are scalars α, β of unit modulus such that $\mu = \alpha\delta_g*\rho m_K$ and $\lambda = \beta\delta_g*\rho m_K$.

COROLLARY 3.1.4. *If $Si = \{\alpha i : |\,\alpha\,| = 1\}$ and if $\Gamma \cap Si = \{i\}$, then for $\mu, \lambda \in \Gamma$ we have $\mu = \lambda$ whenever $s(\mu) = s(\lambda)$.*

PROPOSITION 3.1.5. If G is a locally compact group and Γ is a subgroup of $\Sigma_{M(G)}$ let us write its unit as $i = \rho m_K$, where $\rho \in K^\wedge$, and let $H_0 = \text{supp}(\Gamma)$. Then $K_0 = \{x \in K : \rho(x) = 1\}$ is a compact subgroup of G which is normal in both K and H_0.

Proof. If $\mu \in \Gamma$ and ρ_μ is the unimodular function, Baire measurable on Baire sets in G, such that $\mu = \rho_\mu |\mu|$, then we know that ρ_μ is a translate of ρ to $s(\mu)$, and we also know that $\rho_\mu(x)\rho_\lambda(y) = \rho_{\mu*\lambda}(xy)$ for all $x \in s(\mu)$, $y \in s(\lambda)$, from 3.1.2. Obviously K_0 is normal in K; normality in H_0 is more troublesome.

If $y \in K_0$, $x \in H_0$, then $xyx^{-1} \in K$ and if $x \in s(\mu)$ we get

$$\begin{aligned}\rho(xyx^{-1}) &= \rho_\mu(x)\rho_i(y)\rho_{\tilde{\mu}}^{-1}(x^{-1}) = \rho_\mu(x)\cdot 1 \cdot \rho_{\tilde{\mu}}^{-1}(x^{-1}) \\ &= \rho_{\tilde{\mu}*\mu}^{-1}(xx^{-1}) = \rho(e) = 1\ ,\end{aligned}$$

which $\Rightarrow xyx^{-1} \in K_0$.

PROPOSITION 3.1.6. Let G be a locally compact group, let H_0 be an arbitrary subgroup, let $K \supset K_0$ be a pair of compact subgroups of G which lie within H_0 and are normal therein, and assume that $\rho \in K^\wedge$ is a function such that $K_0 = \text{Ker}\,\rho$. Then we have $\rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0 \Leftrightarrow K$ is central in H_0 mod K_0 (i.e. K/K_0 is a central subgroup of H_0/K_0).

Proof. If K is central in H_0 mod K_0 and $\psi \in C_0(G)$, then

$$\begin{aligned}\langle \rho m_K * \delta_g * \rho m_K, \psi\rangle &= \int_G \Big[\int_G \Big[\int_G \psi(sxt)\rho(s)\rho(t)dm_K(s)\Big]d[\delta_g](x)\Big]dm_K(t) \\ &= \int_G \Big[\int_G \psi(sgt)\rho(st)dm_K(s)\Big]dm_K(t)\ .\end{aligned}$$

But $sg = gs$ mod K_0, so that $sg = gsk$ for some $k \in K_0$, and the last expression becomes

$$\begin{aligned}&= \int_G \Big[\int_G \psi(gskt)\rho(st)dm_K(s)\Big]dm_K(t) \\ &= \int_G \Big[\int_G \psi(gst)\rho(st)dm_K(s)\Big]dm_K(t) \\ &= \int_G \psi(gs)\rho(s)dm_K(s) = \langle \delta_g * \rho m_K, \psi\rangle\ ,\end{aligned}$$

since $(\rho m_K)^2 = \rho m_K$.

If, conversely, $\rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0$, we show that K is central in H_0 mod K_0 as follows. Let

$$\psi_j = \frac{\chi_{gU_j}}{m_K(U_j \cap K)}$$

where $\{U_j : j \in J\}$ is a basis of compact symmetric neighborhoods of the unit in G, and make $\{\psi_j : j \in J\}$ a net of functions in $L^1(G)$ under the obvious partial ordering. Then we have

$$\begin{aligned} 1 = \rho(e) \longleftarrow & \int_K \frac{\chi_{U_j}(s)}{m_K(U_j)} \rho(s) dm_K(s) \\ &= \int_K \frac{\chi_{gU_j}(gs)}{m_K(U_j)} \rho(s) dm_K(s) \\ &= \int_K \psi_j(gs)\rho(s) dm_K(s) \\ &= \int_K \psi_j d[\delta_g * \rho m_K] = \int_K \psi_j d[\rho m_K * \delta_g * \rho m_K] \\ &= \int_K \left[\int_K \psi_j(sgt)\rho(s)\rho(t) dm_K(s) \right] dm_K(t) \\ &= \int_K \left[\int_K \frac{\chi_{gU_j}(gst)}{m_K(U_j)} \rho(gsg^{-1})\rho(t) dm_K(s) \right] dm_K(t) \\ &= \int_K \left[\int_K \frac{\chi_{U_j}(st)}{m_K(U_j)} \rho(gsg^{-1})\rho(t) dm_K(s) \right] dm_K(t) . \end{aligned}$$

But ρ is uniformly continuous on K and hence, given $\varepsilon > 0$ there is an index $j(\varepsilon)$ such that $j > j(\varepsilon)$ in the partial ordering of $J \Rightarrow |\rho(t) - \rho(t')| < \varepsilon$ if $t \in t'U_j$. Hence if $j > j(\varepsilon)$ we get: $\chi_{U_j}(st) \neq 0 \Rightarrow t \in s^{-1}U_j$, which $\Rightarrow |\rho(t) - \rho(s^{-1})| < \varepsilon$. Some trivial computations then show that the last integral is always within ε of the following expression if $j > j(\varepsilon)$.

$$\int_K \rho(gsg^{-1})\rho(s^{-1}) dm_K(s) = \int_K \rho(gsg^{-1})\overline{\rho(s)} dm_K(s) .$$

But $s \longrightarrow \rho(gsg^{-1})$ is a function in $K^\wedge$, and from the known orthogonality of one dimensional representations of K, this integral can be nonzero $\Leftrightarrow \rho(gsg^{-1}) = \rho(s)$ for all $s \in K$. This means that $gs = sg \bmod K_0$ for all $s \in K$, $g \in H_0$.

COROLLARY 3.1.7. *If G is a locally compact group and Γ is a subgroup of $\Sigma_{M(G)}$, let $H_0 = \mathrm{supp}\,(\Gamma)$, and let us write the unit of Γ as $i = \rho m_K$ as in* 2.1.4, *where $K \subset G$ is a compact subgroup and $\rho \in K^\wedge$. Then if $K_0 = \mathrm{Ker}\,\rho$, K must be central in H_0* mod K_0.

Proof. From 3.1.5 we know that K_0 must be normal in H_0. Furthermore, $i * \mu = \mu \Rightarrow \rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0$ (see 3.1.2).

THEOREM 3.1.8. (*Structure Theorem for Subgroups*). *Let G be locally compact group and let Γ be a subgroup of $\Sigma_{M(G)}$ with unit i. Then we have*

(1) $H_0 = \cup \{s(\mu) : \mu \in \Gamma\}$ *is a subgroup of* G.

(2) $i = \rho m_K$ *where* $K \subset G$ *is a compact subgroup and* $\rho \in K^\wedge$.

(3) K *and* $K_0 = \operatorname{Ker} \rho$ *lie within* H_0 *and are normal in* H_0.

(4) K *is central in* $H_0 \bmod K_0$.

(5) $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\}$ *is a subgroup of* $S \times G$ *with* $H_0 = \{g \in G : (\alpha, g) \in \Omega$ *for some* $|\alpha| = 1\}$.

and we have $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$.

Conversely, let H_0 *be a subgroup in* G, *let* $K \subset G$ *be a compact subgroup lying within* H_0, *and let* $\rho \in K^\wedge$ *be chosen such that*

(1) K *and* $K_0 = Ker\ \rho$ *are both normal in* H_0.

(2) K *is central in* H_0 *mod* K_0.

and let Ω *be any subgroup of* $S \times G$ *with* $H_0 = \{g \in G : (\alpha, g) \in \Omega$ *for some* $|\alpha| = 1\}$. *Then* $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ *is a subgroup of* $\Sigma_{M(G)}$ *with* $H_0 = \cup\{s(\mu) : \mu \in \Gamma\}$, *with* $i = \rho m_K$ *as a unit, and with* $\Omega \subset \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} = \Omega \cdot \{(\rho(k), k) : k \in K\}$.

Proof. The first part follows from 3.1.2, 3.1.5, and 3.1.7. Conversely, if K is central in $H_0 \bmod K_0 = \operatorname{Ker} \rho$ in a scheme of this sort we must have $\rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0$. This means that Γ is a group, since the only difficulty in showing this lies in the verification that Γ is closed under convolution. It follows immediately that $H_0 = \cup\{s(\mu) : \mu \in \Gamma\}$ and that $\tau_0 : (\alpha, g) \to \alpha\delta_g * \rho m_K$ is a homomorphism of Ω onto Γ with kernel $\Omega \cap \{(\rho(k), k) : k \in K\}$. Notice that Ω and $\Omega' = \Omega \cdot \{(\rho(k), k) : k \in K\}$ give rise to the same group of measures Γ.

The classical example of a subgroup in $\Sigma_{M(G)}$ is a group of translates of normalized Haar measure $\Gamma = \{\delta_x * m_Q : x \in G_0\}$, where $Q \subset G$ is a compact subgroup, normal in the subgroup G_0. Theorem 3.1.8 can be stated in a form which shows that every subgroup $\Gamma \subset \Sigma_{M(G)}$ corresponds to a subgroup of this type in $\Sigma_{M(S \times G)}$ rather than $\Sigma_{M(G)}$.

Let π_S, π_G be the projection homomorphisms in $S \times G$ and let $\Omega \supset \Omega_0$ be subgroups in $S \times G$ satisfying the conditions

(1) Ω_0 is a compact subgroup of $S \times G$ normal in Ω.

(2) $S \cap \Omega_0 = (1, e)$, so $\pi_G^{-1}(x) \cap \Omega_0$ is a single point if $x \in \pi_G(\Omega_0)$.

If we are given a compact subgroup $K \subset G$ and a function $\rho \in K^\wedge$, we define the mappings

$$\tau_0 : S \times G \longrightarrow M(G)$$
$$\tau^* : C_0(G) \longrightarrow C_0(S \times G)$$
$$\tau^{**} : M(S \times G) \longrightarrow M(G)$$

such that $\tau_0(\alpha, g) = \alpha\delta_g * \rho m_K$, $\tau^*\psi(\alpha, g) = \langle\tau_0(\alpha, g), \psi\rangle$, and $\langle\tau^{**}\mu, \psi\rangle = \langle\mu, \tau^*\psi\rangle$. Clearly $\tau^*\psi \in C_0(S \times G)$ since K is compact, and $\tau^{**}\delta_{(\alpha,g)} =$

$\alpha\delta_g * \rho m_K$ for $(\alpha, g) \in S \times G$. Furthermore, $\tau^{**} : (M(S \times G), (\sigma)) \longrightarrow (M(G), (\sigma))$ is a linear map which is continuous on norm bounded sets since $\mu_j \xrightarrow{(\sigma)} \mu$ in $\Sigma_{M(S\times G)}$, $\psi \in C_0(G) \Longrightarrow$

$$\langle \tau^{**}\mu_j, \psi \rangle = \langle \mu_j, \tau^*\psi \rangle \longrightarrow \langle \mu, \tau^*\psi \rangle = \langle \tau^{**}\mu, \psi \rangle .$$

Also, τ^{**} is a norm decreasing linear map.

Now take $K = \pi_G(\Omega_0)$, where $\Omega \supset \Omega_0$ satisfy (1) and (2) above, and define the function $\rho(k)$ on K such that $(\rho(k), k) \in \Omega_0$ for each $k \in K$. It is clear that $\rho \in K^\wedge$ since Ω_0 is compact. Let us also define $H_0 = \pi_G(\Omega)$, $K_0 = \pi_G(\Omega_0 \cap G)$. Then $K_0 = \operatorname{Ker} \rho$ and it is easily verified that K_0 is normal in both H_0 and K, and that K is central in $H_0 \bmod K_0$, from conditions (1) and (2). Thus $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ is a subgroup in $\Sigma_{M(G)}$ since 3.1.8 applies to the system of objects H_0, K, K_0, ρ.

The mapping $\tau^{**}\delta : (\alpha, g) \longrightarrow \alpha\delta_g * \rho m_K$ is a homomorphism on Ω since

$$\begin{aligned} \tau^{**}\delta_{(\alpha_1\alpha_2, g_1g_2)} &= \alpha_1\alpha_2\delta_{g_1g_2} * \rho m_K = \alpha_1\delta_{g_1} * \rho m_K * \alpha_2\delta_{g_2} * \rho m_K \\ &= \tau^{**}\delta_{(\alpha_1, g_1)} * \tau^{**}\delta_{(\alpha_2, g_2)} . \end{aligned}$$

From normality of Ω_0 in Ω it follows that $\Gamma^\sim = \{\delta_x * m_{\Omega_0} : x \in \Omega\}$ is a subgroup of $\Sigma_{M(S\times G)}$.

LEMMA 3.1.9. *If $\Omega \supset \Omega_0$ satisfy conditions* (1), (2) *above, and if $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ then $\tau_0 : \Omega \longrightarrow \Gamma$ is an epimorphism with kernel Ω_0.*

Proof. We have indicated that τ_0 is an epimorphism. If $\tau_0(\alpha, g) = \rho m_K$, then $g \in K$ and we have $\alpha\delta_g * \rho m_K = \alpha\overline{\rho(g)}\rho m_K = \rho m_K$; hence $\alpha = \rho(g)$ and $(\alpha, g) = (\rho(g), g)$ with $g \in K$, so $(\alpha, g) \in \Omega_0$ by definition of ρ.

THEOREM 3.1.10. *Given subgroups $\Omega \supset \Omega_0$ in $S \times G$ satisfying* (1) *and* (2) *let $K = \pi_G(\Omega_0)$, define $\rho = \pi_S \circ (\pi_G \mid \Omega_0)^{-1}$ on K, and define $\tau^{**} : M(S \times G) \longrightarrow M(G)$ as above. Then $\rho \in K^\wedge$ (so τ_0 and τ^{**} are well defined), $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ and $\Gamma^\sim = \{\delta_x * m_{\Omega_0} : x \in \Omega\}$ are subgroups in $\Sigma_{M(G)}$ and $\Sigma_{M(S\times G)}$ respectively, and τ^{**} is an isomorphism between $\Gamma^\sim$ and Γ. Conversely, if $\Gamma \subset \Sigma_{M(G)}$ is a subgroup with unit $i = \rho m_K$, it arises from a pair of subgroups $\Omega \supset \Omega_0$ in $S \times G$ which satisfy conditions* (1) *and* (2) *by means of the above construction if we take $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\}$ and $\Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = \rho m_K\}$.*

Proof. To establish the first part we will show that $\tau^{**}(\delta_x * m_{\Omega_0}) = \tau^{**}\delta_x = \tau_0(x)$ for any $x \in \Omega$; then from 3.1.9 it is clear that τ^{**} is an isomorphism between $\Gamma^\sim$ and Γ. But Ω_0 is compact, so there exists a net $\{\lambda_j\}$ in $\operatorname{co}[\mathscr{E}_{\Omega_0}]$ with $\lambda_j \xrightarrow{(\sigma)} m_{\Omega_0}$; hence $\delta_x * \lambda_j \xrightarrow{(\sigma)} \delta_x * m_{\Omega_0}$ and

$\tau^{**}(\delta_x * m_{\Omega_0}) \xleftarrow{(\sigma)} \tau^{**}(\delta_x * \lambda_j) \equiv \tau^{**}(\delta_x)$, as required. Conversely, if $\Gamma \subset \Sigma_{M(G)}$ is a subgroup, and if $\Omega \supset \Omega_0$ are formed as indicated, then properties (1) and (2) hold as a consequence of the following lemma, which will be of interest later on. Once this is shown, it is easy to check (see 3.1.8) that $\pi_G(\Omega) = \operatorname{supp}(\Gamma)$, $\pi_G(\Omega_0) = K$, and $\rho = \pi_S \circ (\pi_G \mid \Omega_0)^{-1}$ on K. In the first part we showed that τ^{**} must be an isomorphism of $\{\delta_x * m_{\Omega_0} : x \in \Omega\}$ onto $\{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ and we know that Γ coincides with the latter subgroup of $\Sigma_{M(G)}$ from 3.1.2.

LEMMA 3.1.11. *Let G be a locally compact group and let $\Gamma \subset \Sigma_{M(G)}$ be a subgroup with unit $i = \rho m_K$. Form the pair of subgroups in $S \times G$:*

$$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} \supset$$
$$\Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = \rho m_K\} .$$

Then we have $\Omega_0 = \{(\rho(k), k) : k \in K\}$ and this is a compact subgroup of $S \times G$, normal in Ω. If we define the map $\tau_0 : (\alpha, g) \to \alpha\delta_g * \rho m_K$ for $(\alpha, g) \in S \times G$, then $\tau_0 : S \times G \to (M(G), (\sigma))$ is continuous and $\tau_0 : \Omega \to \Gamma$ is an epimorphism with kernel Ω_0.

Proof. If $\tau_0(\alpha, g) = i$ then $g \in K$ and we have $\alpha\delta_g * \rho m_K = \alpha \cdot \overline{\rho(g)} \cdot \rho m_K$. Hence $\alpha = \rho(g)$ and $(\alpha, g) = (\rho(g), g)$ with $g \in K$. Since $\rho \in K^\wedge$, Ω_0 is a compact subgroup of $S \times G$. Let $H_0 = \operatorname{supp}(\Gamma)$, $K_0 = \operatorname{Ker} \rho$; from 3.1.8, K is central in $H_0 \bmod K_0$, so $\delta_g * \rho m_K = \rho m_K * \delta_g * \rho m_K$ for $g \in H_0$ (see 3.1.6). Thus τ_0 is a homomorphism on Ω (that it is onto is clear from 3.1.2) since

$$\begin{aligned}\tau_0(\alpha_1\alpha_2, g_1g_2) &= \alpha_1\alpha_2\delta_{g_1} * \delta_{g_2} * \rho m_K \\ &= \alpha_1\alpha_2\delta_{g_1} * \rho m_K * \delta_{g_2} * \rho m_K = \tau_0(\alpha_1, g_1) * \tau_0(\alpha_2, g_2)\end{aligned}$$

if $g_1, g_2 \in H_0$. Obviously $\Omega_0 = \operatorname{Ker} \tau_0 \mid \Omega$, so Ω_0 is normal in Ω. The continuity of τ_0 is clear.

4. Norm decreasing homomorphisms on locally compact groups.

Let G be a locally compact group and consider on $M(G)$ the (σ) and (so) topologies defined in § 1. Every norm decreasing homomorphism on $L^1(G)$ extends naturally to a norm decreasing homomorphism on $M(G)$. To appreciate the usefulness of this extension theorem it is helpful to recall 1.1.3.

THEOREM 4.1.1. *Let F, G be locally compact groups and let $\varphi : L^1(F) \to M(G)$ be any norm decreasing homomorphism. Then φ extends uniquely to a norm decreasing homomorphism $\bar{\varphi} : M(F) \to M(G)$ which is continuous on norm bounded sets as a map of $(M(F), (so))$*

into $(M(G), (\sigma))$. If $\{e_j : j \in J\}$ is a left approximate identity of norm one in $L^1(F)$ then the extension is given explicitly by the formula

$$\bar{\varphi}(\mu) = \lim \{\varphi(e_j * \mu) : j \in J\} \qquad \text{all } \mu \in M(F)\ ,$$

where the limit is in the (σ) topology. A similar result holds for right approximate identities.

Proof. Let $B = \varphi(L^1(F))$ and let A be the (σ) closure of B in $M(G)$, so that A and B are subalgebras of $M(G)$.

LEMMA 4.1.2. *Let $\{e_j{}^l : j \in J\}$ be a left approximate identity of norm one and let $\{e_k{}^r : k \in K\}$ be a right approximate identity of norm one in $L^1(F)$. Then in $M(G)$ the (σ) limit points of the nets $\{\varphi(e_j{}^l)\}$ and $\{\varphi(e_k{}^r)\}$ all coincide in a single idempotent $\iota \in M(G)$, so we must have convergence*

$$\varphi(e_j{}^l) \xrightarrow{(\sigma)} \iota$$

$$\varphi(e_k{}^r) \xrightarrow{(\sigma)} \iota$$

in the (σ) topology. If $\varphi \neq 0$ then $\iota \neq 0$ and we have $a = \iota * a = a * \iota$ for all $a \in A$.

Proof. Since $\|\varphi(e_j{}^l)\| \leqq 1$ there is at least one (σ) limit point λ for this net, and for an appropriate subnet we get $\varphi(e_{j(p)}{}^l) \xrightarrow{(\sigma)} \lambda$. Thus if $f \in L^1(F)$ we have

$$\lambda * \varphi f \xleftarrow{(\sigma)} \varphi(e_{j(p)}{}^l) * \varphi f = \varphi(e_{j(p)}{}^l * f) \xrightarrow{\text{norm}} \varphi(f)\ .$$

Hence if $\{\mu_i : i \in I\}$ is a net in B with $\mu_i = \varphi(f_i)$ and $\mu_i \xrightarrow{(\sigma)} \mu$ in A, we have

$$\lambda * \mu \xleftarrow{(\sigma)} \lambda * \mu_i = \lambda * \varphi f_i = \varphi f_i = \mu_i \xrightarrow{(\sigma)} \mu$$

so that $\lambda * a = a$ for all $a \in A$. In particular we have $\lambda \in A$ so $\lambda * \lambda = \lambda$. Similarly if ν is a (σ) limit point of $\{\varphi(e_k{}^r)\}$ then $\nu * \nu = \nu$ and $a * \nu = a$ for all $a \in A$.

If λ, ν are (σ) limit points as above, we have λ, ν in A, which $\Rightarrow$ $\lambda = \lambda * \nu = \nu$; hence $\lambda = \nu$ and all limit points (left or right) coincide in a single idempotent ι such that $\iota * a = a * \iota = a$ if $a \in A$. If $\varphi \neq 0$, clearly $\iota \neq 0$.

The main step in our proof is to show that, if $\{f_j : j \in J\}$ is a norm bounded net in $L^1(F)$ which is (so) convergent to some $\mu \in M(F)$, then the net $\{\varphi(f_j)\}$ converges to a limit λ_μ in the (σ) topology, and this limit depends only on μ, rather than on the particular choice of the net $\{f_j\}$. This can be done for any $\mu \in M(F)$, in view of 1.1.5.

First consider any $f \in L^1(F)$ and notice that $\|f_j * f - \mu * f\| \to 0$, which $\Rightarrow \|\varphi f_j * \varphi f - \varphi(\mu * f)\| \to 0$. Let λ be any (σ) limit point of the norm bounded net $\{\varphi(f_j)\}$; there exists a convergent subnet $\varphi(f_{j(i)}) \xrightarrow{(\sigma)} \lambda$. Then

$$\varphi(\mu * f) \xleftarrow{\text{norm}} \varphi(f_{j(i)} * f) = \varphi f_{j(i)} * \varphi f \xrightarrow{(\sigma)} \lambda * \varphi f$$

for all $f \in L^1(F)$, which means that $\varphi(\mu * f) = \lambda * \varphi f$ for all $f \in L^1(F)$. Clearly $\lambda \in A$ since each $\varphi(f_j) \in B$, and this means that

$$\lambda = \lambda * \iota \xleftarrow{(\sigma)} \lambda * \varphi(e_k{}^r) = \varphi(\mu * e_k{}^r) .$$

Thus we get

$$\lambda = \lim \{\varphi(\mu * e_k{}^r) : k \in K\}$$

in the (σ) topology, and this formula doesn't depend on anything but the choice of $\mu \in M(F)$. Hence if $f_j \xrightarrow{(so)} \mu$, then λ is the only possible limit point of $\{\varphi(f_j)\}$, so if we take $\lambda_\mu = \lim \{\varphi(\mu * e_k{}^r)\}$, we always have $\varphi f_j \xrightarrow{(\sigma)} \lambda_\mu$.

Notice that if $f \in L^1(F)$ we have $\|f * e_k{}^r - f\| \to 0$, which gives

$$\lambda_f \xleftarrow{(\sigma)} \varphi(f * e_k{}^r) \xrightarrow{\text{norm}} \varphi(f)$$

so that $\varphi f = \lambda_f$ for all $f \in L^1(F)$. Now define $\bar{\varphi}(\mu) = \lambda_\mu$ for $\mu \in M(F)$, and verify the properties required. Clearly $\bar{\varphi}(f) = \varphi(f)$ for all $f \in L^1(f)$, so $\bar{\varphi}$ extends φ.

If (γ) is a locally convex topology on $M(F)$ we define the bounded (γ) topology $(b\gamma)$ by taking as a basis of neighborhoods about zero all sets $X \cap Y$ where X is a (γ) neighborhood of zero, and Y is a fixed norm neighborhood of zero. From the discussion above we know that if $\mu \in M(F)$ and if W is a (σ) neighborhood of zero in $M(G)$, then there is an open (bso) neighborhood V of zero in $M(F)$ such that $\bar{\varphi}((\mu + V) \cap L^1(F)) \subset \bar{\varphi}\mu + W$. Now let $W' \subset W$ be a (σ) neighborhood of zero such that $W' = -W'$ and $W' + W' \subset W$, and let U be an open (bso) neighborhood of zero in $M(F)$ such that

$$\bar{\varphi}((\mu + U) \cap L^1(F)) \subset \bar{\varphi}\mu + W' .$$

If $\lambda \in \mu + U$ we can find a (bso) neighborhood U_λ of zero such that

(1) $\bar{\varphi}((\lambda + U_\lambda) \cap L^1(F)) \subset \bar{\varphi}\lambda + W'$

(2) $\lambda + U_\lambda \subset \mu + U$.

Then we have $\bar{\varphi}((\lambda + U_\lambda) \cap L^1(F)) \subset \bar{\varphi}((\mu + U) \cap L^1(F)) \subset \bar{\varphi}\mu + W'$ and $\bar{\varphi}((\lambda + U_\lambda) \cap L^1(F)) \subset \bar{\varphi}\lambda + W'$, which together imply that

$$(\bar{\varphi}\lambda + W') \cap (\bar{\varphi}\mu + W') \neq \emptyset$$

(see 1.1.5), which means that $\bar{\varphi}\lambda \in \bar{\varphi}\mu + W$. Hence $\bar{\varphi}(\mu + U) \subset \bar{\varphi}\mu + W$ as required for continuity.

Clearly $\|\bar{\varphi}\mu\| \leqq \sup\{\|\varphi(\mu * e_k{}^r)\|\} \leqq \|\mu\|$ for $\mu \in M(F)$, and if $\mu, \lambda \in M(F)$ we have

$$\bar{\varphi}(\mu * \lambda) \xleftarrow{(\sigma)} \bar{\varphi}(\mu * (\lambda * e_k{}^r)) = \bar{\varphi}\mu * \varphi(\lambda * e_k{}^r) \xrightarrow{(\sigma)} \bar{\varphi}\mu * \bar{\varphi}\lambda$$

since $\lambda * e_k{}^r \xrightarrow{(so)} \lambda$. Hence $\bar{\varphi}$ is a norm decreasing homomorphism.

EXAMPLE. In 4.1.1 we cannot replace the (so) topology with the (σ) topology in $M(F)$. Indeed, if $Z =$ integers, $S =$ circle, and if β is some irrational number, then $\varphi(\sum_{n=1}^\infty \alpha_n \delta_{x_n}) = \sum_{n=1}^\infty \alpha_n \delta_{(e^{ix_n\beta})}$ gives a norm decreasing homomorphism $\varphi : L^1(Z) \to M(S)$. This map coincides with its extension $\bar{\varphi}$. The sequence $\{\mu_n = \delta_{(n)} : n = 1, 2 \cdots\}$ is (σ) convergent to zero, while $\varphi(\mu_n)$ is not (σ) convergent in $M(S)$.

REMARK. The proof of 4.1.1 is also valid for any *bounded* homomorphism $\varphi : L^1(F) \to M(G)$, which means that the structure of a bounded homomorphism is determined once we know the structure of the bounded group of measures $\bar{\varphi}(\mathscr{E}_F)$; however, the structure of the bounded subgroups in $M(G)$ is generally not known unless G is abelian or the subgroup lies within $\Sigma_{M(G)}$.

4.2. The structure of norm decreasing homomorphisms. If $\bar{\varphi}$ extends the norm decreasing homomorphism $\varphi : L^1(F) \to M(G)$, as in 4.1.1, then $\Gamma = \bar{\varphi}(\mathscr{E}_F)$ is a subgroup of the unit ball in $M(G)$. Using the continuity properties of $\bar{\varphi}$ demonstrated in 4.1.1 and our knowledge of the structure of Γ we can determine φ completely (see 1.1.3).

Let $i = \rho m_K$ be the unit of Γ and denote $H_0 = \operatorname{supp}(\Gamma)$, $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_k \in \Gamma\} \supset \Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = \rho m_K\}$, and $K_0 = \operatorname{Ker} \rho$. Let $\pi : S \times G \to (S \times G/\Omega_0)_r$ be the canonical map onto the space of right cosets of Ω_0, so π is a homomorphism when restricted to $S \times H_0$, and let $\tau_0 : (\alpha, g) \to \alpha\delta_g * \rho m_K$ for $(\alpha, g) \in S \times G$. Then define $\theta : F \to (\Omega/\Omega_0) \subset (S \times G/\Omega_0)_r$ to be $\theta = \pi \circ \tau_0{}^{-1} \circ \bar{\varphi} \circ \delta$, so that $\theta(x) = \pi(\alpha, g)$ if and only if $\bar{\varphi}(\delta_x) = \alpha\delta_g * \rho m_K$ in $M(G)$. The mappings involved are shown in the following (commutative) diagram.

$$\begin{array}{ccc} S \times G \supset \Omega & \xleftarrow{\tau_0{}^{-1}} & \Gamma \\ \pi \big\downarrow & & \big\uparrow \bar{\varphi}\circ\delta \\ (S \times G/\Omega_0)_r \supset \Omega/\Omega_0 & \xleftarrow{\theta} & F \end{array}$$

Figure 1

PROPOSITION 4.2.1. The map $\theta: F \to \Omega/\Omega_0$ is an epimorphism and is continuous as a mapping $\theta: F \to (S \times G/\Omega_0)_r$.

Proof. Let $x_1, x_2 \in F$ and let $(\alpha_1, g_1), (\alpha_2, g_2) \in \Omega$ be chosen such that $\pi(\alpha_i, g_i) = \theta(x_i)$, and let $(\alpha, g) \in \Omega$ be chosen such that $\pi(\alpha, g) = \theta(x_1 x_2)$. Thus $\bar{\varphi}(\delta_{x_i}) = \alpha_i \delta_{g_i} * \rho m_K$. We have $\theta(x_1 x_2) = \theta x_1 \cdot \theta x_2$ if $\pi(\alpha, g) = \pi(\alpha_1, g_1)\pi(\alpha_2, g_2)$, which happens if $(\alpha_1\alpha_2, g_1 g_2) \in (\alpha, g)\Omega_0$. This follows since

$$\begin{aligned}\bar{\alpha}\alpha_1\alpha_2\delta_{(g^{-1}g_1g_2)} * \rho m_K &= (\bar{\alpha}\delta_{g^{-1}} * \rho m_K) * (\alpha_1\delta_{g_1} * \rho m_K) * (\alpha_2\delta_{g_2} * \rho m_K) \\ &= \bar{\varphi}(\delta_{x_1x_2})^{-1} * \bar{\varphi}(\delta_{x_1}) * \bar{\varphi}(\delta_{x_2}) = \rho m_K .\end{aligned}$$

We want to show $\theta: F \to (S \times G/\Omega_0)_r$ is continuous. Because $\bar{\varphi} \circ \delta$ and π are continuous it suffices to show that $\tau_0: S \times G \to N = \tau_0(S \times G)$ is an open map when N has the restricted (σ) topology. Let $(\alpha_0, g_0) \in S \times G$ and let $U \times V$ be a product of open sets in S, G with $\alpha_0 \in U$, $g_0 \in V$. It suffices to show that $\tau_0(U \times V)$ is always a (σ) neighborhood of $\tau_0(\alpha_0, g_0)$ in N. If this set fails to be a neighborhood there is a net $\{(\alpha_j, g_j)\}$ such that $\mu_j = \tau_0(\alpha_j, g_j) = \alpha_j\delta_{g_j} * \rho m_K \xrightarrow{(\sigma)} \alpha_0\delta_{g_0} * \rho m_K$, while $\mu_j \notin \tau_0(U \times V)$. We can assume $g_j \in g_0 W K$ for some compact neighborhood W of g_0, and, by taking subnets, we get $g_j \to g_1 \in g_0 K$, $\alpha_j \to \alpha_1 \in S$.

If we let $g_j^* = g_j(g_1^{-1}g_0)$, then $g_j^* \to g_0$. Let $\alpha_j^* = \alpha_j \rho(g_1^{-1}g_0)$; this makes sense because $g_0 g_1^{-1} \in K$. Then we have

$$\begin{aligned}\tau_0(\alpha_j^*, g_j^*) &= \alpha_j^*\delta_{(g_j^*)} * \rho m_K = \alpha_j\rho(g_1^{-1}g_0)\delta_{g_j} * \delta_{(g_1^{-1}g_0)} * \rho m_K \\ &= \alpha_j\delta_{g_j} * \rho m_K = \tau_0(\alpha_j, g_j) \xrightarrow{(\sigma)} \alpha_0\delta_{g_0} * \rho m_K .\end{aligned}$$

Since $g_j^* \to g_0$ we must have $\alpha_j^* \to \alpha_0$ and α_j^* is eventually in U; hence $\tau_0(\alpha_j^*, g_j^*) = \tau_0(\alpha_j, g_j)$ is eventually in $\tau_0(U \times V)$, a contradiction.

Let $\tau^*\psi(\alpha, g) = \langle \alpha\delta_g * \rho m_K, \psi \rangle$ for $\psi \in C_0(G)$. Then $\tau^*\psi \in C_0(S \times G)$ since K is compact, and in fact $\tau^*\psi$ is constant on right cosets of Ω_0 in $S \times G$ since $\Omega_0 = \{(\rho(k), k) : k \in K\}$. If $\Psi \in C_0(S \times G)$ and is constant on right cosets of Ω_0, let us identify it with a function

$$\pi^*\Psi \in C_0((S \times G/\Omega_0)_r) .$$

This function vanishes at infinity since Ω_0 is compact. We can give an integral representation for norm decreasing homomorphisms as follows.

THEOREM 4.2.2. *Let F, G be locally compact groups and let $\varphi: L^1(F) \to M(G)$ be a nonzero norm decreasing homomorphism with extension $\bar{\varphi}$ to $M(F)$, as in* 4.1.1. *Denote*

(1) $\Gamma = \bar{\varphi}(\mathscr{E}_F)$

(2) $i = \rho m_K$ *the unit of* Γ

(3) $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\}$

(4) $\Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = i\}$.
Define the maps

$$\tau_0 : S \times G \longrightarrow \Gamma$$
$$\tau^* : C_0(G) \longrightarrow C_0(S \times G)$$
$$\pi^*\tau^* : C_0(G) \longrightarrow C_0((S \times G/\Omega_0)_r)$$
$$\theta : F \longrightarrow \Omega/\Omega_0$$

as indicated above. Then we have the representation

$$(*) \qquad \langle \bar{\varphi}\mu, \psi \rangle = \langle \mu, (\pi^*\tau^*\psi) \circ \theta \rangle$$

for all $\mu \in M(F)$ *and* $\psi \in C_0(G)$.

REMARK. Since $\theta : F \rightarrow \Omega/\Omega_0$ is a continuous homomorphism and $\pi^*\tau^*\psi \in C_0((S \times G/\Omega_0)_r)$, it follows that $(\pi^*\tau^*\psi) \circ \theta$ is a uniformly continuous and bounded function on F. Thus the right hand side of (*) is uniquely determined. We will want to make use of 1.1.1 in the following discussion.

Proof. We have $\langle \bar{\varphi}(\delta_x), \psi \rangle = \langle \delta_{\theta x}, \pi^*\tau^*\psi \rangle = \langle \delta_x, (\pi^*\tau^*\psi) \circ \theta \rangle$ if $x \in F$. If $\mu \in M(F)$ is of norm one then there exists a net $\{\sigma_j : j \in J\}$ in the convex span of the extreme points of $\Sigma_{M(F)}$ such that $\|\sigma_j\| = 1$ and $\sigma_j \xrightarrow{(so)} \mu$ (see 1.1.3). If we write $\sigma_j = \sum \lambda(j, x)\delta_x$ (finite sum), we can apply 1.1.1. to get

$$\begin{aligned} \langle \bar{\varphi}\mu, \psi \rangle \longleftarrow \langle \bar{\varphi}(\sigma_j), \psi \rangle &= \sum \lambda(j, x) \langle \bar{\varphi}(\delta_x), \psi \rangle \\ &= \sum \lambda(j, x) \langle \delta_x, (\pi^*\tau^*\psi) \circ \theta \rangle \\ &= \langle \sigma_j, (\pi^*\tau^*\psi) \circ \theta \rangle \longrightarrow \langle \mu, (\pi^*\tau^*\psi) \circ \theta \rangle . \end{aligned}$$

Thus $\langle \bar{\varphi}\mu, \psi \rangle = \langle \mu, (\pi^*\tau^*\psi) \circ \theta \rangle$.

As a converse we have the following theorem which classifies all norm decreasing homomorphisms.

THEOREM 4.2.3. *Let* F, G *be locally compact groups and let* Γ *be a subgroup of* $\Sigma_{M(G)}$ *with unit* $i = \rho m_K$ *and with*

$$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} ,$$

$\Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = i\}$. *Then if* $\theta : F \rightarrow \Omega/\Omega_0$ *is any continuous epimorphism* (Ω/Ω_0 *is given the restricted topology from* $(S \times G/\Omega_0)_r$), *the relation*

$$(*) \qquad \langle \bar{\varphi}\mu, \psi \rangle = \langle \mu, (\pi^*\tau^*\psi) \circ \theta \rangle$$

for $\mu \in M(F)$, $\psi \in C_0(G)$ *defines a norm decreasing homomorphism*

$\bar{\varphi}: (M(F), (so)) \to (M(G), (\sigma))$ *which is continuous on norm bounded sets, and we have* $\bar{\varphi}(\mathscr{E}_F) = \Gamma$.

REMARK. If $\bar{\varphi}$ has the above continuity properties it is clear that $\bar{\varphi}$ is obtained, as in 4.1.1, by extending the norm decreasing homomorphism $\bar{\varphi} \mid L^1(F)$.

Proof. From 3.1.11 we see that Ω_0 is a compact subgroup of $S \times G$ which is normal in Ω, so Ω/Ω_0 is well defined. We have also noted that $\tau^*\psi(\alpha, g) = \langle \alpha\rho_g * \rho m_K, \psi \rangle$ is in $C_0(S \times G)$ and

$$\pi^*\tau^*\psi \in C_0((S \times G/\Omega_0)_r) ,$$

so $(\pi^*\tau^*\psi)\circ\theta$ is bounded and uniformly continuous on F. Hence $(*)$ is always well defined.

Clearly $\bar{\varphi}: (M(F), (so)) \to (M(G), (\sigma))$ is a norm decreasing linear map, and continuity on norm bounded sets follows from 1.1.1. Now $\bar{\varphi}(\delta_x) = \alpha\delta_g * \rho m_K$ for all $(\alpha, g) \in \pi^{-1}\theta(x)$, so $\bar{\varphi}\circ\delta = (\tau_0 \circ \pi^{-1})\circ\theta$; thus, $\bar{\varphi}(\mathscr{E}_F) = \Gamma$ and $\bar{\varphi}$ is a continuous homomorphism of $(\mathscr{E}_F, (so))$ into $(M(G), (\sigma))$. Convolution is a jointly (so) continuous operation in $\Sigma_{M(F)}$, so $\bar{\varphi}$ is a norm decreasing homomorphism of $M(F)$ in view of the density theorems 1.1.3, 1.1.4.

A norm decreasing homomorphism $\varphi: L^1(F) \to M(G)$ is order preserving if $\mu \geqq 0 \Rightarrow \varphi(\mu) \geqq 0$. From the continuity properties given in 4.1.1 and the structure theorem 3.1.8 it follows that φ is order preserving $\Leftrightarrow \bar{\varphi}(\mathscr{E}_F)$ is a group of translates of Haar measure $\{\delta_x * m_Q : x \in G_0\}$, where $Q \subset G$ is a compact subgroup, normal in the subgroup G_0. Every norm decreasing homomorphism φ is closely related to an order preserving norm decreasing homomorphism of $L^1(F)$ into $M(S \times G)$.

If $\Omega \supset \Omega_0$ are two subgroups in $S \times G$ satisfying conditions (1) and (2) in the discussion following 3.1.8, define the maps $\tau_0, \cdots, \tau^{**}$ as indicated there.

THEOREM 4.2.4. *If* $\bar{\varphi}: (M(F), (so)) \to (M(G), (\sigma))$ *is a norm decreasing homomorphism, continuous on norm bounded sets, and if* $\Gamma = \bar{\varphi}(\mathscr{E}_F)$ *has unit* $i = \rho m_K$, *then the subgroups*

$$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} \supset \Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = \rho m_K\}$$

satisfy conditions (1) *and* (2) *of* 3.1.9 *and we can factor* $\bar{\varphi} = \tau^{**}\circ\Phi$ *where* Φ *is some order preserving norm decreasing homomorphism of* $M(F)$ *into* $M(S \times G)$. *Here* Φ *maps* $\mathscr{E}_F$ *to the group of measures* $\{\delta_x * m_{\Omega_0} : x \in \Omega\}$ *and* τ^{**} *is a homomorphism on the range of* Φ.

Conversely, if $\Phi: M(F) \rightarrow M(S \times G)$ *is any order preserving norm decreasing homomorphism, let* $\Omega = \operatorname{supp}(\Phi(\mathscr{E}_F)) \supset \Omega_0 = s(\Phi(\delta_e))$. *If* Ω, Ω_0 *satisfy conditions* (1), (2) *in* 3.1.9, *then* $\varphi = \tau^{**} \circ \Phi: (M(F), (so)) \rightarrow (M(G), (\sigma))$ *is a norm decreasing homomorphism, continuous on norm bounded sets.*

Proof. Let $\Omega \supset \Omega_0$ satisfy (1) and (2) of 3.1.9 and define M_Ω to be the subspace of measures in $M(S \times G)$ whose intersection with $\Sigma_{M(S\times G)}$ is $co[S\mathscr{E}_\Omega : \sigma]$. We assert that

(*) M_Ω is a subalgebra in $M(S \times G)$, $\tau^{**}(\delta_x * m_{\Omega_0}) = \tau^{**}(\delta_x) = \tau_0(x)$ for $x \in \Omega$, and τ^{**} is a norm decreasing homomorphism on M_Ω.

Clearly $M_\Omega = \{\mu : s(\mu) \subset \bar{\Omega}\}$, and is a subalgebra. We have already shown (in discussing 3.1.10) that $m_{\Omega_0} \in M_\Omega$ and $\tau^{**}(\delta_x * m_{\Omega_0}) = \tau^{**}(\delta_x) = \tau_0(x)$ for all $x \in \Omega$. Thus τ^{**} is multiplicative on $S\mathscr{E}_\Omega$, and since convolution is separately (σ) continuous we can show that $\tau^{**}(\delta_x * \mu) = \tau^{**}(\delta_x) * \tau^{**}(\mu)$ for $\mu \in M_\Omega$, $x \in \bar{\Omega}$. Then if $\lambda_j \xrightarrow{(\sigma)} \lambda$ for $\|\lambda\| = 1$, where $\lambda_j \in co[S\mathscr{E}_\Omega]$ we use the same idea once more to get

$$\tau^{**}(\lambda * \mu) \xleftarrow{(\sigma)} \tau^{**}(\lambda_j * \mu) = \tau^{**}(\lambda_j) * \tau^{**}(\mu) \xrightarrow{(\sigma)} \tau^{**}(\lambda) * \tau^{**}(\mu),$$

so $\tau^{**} \mid M_\Omega$ is a homomorphism.

Now Φ maps $S\mathscr{E}_F$ into M_Ω, and if $\mu \in M(F)$, $\|\mu\| = 1$, there exists a net $\{\mu_j\} \subset co[S\mathscr{E}_F]$ such that $\mu_j \xrightarrow{(so)} \mu$. This means $\Phi\mu_j \xrightarrow{(\sigma)} \Phi\mu$ while $\Phi\mu_j \in \Sigma_{M(S\times G)} \cap M_\Omega$, so $\Phi\mu \in M_\Omega$ and Φ maps $M(F)$ into M_Ω. Thus $\tau^{**}\Phi$ is well defined and is a norm decreasing homomorphism with the desired continuity properties (τ^{**} is (σ) continuous on $M(S \times G)$).

Conversely, let $\bar{\varphi}$ be given; then Ω, Ω_0 defined above satisfy (1) and (2), as shown in 3.1.11. The homomorphism $\theta: F \rightarrow \Omega/\Omega_0$, associated with $\bar{\varphi}$ as in 4.2.2, is continuous, so $\theta^*\psi = \psi \circ \theta$ is uniformly continuous and bounded (UCB) on F and we can consider the dual maps.

$$\theta^*: C_0((S \times G/\Omega_0)_r) \longrightarrow UCB(F)$$
$$\theta^{**}: M(F) \longrightarrow M((S \times G/\Omega_0)_r).$$

For $\psi \in C_0(S \times G)$ define $\pi^*\psi \in C_0((S \times G/\Omega_0)_r)$ by lifting the function $\pi^*\psi(x) = \int \psi(xt)dm_{\Omega_0}(t)$ (constant on right cosets of Ω_0) over to the coset space $(S \times G/\Omega_0)_r$. The desired map Φ is given by

$$\langle \Phi\mu, \psi \rangle = \langle \theta^{**}\mu, \pi^*\psi \rangle = \langle \mu, (\pi^*\psi) \circ \theta \rangle$$

for $\psi \in C_0(S \times G)$. It is easy to verify that $\Phi(\delta_x) = \delta_{(\alpha, g)} * m_{\Omega_0}$ for all $(\alpha, g) \in \pi^{-1}\theta(x)$; therefore, as indicated in 3.1.10, we have $\langle \tau^{**}\Phi(\delta_x), \psi \rangle = \langle \tau^{**}(\delta_{(\alpha, g)} * m_{\Omega_0}), \psi \rangle = \langle \tau^{**}(\delta_{(\alpha, g)}), \psi \rangle = \langle \alpha\delta_g * \rho m_K, \psi \rangle = \langle \bar{\varphi}(\delta_x)\psi \rangle$ for all $x \in F$. Thus $\bar{\varphi} = \tau^{**}\Phi$ on $S\mathscr{E}_F$. But from 1.1.1 we see that Φ

defined above is continuous on norm bounded sets mapping from the (so) to the (σ) topology; clearly, then $\tau^{**}\Phi : (M(F), (so)) \longrightarrow (M(G), (\sigma))$ is continuous on norm bounded sets. Now $\bar{\varphi}$ and $\tau^{**}\Phi$ both enjoy this continuity property and coincide on $S\mathscr{E}_F$; from 1.1.3 it follows that they coincide on all of $M(F)$, and this is the desired factorization of $\bar{\varphi}$.

5. Examples and applications. In 5.1 we analyze the special structure of norm decreasing monomorphisms $\varphi : L^1(F) \longrightarrow M(G)$ between locally compact groups F and G; then in § 5.2 we give the structure of all norm decreasing homomorphisms φ which map $L^1(F)$ onto $L^1(G)$. Maps in the latter class have very simple structure.

5.1. Norm decreasing monomorphisms. Let us denote $\mathscr{F} = S\mathscr{E}_F = \{\alpha\delta_x : |\alpha| = 1,\ x \in F\}$ and $\mathscr{F}_0 = \mathscr{E}_F$ throughout this discussion.

LEMMA 5.1.1. *If $\varphi : L^1(F) \longrightarrow M(G)$ is a norm decreasing monomorphism, and if $\bar{\varphi}$ is its extension to $M(F)$ as in* 4.1.1, *then φ is a monomorphism of $M(F)$ into $M(G)$. Furthermore $\bar{\varphi}(\mathscr{F}_0) \cap Si = \{i\}$, where $i = \bar{\varphi}(\delta_e)$, and $\mu = \lambda$ in $\bar{\varphi}(\mathscr{F}_0)$ whenever $s(\mu) = s(\lambda)$.*

Proof. If $\mu, \lambda \in M(F)$ have $\bar{\varphi}\mu = \bar{\varphi}\lambda = \xi$ and $\mu \neq \lambda$, then there is some $f \in L^1(F)$ such that $\mu * f \neq \lambda * f$ while $\varphi(\mu * f) = \varphi(\lambda * f) = \xi * \varphi f$, a contradiction. Hence $\bar{\varphi}(\mathscr{F}_0) \cap Si = \{i\}$ and the last property follows from 3.1.4.

We propose to study the structure of all norm decreasing homomorphisms φ whose extensions $\bar{\varphi}$ have the special property $\Gamma_0 \cap Si = \{i\}$, where $\Gamma_0 = \bar{\varphi}(\mathscr{F}_0)$ and $i = \bar{\varphi}(\delta_e)$ is the unit in Γ_0. This discussion will apply to norm decreasing monomorphisms as a particular case. Hereafter we will denote $\Gamma = \bar{\varphi}(\mathscr{F})$, $\Gamma_0 = \bar{\varphi}(\mathscr{F}_0)$ (writing the unit of these groups as $i = \rho m_K$), $H_0 = \operatorname{supp}(\Gamma)$, and

$$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} .$$

Let $\pi : G \longrightarrow (G/K)_r$ be the canonical map onto the right coset space, so $\pi : H_0 \longrightarrow H_0/K$ is the corresponding canonical homomorphism. Let γ_F, γ_G be the topologies on F, G and, if γ is a group topology on G, let γ/π denote the quotient space topology on $(G/K)_r$ (notice $\gamma/\pi = \pi(\gamma)$). The restriction of γ_G to a subset $N \subset G$ is $\gamma_G | N$. We will speak interchangeably of a topology γ and the collection of open sets it specifies.

The following lemma holds for all locally compact groups; notation is chosen so its meaning in the present context is clear.

LEMMA 5.1.2. *Let F, G be locally compact groups and consider any system of subgroups $K \subset H_0 \subset G$ with K a compact subgroup in G which is normal in H_0. Let $\pi : G \to (G/K)_r$ be the canonical map onto the right coset space. If $\zeta : F \to (H_0/K, \gamma_G/\pi)$ is a continuous epimorphism then $\zeta(\gamma_F)$ and $\pi^{-1} \circ \zeta(\gamma_F)$ are topologies in H_0/K and H_0 respectively; moreover, if γ is the common refinement in H_0 of $(\gamma_G \mid H_0)$ and $\pi^{-1} \circ \zeta(\gamma_F)$ then (H_0, γ) is a locally compact topological group, $\gamma/\pi = \zeta(\gamma_F)$, and $\zeta : F \to (H_0/K, \gamma/\pi)$ is an open, continuous epimorphism.*

REMARK. Unless K is trivial, $\pi^{-1} \circ \zeta(\gamma_F)$ will not be a Hausdorff topology, but in all other respects (homogeneity, joint continuity of multiplication, etc.) it is like a group topology.

Proof. The topology axioms for $\pi^{-1} \circ \zeta(\gamma_F)$ follow if we can verify them for $\zeta(\gamma_F)$. Only the finite intersection property is nontrivial. If $V_1, V_2 \in \gamma_F$ let $U_i = V_i \cdot \operatorname{Ker} \zeta$ and notice that $\zeta^{-1}(x) \cap U_i \neq \varnothing$ implies that $\zeta^{-1}(x) \subset U_i$. Thus

$$\begin{aligned}\zeta(V_1) \cap \zeta(V_2) = \zeta(U_1) \cap \zeta(U_2) &= \{x : \zeta^{-1}(x) \cap U_i \neq \varnothing,\ i = 1, 2\} \\ &= \zeta(U_1 \cap U_2) \in \zeta(\gamma_F)\ .\end{aligned}$$

Now (H_0, γ) is a Hausdorff space and the collection of sets $\mathscr{U} = \{U \cap V : U = W \cap H_0,\ W \in \gamma_G;\ V = \pi^{-1} \circ \zeta(X),\ X \in \gamma_F\}$ is a base for γ. If $U \cap V \in \mathscr{U}$ then $(U \cap V)^{-1} = U^{-1} \cap V^{-1} \in \mathscr{U}$, so the inverse mapping is bicontinuous. It is quite easy to verify that γ is homogeneous, in the sense that $\gamma = \{xU : U \in \gamma\}$ for any $x \in H_0$, so joint continuity of multiplication will only be proved at the identity $e \in H$. If e lies within $U \cap V \in \mathscr{U}$ there exist $U_0 \in \gamma_G \mid H_0$ and $V_0 \in \pi^{-1} \circ \zeta(\gamma_F)$, which contain e, such that $U_0^2 \subset U$ and $V_0^2 \subset V$; hence $(U_0 \cap V_0) \times (U_0 \cap V_0)$ is an open neighborhood of (e, e) in $(H_0, \gamma) \times (H_0, \gamma)$ which maps into $U \cap V$ under the product mapping.

Clearly $\gamma \supset \pi^{-1} \circ \zeta(\gamma_F)$, so that $\gamma/\pi = \pi(\gamma) \supset \pi \circ \pi^{-1}(\zeta(\gamma_F)) = \zeta(\gamma_F)$. For the converse inclusion, we first make a few simple assertions:

(1) If $A \subset H_0$ is a union of K-cosets and if B is any subset of H_0, then $(A \cap B) \cdot K = A \cap (B \cdot K)$;

(2) If $A_\alpha \subset H_0$ for indices $\alpha \in I$, then $(\bigcup_{\alpha \in I} A_\alpha) \cdot K = \bigcup_{\alpha \in I} (A_\alpha \cdot K)$.

Now a typical element in γ has the form $X = \bigcup_{\alpha \in I} A_\alpha \cap B_\alpha$ where $A_\alpha = \pi^{-1} \circ \zeta(U_\alpha)$ for some $U_\alpha \in \gamma_F$, and $B_\alpha = V_\alpha \cap H_0$ for some $V_\alpha \in \gamma_G$. Evidently $A_\alpha = A_\alpha \cdot K$ and $B_\alpha \cdot K = \pi^{-1} \circ (\pi B_\alpha)$, so we get

$$(*) \qquad \begin{aligned}\pi(X) = \pi(X \cdot K) &= \pi(\textstyle\bigcup_{\alpha \in I} (A_\alpha \cap B_\alpha) \cdot K) \\ &= \pi(\textstyle\bigcup_{\alpha \in I} A_\alpha \cap (B_\alpha \cdot K)) \\ &= \pi(\textstyle\bigcup_{\alpha \in I} \pi^{-1} \zeta(U_\alpha) \cap \pi^{-1} \pi(B_\alpha)) \\ &= \textstyle\bigcup_{\alpha \in I} \zeta(U_\alpha) \cap \pi(B_\alpha)\ .\end{aligned}$$

But continuity of ζ implies that $\zeta(\gamma_F) \supset (\gamma_G/\pi \mid H_0/K)$ and it is easily verified that the latter collection of sets is just $\pi(\gamma_G \mid H_0)$; hence the last term in $(*)$ is in $\zeta(\gamma_F)$, giving $\pi(\gamma) \subset \zeta(\gamma_F)$. Clearly $\gamma/\pi = \zeta(\gamma_F) \Rightarrow \zeta : F \rightarrow (H_0/K, \gamma/\pi)$ is an open mapping; so $(H_0/K, \gamma/\pi)$ is topologically isomorphic to the locally compact quotient group $F/\mathrm{Ker}(\zeta)$. To see the local compactness of (H_0, γ), notice that $K \subset H_0$ is γ compact because $\gamma \mid K = \gamma_G \mid K$. A result due to Mackey gives local compactness (see Montgomery-Zippen [7], p. 52).

If $x \in F$ write $s(x) = s(\bar{\varphi}(\delta_x))$, a coset of K in H_0. The map $\zeta = \pi \circ s : F \rightarrow ((G/K)_r, \gamma_G/\pi)$ carries F onto H_0/K, is a homomorphism (see 2.1.1), and is continuous since $x_j \rightarrow x \Rightarrow \delta_{x_j} \xrightarrow{(so)} \delta_x \Rightarrow \bar{\varphi}(\delta_{x_j}) \rightarrow \bar{\varphi}(\delta_x) \Rightarrow \pi \circ s(x_j) \rightarrow \pi \circ s(x)$. If $\mu \in \Gamma_0$ and $g \in s(\mu)$, then we can write $\mu = \rho_\mu \mid \mu \mid = \rho_\mu(\delta_g * m_K)$ and we can take ρ_μ to be a unique continuous function on the coset $s(\mu) \subset H_0$. Assigning ρ_μ in this manner for each $\mu \in \Gamma_0$ we have $\rho_{\mu*\lambda}(st) = \rho_\mu(s)\rho_\lambda(t)$ for $s \in s(\mu)$, $t \in s(\lambda)$, as indicated in the proof of 3.1.2. Define ρ on all of H_0 such that $\rho(x) = \rho_\mu(x)$ if $x \in s(\mu)$, $\mu \in \Gamma_0$; this is unambiguous since $s(\mu) = s(\lambda) \Rightarrow \mu = \lambda$ (we assume $\Gamma_0 \cap Si = \{i\}$, so 3.1.4 applies).

Consider the group topology γ on H_0 constructed as in 5.1.2 for the epimorphism $\zeta = \pi \circ s : F \rightarrow (H_0/K, \gamma_G/\pi)$.

PROPOSITION 5.1.3. (H_0, γ) is a locally compact Hausdorff group and $\rho \in (H_0, \gamma)^\wedge$.

Proof. Clearly ρ is a unimodular multiplicative function on H_0 which is continuous on cosets of K (see proof of 3.1.2). The topological group properties of (H_0, γ) were verified in 5.1.2. Given $\varepsilon > 0$ we can find a γ_G neighborhood V of the unit in G such that $\mid \rho(g_1) - \rho(g_2) \mid < \varepsilon$ for all $g_1, g_2 \in H_0$ with $g_1 = g_2 \bmod K$ and $g_1^{-1}g_2 \in V$. This is clear since $\rho_\mu{}^g(s) = \rho_\mu(gs) = \alpha\rho(s)$ for some $\mid \alpha \mid = 1$, whenever $s \in K$, $g \in s(\mu)$, $\mu \in \Gamma$ (see proof of 3.1.2), and we know ρ is uniformly continuous on K.

Let $g \in H_0$; then $g_0 \in s(x_0)$ for some $x_0 \in F$, and if U is a compact γ_F neighborhood of x_0, $N = s(U)$ is a neighborhood of g_0 in (H_0, γ). N is compact since continuity of $\pi \circ s : F \rightarrow (H/K, \gamma/\pi) \Rightarrow \pi \circ s(U)$ is compact, and since K is a γ compact subgroup in H_0. If ρ fails to be γ continuous at g_0 we can find a net $\{g_j\} \subset N$ such that $g_j \xrightarrow{(\gamma)} g_0$ while $\rho(g_j) \rightarrow \beta \neq \beta_0 = \rho(g_0)$. For each index j there exists an $x_j \in U$ such that $g_j \in s(x_j)$; we can assume that the net $\{x_j\}$ is γ_F convergent to some $x_1 \in U$, which will $\Rightarrow \mu_j = \bar{\varphi}(\delta_{x_j}) \xrightarrow{(\sigma)} \bar{\varphi}(\delta_{x_1}) = \mu_1$. But this $\Rightarrow s(\mu_1) = g_0K = s(\mu_0)$, since $g_j \xrightarrow{(\gamma_G)} g_0$, so $\mu_1 = \mu_0 = \bar{\varphi}(\delta_{x_0})$. Recall that $\mu_j = \rho(\delta_{g_j} * m_K)$ and $\mu_0 = \rho(\delta_{g_0} * m_K)$ from the definition of ρ.

If $\psi \in C_0(G)$ has sup norm one and $\langle \mu_0, \psi \rangle \neq 0$, then $\Psi(s) =$

$\int \psi(ts)\rho(t)dm_K(t)$ is in $C_0(G)$, $\|\Psi\|_\infty \leqq 1$, and $\Psi(s) = \alpha\overline{\rho(s)}$ for all $s \in g_0K$, where $\alpha = \langle \mu_0, \psi \rangle$ (a constant $\neq 0$). Furthermore,

$$\begin{aligned}\langle \mu_0, \Psi \rangle &= \int\left[\int \psi(ts)\rho(t)dm_K(t)\right]\rho(s)d[\delta_{g_0} * m_K](s) \\ &= \int\left[\int \psi(ts)\rho(ts)dm_K(t)\right]d[\delta_{g_0} * m_K](s) \\ &= \int \psi(x)\rho(x)d[m_K * \delta_{g_0} * m_K](x) \\ &= \int \psi(x)\rho(x)d[\delta_{g_0} * m_K](x) = \langle \mu_0, \psi \rangle .\end{aligned}$$

If $\varepsilon > 0$ we can insure that $|\Psi(g_js) - \Psi(g_0s)| < \varepsilon$ for all $s \in K$ and $j \geqq j_\varepsilon$ since $g_j \xrightarrow{(\gamma_G)} g_0$, and this means that $\alpha = \langle \mu_0, \Psi \rangle \longleftarrow \langle \mu_j, \Psi \rangle = \int \Psi\rho d[\delta_{g_j} * m_K] = \int \Psi(g_js)\rho(g_js)dm_K(s)$. The last integral is eventually within ε of

$$\begin{aligned}\int \Psi(g_0s)\rho(g_js)dm_K(s) &= \alpha \int \overline{\rho(g_0s)}\rho(g_js)dm_K(s) \\ &= \alpha \int \overline{\rho(g_0)}\rho(g_j)dm_K(s) \longrightarrow \alpha\beta_0\overline{\beta} .\end{aligned}$$

Since this is true for all $\varepsilon > 0$, and $\beta_0 \neq \beta$, we have a contradiction.

COROLLARY 5.1.4. *If F is a compact group and $\varphi : L^1(F) \longrightarrow M(G)$ is a norm decreasing monomorphism, then in* 5.1.2 $\Gamma = \bar{\varphi}(\mathscr{F})$ *is a* (σ) *compact subgroup of* $\sum_{M(G)}$, $H_0 = \operatorname{supp}(\Gamma)$ *is a compact subgroup in* G, *and* $\gamma = \gamma_G \mid H_0$ *in* H_0. *Thus if ρ is defined as above,* $\rho \in (H_0, \gamma_G)^\wedge$.

Proof. Clearly Γ is compact; H_0 is then γ_G compact since $(H_0/K, \gamma_G/\pi)$ is compact (recall $\pi \circ s : F \longrightarrow (H_0/K, \gamma_G/\pi)$ is a continuous epimorphism). By definition of γ the map $\pi \circ s : F \longrightarrow (H_0/K, \gamma/\pi)$ is continuous and we know that $K \subset H_0$ is γ compact; thus H_0 is γ compact as well as γ_G compact. Since γ is finer than γ_G, these must be equivalent topologies on H_0.

Consider the following maps between measure algebras.

(1) Let H, G be locally compact groups and let $j : H \longrightarrow G$ be a continuous monomorphism. Define $j^{**} : (M(H), (so)) \longrightarrow (M(G), (\sigma))$ such that $\langle j^{**}\mu, \psi \rangle = \langle \mu, \psi \circ j \rangle$ for $\psi \in C_0(G)$.

(2) Let H be a locally compact group and let $\rho \in H^\wedge$. Define $A_\rho : (M(H), (so)) \longrightarrow (M(H), (so))$ such that $A_\rho(\mu) = \rho\mu$, so $\langle A_\rho\mu, \psi \rangle = \langle \mu, \rho\psi \rangle$.

(3) Let F, H be locally compact groups, let K be a compact

normal subgroup in H, and let $\zeta: F \to (H/K, \gamma_H/\pi)$ be an open continuous epimorphism, where $\pi: H \to H/K$ is the canonical homomorphism. Then define $\Phi: (M(F),(so)) \to (M(H),(so))$ such that $\langle \Phi\mu, \psi\rangle = \langle \mu, (\pi^*\psi)\circ\zeta\rangle$, where the function $\pi^*\psi(x) = \int \psi(xt)dm_K(t)$, constant on cosets of K, is considered as a function in $C_0(H/K)$.

We assert that the maps in (1) $\cdots$ (3) are all norm decreasing homomorphisms, continuous on norm bounded sets with respect to the topologies indicated. Since $(A_\rho\mu)*f = \rho(\mu * \bar{\rho}f)$, this assertion is clear for (2), and follows easily from 1.1.1 for (1), because $\psi \circ j$ is uniformly continuous and bounded on H; we momentarily put off verification of (3). Once this assertion has been checked we can prove the following structure theorem.

THEOREM 5.1.5. *If we are given groups and maps as in* (1) $\cdots$ (3) *then the map* $\bar{\varphi} = j^{**}\circ A_\rho \circ \Phi: (M(F),(so)) \to (M(G),(\sigma))$ *is a norm decreasing homomorphism, continuous on norm bounded sets, with the special property that* $\Gamma_0 \cap Si = \{i\}$, *where* $\Gamma_0 = \bar{\varphi}(\mathscr{F}_0)$ *and* $i \in \Gamma$ *is its unit. Conversely, let* $\varphi: L^1(F) \to M(G)$ *be a norm decreasing homomorphism whose extension* $\bar{\varphi}$ *(as described in* 4.1.1) *has the special property* $\Gamma_0 \cap Si = \{i\}$, *where* $\Gamma_0 = \bar{\varphi}(\mathscr{F}_0)$ *and* $i = \rho m_K$ *is its unit. If* $H_0 = \operatorname{supp}(\Gamma)$, *then we get* $\bar{\varphi} = j^{**}\circ A_\rho\circ\Phi$ *by taking groups* $H = (H_0, \gamma) \supset K = (K, \gamma)$ *and maps* $\zeta = \pi \circ s: F \to (H_0/K, \gamma/\pi)$, $j = id: (H_0, \gamma) \to (G, \gamma_G)$, *where* $\rho \in (H_0, \gamma)^\wedge$ *is the unique function on* H_0, *continuous on cosets of* K, *with the property* $\mu = \rho|\mu|$ *for all* $\mu \in \Gamma_0$.

REMARK. In the first part, $\bar{\varphi}$ is clearly the extension of $\varphi = \bar{\varphi}\,|\,L^1(F)$. Furthermore, the unit of Γ will be $i = \rho m_K$ and $\operatorname{supp}(\Gamma) = H$, when H and K are regarded as subgroups in G. In the second part the γ topology in H_0 is defined as in 5.1.2.

Proof. In the first part consider H and K as subgroups of G (with new group topologies) and j as the identity injecting H into G; H has a topology finer than $\gamma_G|H$, but since j is continuous, it is a homeomorphism on compacta and on cosets of K in particular. If $x \in F$ it is easy to verify that $\bar{\varphi}(\delta_x) = \rho(\delta_g * m_K)$ for any $g \in \bar{\pi}^{-1}\circ\zeta(x)$. From this it is clear that Γ has unit $i = \rho m_K$, and that $\Gamma_0 \cap Si = \{i\}$.

Conversely let $\varphi: L^1(F) \to M(G)$ be given. If we take $H = (H_0, \gamma)$, $K = (K, \gamma)$ and let $\zeta = \pi\circ s$, $j = id: (H_0, \gamma) \to (G, \gamma_G)$, we see that H is a locally compact group and that $\zeta: F \to (H_0/K, \gamma/\pi)$ is an open, continuous homomorphism (5.1.2); thus, the maps j^{**}, A_ρ, Φ are well defined. We know $\rho \in (H_0, \gamma)^\wedge$ from 5.1.3.

If $x \in F$ then $|\bar{\varphi}(\delta_x)| = \delta_g * m_K$ for any $g \in s(x)$ and $\bar{\varphi}(\delta_x) = \rho(\delta_g * m_K)$

by definition of ρ. It is a simple matter to verify that

$$id^{**} \circ A_\rho \circ \Phi(\delta_x) = \rho(\delta_g * m_K)$$

for any $g \in s(x)$, so that $\bar{\varphi} = id^{**} \circ A_\rho \circ \Phi$ on $\mathscr{F}$. Since the maps on each side of this identity are continuous on norm bounded sets, as maps of $(M(F), (so))$ into $(M(G), (\sigma))$, we get $\bar{\varphi} = id^{**} \circ A_\rho \circ \Phi$ on all of $M(F)$ from 1.1.3.

COROLLARY 5.1.6. *A norm decreasing homomorphism* $\varphi : L^1(F) \to M(G)$ *is a monomorphism* $\Leftrightarrow$ *its extension has the structure* $\bar{\varphi} = id^{**} \circ A_\rho \circ \Phi$, *as in* 5.1.5, *where the map* $\zeta = \pi \circ s$ *which induces* Φ *is an isomorphism of* F *onto* H_0/K.

Proof. If φ is a monomorphism, so is $\bar{\varphi} \mid F$ (see 5.1.1); now 5.1.5 applies and it is clear that $\zeta = \pi \circ s$ is an isomorphism, as required for ($\Rightarrow$). Notice that the maps A_ρ and id^{**} are always monomorphisms in (2) and (3) above. Conversely, in (3) we have $\Phi = \pi^{**} \circ \zeta^{**}$, where $\langle \zeta^{**}\mu, \psi \rangle = \langle \mu, \psi \circ \zeta \rangle$ and $\langle \pi^{**}\mu, \psi \rangle = \langle \mu, \pi^*\psi \rangle$ define maps

$$M(F) \xrightarrow{\zeta^{**}} M(H_0/K, \gamma/\pi) \xrightarrow{\pi^{**}} M(H_0, \gamma) \ .$$

Since $\zeta = \pi \circ s : F \to (H_0/K, \gamma/\pi)$ is a topological isomorphism if ζ is $1:1$, ζ^{**} is a monomorphism. It is easy to verify that $\pi^*(C_0(H_0, \gamma))$ is sup norm dense in $C_0(H_0/K, \gamma/\pi)$; hence π^{**} is always a monomorphism.

In the following paragraphs we digress to study the map defined in (3) and prove the assertions about it which were used to prove 5.1.5. Then in 5.2, we will use these observations to study the structure of special norm decreasing homomorphisms.

THEOREM 5.1.7. *Let* F *and* H *be locally compact groups, let* $K \subset H$ *be a compact normal subgroup, and let* $\zeta : F \to H/K$ *be an open, continuous epimorphism. Then the map* $\Phi : (M(F), (so)) \to (M(H), (so))$, *defined such that* $\langle \Phi\mu, \psi \rangle = \langle \mu, (\pi^*\psi) \circ \zeta \rangle$, *is a norm decreasing homomorphism, continuous on norm bounded sets, if we identify* $\pi^*\psi(x) = \int \psi(xt)dm_K(t)$ (*constant on cosets of* K) *with a function in* $C_0(H/K)$ *for each* $\psi \in C_0(H)$.

Proof. Consider the maps shown in Figure 2,

$$\begin{array}{ccc} (M(F), (so)) & \xrightarrow{\Phi} & (M(H), (so)) \\ \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \pi^{**}} \\ (M(F/F_0), (so)) & \xrightarrow{\Phi_2} & (M(H/K), (so)) \end{array}$$

Figure 2

where $F_0 = \mathrm{Ker}\,(\zeta)$, $\langle \Phi_1\mu, \psi\rangle = \langle \mu, \psi\circ\pi_0\rangle$ ($\pi_0 : F \to F/F_0$ is the canonical homomorphism), $\langle \Phi_2\mu, \psi\rangle = \langle \mu, \psi\circ\zeta\circ\pi_0^{-1}\rangle$, and where $\langle \pi^{**}\mu, \psi\rangle = \langle \mu, \pi^*\psi\rangle$. Clearly Φ_2 is bicontinuous with respect to the topologies in Figure 2. Continuity of Φ follows from the lemmas below, since we can verify by direct computation that $\Phi = \pi^{**}\circ\Phi_2\circ\Phi_1$ on $M(F)$.

LEMMA 5.1.8. *Let Q be a locally compact group with $Q_0 \subset Q$ a closed, normal subgroup, and let $\pi_0 : Q \to Q/Q_0$ be the canonical homomorphism. Define $\Phi : (M(Q), (so)) \to (M(Q/Q_0), (so))$ such that $\langle \Phi\mu, \psi\rangle = \langle \mu, \psi\circ\pi_0\rangle$ for $\psi \in C_0(Q/Q_0)$. Then Φ is a norm decreasing homomorphism, continuous on norm bounded sets.*

Proof. It is easy to verify that Φ is a norm decreasing homomorphism. We assert that $\Phi(M(Q)) = M(Q/Q_0)$, and in fact $\Phi(\Sigma_{M(Q)}) = \Sigma_{M(Q/Q_0)}$; from this it will follow that $\Phi(L^1(Q))$ is a two sided ideal in $M(Q/Q_0)$ since $L^1(Q)$ is a two sided ideal in $M(Q)$. If $\Sigma_X = \{\mu : \|\mu\| \leqq 1,\ s(\mu) \subset X\}$ for $X \subset Q$, we will show that $\Phi(\Sigma_K) = \Sigma_{\pi_0 K}$ for all compacta $K \subset Q$; since π_0 is open, this means that every μ with compact support in $M(Q/Q_0)$ is the Φ-image of some $\mu \in M(Q)$ with $\|\mu\| = \|\lambda\|$. Clearly $\Phi(\Sigma_K) \subset \Sigma_{\pi_0 K}$, and K compact $\Rightarrow$ the map

$$\Phi : (\Sigma_K, (\sigma)) \longrightarrow (M(Q/Q_0), (\sigma))$$

is continuous, in fact if $\{\mu_j\} \subset \Sigma_K$ with $\mu_j \xrightarrow{(\sigma)} \mu$ and if $f \in C_0(Q)$ has $f = 1$ on K, then for any $\psi \in C_0(Q/Q_0)$ we get $\langle \Phi\mu_j, \psi\rangle = \langle \mu_j, \psi\circ\pi_0\rangle = \langle \mu_j, f\cdot(\psi\circ\pi_0)\rangle \to \langle \mu, f\cdot(\psi\circ\pi_0)\rangle = \langle \Phi\mu, \psi\rangle$. Now Σ_K is precisely the (σ)-closed convex span of $\{\alpha\delta_x : |\alpha| = 1,\ x \in K\}$, so $\Phi(\Sigma_K)$ is (σ)-compact; since $\Phi(\delta_x) = \delta_{\pi_0 x}$ we have $\Phi(\Sigma_K) \supset co\,\{\alpha\delta_{\pi_0 x} : |\alpha| = 1,\ x \in K\}$, which gives the converse inclusion.

Now if $\lambda \in M(Q/Q_0)$ there are measures λ_n with compact support such that $\|\lambda_n - \lambda\| \to 0$ and $\|\lambda\| = \|\lambda_1\| + \sum_{j=1}^{\infty} \|\lambda_{n+1} - \lambda_n\|$ (restrict λ to increasingly large compacta). Then there exist $\mu_n \in M(Q)$ with $\Phi(\mu_1) = \lambda_1$, $\|\mu_1\| = \|\lambda_1\|$ and $\Phi(\mu_{n+1}) = (\lambda_{n+1} - \lambda_n)$, $\|\mu_{n+1}\| = \|\lambda_{n+1} - \lambda_n\|$ for $n \geqq 1$; hence $\mu = \sum_{n=1}^{\infty} \mu_n$ converges in $M(Q)$, $\|\mu\| = \|\lambda\|$, and $\Phi(\mu) = \lambda$ as required.

Next we show $\Phi(L^1(Q)) \subset L^1(Q/Q_0)$; in fact, if $\mu = \Phi f$ and $x \in Q/Q_0$, then given $\varepsilon > 0$ and $g \in \pi_0^{-1}(x)$ we can find a compact neighborhood V of g with $\|\delta_h * f - f\| < \varepsilon$ for all $h \in V$. Thus $\|\Phi(\delta_h) * \Phi f - \Phi f\| = \|\delta_y * \mu - \mu\| < \varepsilon$ for $y \in \pi_0 V$. Since π_0 is open and continuous, this means $\mu = \Phi f \in L^1(Q/Q_0)$ (see Rudin [9], p. 230; the abelian hypothesis used there is superfluous). To prove 5.1.8 it is now sufficient to show that $\Phi(L^1(Q))$ is norm dense in $L^1(Q/Q_0)$. To prove density, let $\{e_j\} \subset L^1(Q)$ be a left approximate identity such that $e_j > 0$, $\|e_j\| = 1$, and $s(e_j)$ are compacta which are eventually within any fixed neighborhood of the unit in Q. Then $s(\Phi e_j)$ are compacta shrinking to the unit in Q/Q_0. But $\langle \Phi e_j, \psi\rangle \rightarrow \psi(e)$ for all $\psi \in C_0(Q/Q_0)$, hence $\Phi e_j \xrightarrow{(\sigma)} \delta_e$ and $\|\Phi e_j\| \leqq 1$; these facts together imply that $\lim\{\|\Phi e_j\|\} = 1$. Since $\Phi e_j \geqq 0$ (clear), $\{\Phi e_j\} \subset L^1(Q/Q_0)$ is an approximate identity for $L^1(Q/Q_0)$. Since $\Phi(L^1(Q))$ is an ideal in $M(Q/Q_0)$, norm density of $\Phi(L^1(Q))$ in $L^1(Q/Q_0)$ follows.

LEMMA 5.1.9. *If Q is a locally compact group, $K \subset Q$ a compact normal subgroup with canonical homomorphism $\pi: Q \rightarrow Q/K$, define $\pi^{**}: (M(Q/K), (so)) \rightarrow (M(Q), (so))$ such that $\langle \pi^{**}\mu, \psi\rangle = \langle \mu, \pi^*\psi\rangle$ where $\pi^*\psi(x) = \int \psi(xt)dm_K(t)$ (constant on cosets of K) is regarded as a function in $C_0(Q/K)$. Then π^{**} is a norm decreasing homomorphism, continuous on norm bounded sets.*

Proof. Normality of K in $Q \Rightarrow h*m_K = m_K*h = m_K*h*m_K$ for all $h \in L^1(Q)$. Define $\xi: M(Q) \rightarrow M(Q/K)$ such that $\langle \xi\mu, \psi\rangle = \langle \mu, \psi\circ\pi\rangle$. It is a simple matter to verify that (1) $\pi^{**}\xi(\mu) = \mu*m_K$ for all $\mu \in M(Q)$, and (2) $\xi\pi^{**}(\mu) = \mu$ for all $\mu \in M(Q/K)$. One can also verify by direct computation that $\pi^{**}\mu = (\pi^{**}\mu)*m_K$ for $\mu \in M(Q/K)$. From (1) we see that $\pi^{**}(M(Q/K)) = M(Q)*m_K$, so that $\pi^{**}(L^1(Q/K))$ is closed under right or left multiplication by elements of $M(Q)*m_K$. Finally, $\pi^{**}(L^1(Q/K)) \subset L^1(Q)$; for if $\varepsilon > 0$ and $x \in Q$, and if $f \in L^1(Q/K)$, we can find a neighborhood V of $\pi(x)$ with $\|\delta_u f - f\| < \varepsilon$ whenever $u \in V$. Thus, if W is a neighborhood of x such that $\pi(W) \subset V$, we have $\|\delta_y*(\pi^{**}f) - \pi^{**}f\| \leqq \|\xi(\delta_y)*\xi(\pi^{**}f) - \xi(\pi^{**}f)\| = \|\delta_{\pi y}*f - f\| < \varepsilon$ for all $y \in W$. Thus $\pi^{**}f \in L^1(Q)$ (again see Rudin [9], p. 230). If $\{e_j\}$ is a norm one approximate identity in $L^1(Q/K)$, then $e_j \xrightarrow{(so)} \delta_e$ and it is easy to show that $\pi^{**}e_j \xrightarrow{(\sigma)} m_K = \pi^{**}(\delta_e)$ from 1.1.1. We can arrange that the supports $s(\pi^{**}e_j)$ shrink to $s(m_K) = K$, a compact set; thus we get $\pi^{**}e_j \xrightarrow{(so)} m_K$ by applying 1.1.2. Since $\pi^{**}(L^1(Q/K))$ is closed under right multiplication by elements of $m_K*M(Q) = M(Q)*m_K$, we get (for any $h \in L^1(Q)$) $\|(\pi^{**}e_j)*m_K*h - m_K*m_K*h\| \rightarrow 0$, which $\Rightarrow \pi^{**}(L^1(Q/K))$ is norm dense in $m_K*L^1(Q)$.

Consider $\mu_j \xrightarrow{(so)} \mu$ in $M(Q/K)$ with $\|\mu_j\| \leqq 1$; if $h \in L^1(Q)$ then $(\pi^{**}\mu_j)*h = (\pi^{**}\mu_j)*m_K*h$. But we can approximate m_K*h in norm by some $\pi^{**}f (f \in L^1(Q/K))$ and we know that $(\pi^{**}\mu_j)*(\pi^{**}f) = \pi^{**}(\mu_j*f) \xrightarrow{\text{norm}} (\pi^{**}\mu)*(\pi^{**}f)$.

5.2. **Norm decreasing homomorphisms which map $L^1(F)$ to $L^1(G)$.** Suppose φ actually maps $L^1(F)$ onto $L^1(G)$, then the structure of φ is exceedingly simple. First recall that if φ is a norm decreasing isomorphism of $L^1(F)$ onto $L^1(G)$ it is actually an isometry; furthermore, an isometric isomorphism has the special structure

$$\langle \varphi f, \psi \rangle = \int_F \rho \circ s(x) \psi \circ s(x) df(x)$$

where $s: F \to G$ is any topological isomorphism and $\rho \in G^\wedge$, as was first proved by Wendel [10], [11]. Although the structure theorem 5.1.4 could be used as the basis for a direct proof of these results, it only gives conditions on the structure of φ which are necessary (but not sufficient) if we are to have $\varphi(L^1(F)) = L^1(G)$. To identify these norm decreasing isomorphisms (or isometries) precisely we would have to retrace some of Wendel's analysis rather than do this we use Wendel's analysis as a starting point.

THEOREM 5.2.1. *Let* $\varphi: L^1(F) \to L^1(G)$ *be a norm decreasing epimorphism. Then there exists a closed normal subgroup* $F_0 \subset F$, *an isometric isomorphism* $\Lambda: L^1(F/F_0) \to L^1(G)$, *and* $\beta \in F^\wedge$ *with* $\operatorname{Ker} \beta \subset F_0$ *such that* $\varphi = \Lambda \circ (\pi^* A_\beta)$, *where* $A_\beta(\mu) = \beta\mu$, *and the canonical homomorphism* $\pi: F \to F/F_0$ *gives* $\langle \pi^*(\mu), \psi \rangle = \langle \mu, \psi \circ \pi \rangle$ *for* $\psi \in C_0(F/F_0)$.

Proof. First notice that, if $s(x) = s(\bar{\varphi}(\delta_x))$, then $s: F \to G$ is a continuous homomorphism; in fact, $\bar{\varphi}(\delta_e) = \rho m_K$ for compact subgroup $K \subset G$ and $\rho \in K^\wedge$, and if $h \in L^1(G)$ we can write $h = \varphi f$ for some $f \in L^1(F)$. Thus $h * \rho m_K = \varphi(f) * \bar{\varphi}(\delta_e) = \varphi f = h$, which is impossible for all h unless $K = \{e\}$, so $\bar{\varphi}$ maps $\mathscr{E}_{M(F)}$ into $\mathscr{E}_{M(G)}$. For continuity of s see remarks preceding 5.1.3. Hence $F_0 = \{x \in F: s(x) = e \text{ in } G\}$ is a closed normal subgroup in F. If we define $\beta(x) = \alpha \in S \Leftrightarrow \bar{\varphi}(\delta_x) = \alpha\delta_{s(x)}$, the continuity properties of $\bar{\varphi}$ (see 4.1.1) insure that $\beta \in F^\wedge$; thus $A_\beta: \mu \to \beta\mu$ is an isometric automorphism of $M(F)$. The map $\pi^*: M(F) \to M(F/F_0)$ has been discussed in 5.1.8; we assert that π^* has the following properties (which will be verified at the end of this proof):

(1) $\pi^* L^1(F) = L^1(F/F_0)$, and

(2) $\|\pi^*(\mu)\| = \inf\{\|\mu + n\| : n \in \operatorname{Ker}(\pi^*)\}$,

the quotient norm in $M(F)/\operatorname{Ker}(\pi^*)$.

Clearly $\mu \in \operatorname{Ker}(\pi^* A_\beta) \Leftrightarrow \langle \pi^* A_\beta(\mu), \psi \rangle = \int_F \beta(x)\psi(\pi x) d\mu(x) = 0$ for all $\psi \in C_0(F/F_0)$; it is not hard to show that $\mu \in \operatorname{Ker} \bar{\varphi} \Leftrightarrow \langle \bar{\varphi}\mu, \psi \rangle = \int_F \langle \bar{\varphi}(\delta_x), \psi \rangle d\mu(x) = \int_F \beta(x) \langle \delta_{s(x)}, \psi \rangle d\mu(x) = 0$ for $\psi \in C_0(G)$. The non-trivial first equality here can be seen from 4.2.2, or directly by looking at the action of $\bar{\varphi}$ on finite sums of point masses and using the (so) continuity of $\bar{\varphi}$. We assert that $\operatorname{Ker} \bar{\varphi} \supset \operatorname{Ker}(\pi^* A_\beta)$, so

$$\Lambda = \bar{\varphi} \circ (\pi^* A_\beta)^{-1}: M(F/F_0) \longrightarrow M(G)$$

is a well defined homomorphism.

LEMMA 5.2.2. *If $\{f_j\}$ is a net of bounded functions in $C(F)$ with $M = \sup\{\|f_j\|_\infty\} < \infty$ and $f_j \to f$ uniformly on compacta, then $\langle \mu, f_j\rangle \to \langle \mu, f\rangle$ for all $\mu \in M(F)$.*

Proof. As usual, for bounded $f \in C(F)$ we define $\langle \mu, f\rangle = \langle \mu, \chi_E f\rangle$ where $E \in \boldsymbol{B}(F)$ and $E \supset s(\mu)$. For K compact in F we obviously have $\langle(\mu \mid K), f_j\rangle \to \langle(\mu \mid K), f\rangle$ and for suitably chosen compacta $K_n \subset s(\mu)$ we have $\|\mu - (\mu \mid K_n)\| \to 0$; hence $\langle \mu, f_j\rangle \to \langle \mu, f\rangle$.

Each function $\Psi(x) = \langle \delta_{s(x)}, \psi\rangle$ is continuous, bounded, and constant on cosets of F_0 in F, if $\psi \in C_0(G)$. But any bounded $f \in C(F)$ which is constant on cosets of F_0 can be approximated uniformly on compacta by a uniformly bounded net of functions selected from $\{h \circ \pi : h \in C_0(F/F_0)\}$; in fact, if $K \subset F$ is compact so is πK, and if U is a relatively compact open neighborhood of πK, we can find continuous h such that $h \equiv 1$ on πK, $h \equiv 0$ outside U and $0 \leqq h \leqq 1$. Then $f\cdot(h \circ \pi)$ coincides with $h_1 \circ \pi$ on F, where $h_1(x) = h(x)\cdot f(\pi^{-1}(x)) \in C_0(F/F_0)$; we have $h_1 \circ \pi \equiv f$ on K and $\|h_1 \circ \pi\|_\infty \leqq \|f\|_\infty$ as desired. Taking Ψ as the uniform on compacta limit of uniformly bounded net $\{h_j \circ \pi\}$ we get

$$\langle \bar{\varphi}\mu, \psi\rangle = \langle \beta\mu, \bar{\Psi}\rangle = \lim\{\langle \beta\mu, h_j \circ \pi\rangle\} = \lim\{\langle \pi^* A_\beta(\mu), h_j\rangle\} = 0$$

if $\mu \in \operatorname{Ker}(\pi^* A_\beta)$, so $\operatorname{Ker}\bar{\varphi} \supset \operatorname{Ker}(\pi^* A_\beta)$.

Now $\|\Lambda\| \leqq 1$ since (2) insures that

$$\begin{aligned}\|\pi^* A_\beta(\mu)\| &= \inf\{\|\mu + n\| : n \in \operatorname{Ker}(\pi^* A_\beta)\} \\ &\geqq \inf\{\|\mu + n\| : n \in \operatorname{Ker}\bar{\varphi}\} \\ &\geqq \inf\{\|\bar{\varphi}\mu + \bar{\varphi}n\| = \|\bar{\varphi}\mu\|\} = \|\bar{\varphi}\mu\|\end{aligned}$$

for $\mu \in M(F)$. Since $\pi^* : (M(F), (so)) \to (M(F/F_0), (so))$ is continuous on norm bounded sets (see 5.1.8), $(\pi^*)^{-1}$ is open on $\Sigma_{M(F/F_0)}$ relative to the (so) topologies; hence $\Lambda : (M(F/F_0), (so)) \to (M(G), (\sigma))$ is continuous on norm bounded sets. From (1) we see that Λ maps $L^1(F/F_0)$ onto $L^1(G)$, so Λ on $M(F/F_0)$ coincides with the extension $\bar{\Lambda}$ from $L^1(F/F_0)$ discussed in 4.1.1. Furthermore,

$$\Lambda(\mathscr{E}_{F/F_0}) = \bar{\varphi} \circ (\pi^* A_\beta)^{-1}(\mathscr{E}_{F/F_0}) = \bar{\varphi}\{\overline{\beta(x)}\delta_x : x \in F\} = \{\delta_{s(x)} : x \in F\}\,,$$

so $\Lambda(\mathscr{E}_{F/F_0}) \cap S\{\delta_e\} = \{\delta_e\}$ in $\mathscr{E}_{M(G)}$ and the analysis of 5.1 applies; i.e. we can write $\Lambda = j^{**} \circ A_\rho \circ \zeta^* : M(F/F_0) \to M(G)$ as in 5.1.5. In our present context some of these maps are trivial since $\Lambda(\delta_x) = \delta_{s(\pi^{-1}(x))}$ for $x \in F/F_0$; indeed, ρ and K are trivial, j is the injection of $H = \{s(x) : x \in F\}$ into G, and $\zeta : F/F_0 \to H$ is given by $\zeta(x) = s(\Lambda(\delta_x)) = s(\pi^{-1}(x))$. But $F_0 = \operatorname{Ker} s = \operatorname{Ker}\pi$ in F, so ζ is an isomorphism of F/F_0 onto H; hence, as indicated in 5.1.6, Λ must be a monomorphism on $M(F/F_0)$. Thus A is a norm decreasing isomorphism between $L^1(F/F_0)$ and $L^1(G)$, and Wendel's analysis applies to A.

In proving 5.1.8 we showed that $\pi^*(M(F)) = M(F/F_0)$ and that $\pi^*(\Sigma_{M(F)}) = \Sigma_{M(F/F_0)}$. The latter identity proves assertion (2) above. Furthermore, we showed $\pi^*(L^1(F)) \subset L^1(F/F_0)$ is norm dense, and that a right approximate identity $\{e_j\}$ of norm one in $L^1(F)$ is mapped to the same sort of approximate identity $\{\pi^* e_j\}$ in $L^1(F/F_0)$. Let $f \in L^1(F/F_0)$, say with $\|f\| = 1$, and let $\mu \in M(F)$ be chosen with $\|\mu\| = 1$, $\pi^*\mu = f$; then $\pi^*(\mu * e_j) = (\pi^*\mu)*(\pi^* e_j) = f*(\pi^* e_j) \xrightarrow{\text{norm}} f$ and $\mu * e_j \in L^1(F)$ with $\|\mu * e_j\| \leqq \|\mu\| = \|f\|$. Hence we see $\pi^*(\Sigma_{L^1(F)})$ is norm dense in $\Sigma_{L^1(F/F_0)}$. We can find $g_1 \in L^1(F)$ with $\|g_1\| \leqq 1$ and $\|\pi^* g_1 - f\| \leqq 1/2$. Since $\pi^* g_1 - f \in L^1(F/F_0)$, there exists $g_2 \in L^1(F)$ with $\|g_2\| \leqq 1/2$ and $\|\pi^* g_2 - (f - \pi^* g_1)\| < (1/2)^2$. By continuing this selection we get $g_n \in L^1(F)$ with $\|g_n\| \leqq (1/2)^{n-1}$ and $\|\pi^* g_n - (f - \sum_{j=1}^{n-1} \pi^* g_j)\| < (1/2)^n$. Thus $g = \sum_{n=1}^{\infty} g_n$ converges in $L^1(F)$ and $\pi^* g = \sum_{n=1}^{\infty} \pi^* g_n = f$, proving assertion (1) above.

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THE 2-LENGTH OF A FINITE SOLVABLE GROUP

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One measure of the structure of a finite solvable group G is its p-length $l_p(G)$. A problem connected with this measure is to obtain an upper bound for $l_p(G)$ in terms of $e_p(G)$, which is a numerical invariant of the Sylow p-subgroups of G. This problem has been solved but the best-possible result is not known for $p = 2$. The main result of this paper is that $l_2(G) \leqq 2e_2(G) - 1$, which is an improvement on earlier results. A secondary objective of this paper is to investigate finite solvable groups in which the Sylow 2-group is of exponent 4. In particular it is proved that if G is a finite group of exponent 12, then the 2-length is at most 2.

Introduction and discussion of results. The object of this paper is to obtain bounds for the 2-length of a finite solvable group. Following Hall and Higman [4], we call a finite group G p-solvable if it possesses a normal series such that each factor group is either a p-group or a p'-group. The p-length, $l_p(G)$, of such a group is the smallest number of p-groups which can occur as factor groups in such a normal series. $e_p(G)$ is defined to be the smallest n such that $x^{p^n} = 1$ for all x belonging to a Sylow p-subgroup of G.

For an odd prime p, it is proved in [4] that $l_p(G) \leqq e_p(G)$ if p is not a Fermat prime and $l_p(G) \leqq 2e_p(G)$ if p is a Fermat prime. Furthermore these results are best-possible. A. H. M. Hoare [6] then proved that in a 2-solvable group G, $l_2(G) \leqq 3e_2(G) - 2$ provided that $l_2(G) \geqq 1$. The primary purpose of this paper is to prove the following improvement:

THEOREM A. *If G is a finite solvable group and $l_2(G) \geqq 1$, then $l_2(G) \leqq 2e_2(G) - 1$.*

Feit and Thompson [1] have proved that solvability and 2-solvability are equivalent notions for finite groups. Thus no loss of generality is involved in requiring G to be solvable in the theorem.

Theorem A will be shown to be an easy consequence of the following theorem about linear groups:

THEOREM B. *Let G be a finite solvable linear group over a field*

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F of characteristic 2 and assume G has no nontrivial normal 2-subgroup. Then if N is the largest normal 2'-subgroup of G and if g is an exceptional element of order 2^m in G, it follows that $g^{2^{m-1}}$ is in the largest normal 2-subgroup of G/N.

Here, following [4], an element x of order p^n in a linear group over a field of characteristic p is said to be exceptional if $(x-1)^{p^{n-1}} = 0$.

Whether or not Theorem A represents a best-possible result is not known, but it seems likely that further improvements can be made. Indeed, the author knows of no group whose 2-length exceeds its 2-exponent. In the special case of finite solvable groups satisfying $e_2(G) = 2$, i.e., solvable groups whose Sylow 2-subgroups are of exponent 4, I think it likely that $l_2(G) \leqq 2$ instead of the bound $l_2(G) \leqq 3$ furnished by Theorem A.

In § 4 of this paper, groups satisfying $e_2(G) = 2$ are studied in more detail. A sufficient condition for $l_2(G) \leqq 2$ in this special case is established, and, as an application, we prove that $l_2(G) \leqq e_2(G)$ if G is a finite group of exponent 12.

2. Proof of Theorem A from Theorem B. For the rest of this paper we adopt the convention that all groups referred to are assumed finite, and, if G is such a group, then $|G|$ denotes its order. If H is a normal subgroup of G, we write $H \lhd G$.

We now recall the definition of the upper 2-series of the solvable group G:

$$1 = P_0 \leqq N_0 < P_1 < N_1 < \cdots < P_l \leqq N_l = G .$$

Here N_k/P_k is defined to be the greatest normal 2'-subgroup of G/P_k and P_{k+1}/N_k the greatest normal 2-subgroup of G/N_k. The least integer l such that $N_l = G$ is the 2-length $l_2(G)$. (If there is no danger of confusion we write simply l_2.)

It is proved in [4] that the automorphisms of P_1/F, where F/N_0 is the Frattini subgroup of P_1/N_0, induced by G represent G/P_1 faithfully. Thus G/P_1 is faithfully represented as a linear group operating on P_1/F (P_1/F is an elementary abelian 2-group and so is considered as a vector space over the field with 2 elements).

Now if $l_2(G)=1$, the conclusion of A is trivial. Also the p-length group is at most equal to the class of a Sylow p-subgroup [4, Theorem 1.2.6]. An immediate consequence of this is that if G is solvable and $e_2(G) = 1$, then $l_2(G) = 1$. Thus $l_2 = 2$ implies that $e_2 \geqq 2$ so the result again follows. Now if $l_2 > 2$, then $l_2(G/P_2) = l_2(G) - 2 \geqq 1$ so that Theorem A would follow by induction on l_2 if we could prove that

$e_2(G/P_2) \leqq e_2(G) - 1$.

Now suppose g is an element of maximal order 2^m in a Sylow 2-Sylow subgroup of G/P_1. If g is not exceptional, then [4, Lemma 3.1.2] we have $e_2(G) \geqq m + 1$. If g is exceptional, then, since G/P_1 satisfies the hypothesis of Theorem B, $g^{2^{m-1}}$ is in P_2/P_1 if Theorem B is true. Thus, assuming the validity of B, we obtain in all cases $e_2(G/P_2) \leqq e_2(G) - 1$ and Theorem A follows.

3. Proof of Theorem B. Neither the hypothesis nor the conclusion of the theorem is affected by an extension of the field F. Hence, without loss of generality, we assume that F is algebraically closed. Since an element of order 2 cannot be exceptional, m must be greater than 1. Let $h = g^{2^{m-2}}$ and so $h^2 = g^{2^{m-1}}$.

In proving B we will define subgroups H and H_1 such that $H \lhd G$, $H_1 \lhd H$, $h^2 \in H_1$, and g normalizes H_1. It then will be shown that if x is any element in the largest normal 2-subgroup of $H_1/H_1 \cap N$ then $(h^2, x) = (h, x)^2$. From this it will follow that h^2 is in the largest normal 2-subgroup of $H_1/H_1 \cap N$, and, finally, from this the theorem will follow.

First we need two lemmas which are of use later and which motivate the definition of H. Here, and elsewhere, we denote the space on which G operates by V.

LEMMA 3.1. *If Q is any $2'$-subgroup of G which is normalized by g, then h^2 fixes every minimal characteristic $F - Q$ submodule of V.*

Proof. A minimal characteristic $F - Q$ submodule is simply the join of all those $F - Q$ submodules operator isomorphic to a given irreducible $F - Q$ submodule. Now if Q is a $2'$-group, V can be written as the direct sum of the minimal characteristic $F - Q$ submodules. g normalizes Q so g must permute the minimal characteristic $F - Q$ submodules. If the lemma were not true, then g, as a permutation of these submodules, would have a cycle of length 2^m which would contradict the assumption that g is exceptional.

LEMMA 3.2. *If Q is any abelian $2'$-subgroup of G and x is any element of G normalizing Q and fixing every minimal characteristic $F - Q$ submodule of V, then x centralizes Q.*

Proof. Let V_i be any minimal characteristic $F - Q$ submodule of V. Since Q is abelian and F is algebraically closed, Q operates on V_i as a scalar multiplication, i.e., if $y \in Q$ and $v \in V_i$ then $yv = \chi_i(y)v$ where $\chi_i(y)$ is a scalar. We now obtain

$$\chi_i(x^{-1}yx)v = x^{-1}y(xv) = x^{-1}\chi_i(y)xv = \chi_i(y)v .$$

Thus (y, x) is the identity on V_i for all $y \in Q$ and the lemma follows.

Now let H be the normal subgroup of G consisting of all elements which fix every minimal characteristic $F - Q$ submodule for every normal $2'$-subgroup Q. Since the largest normal 2-subgroup and the largest normal $2'$-subgroup of H are normal in G, we see that H has no normal 2-subgroup greater than the identity and the largest normal $2'$-subgroup of H is $H \cap N$. By Lemma 3.1 h^2 must belong to H.

Let M be the largest normal nilpotent subgroup of H. Clearly M is a $2'$-group and $M \lhd G$. Furthermore, since H is solvable, M contains its own centralizer in H [2].

LEMMA 3.3. *M is of class 2.*

Proof. Since $h^2 \in H$, h^2 does not centralize M. Thus by Lemmas 3.1 and 3.2, M is not abelian. Now let c be the class of M and suppose $c \geqq 3$. Then if $\Gamma_i(M)$ is the ith term in the lower central series of M ($\Gamma_1(M) = M$ and $\Gamma_{i+1}(M) = (\Gamma_i(M), M)$) and if d is the first integer $\geqq (c+1)/2$, we have [3, Chap. 10]

$$(\Gamma_d(M), M) = \Gamma_{d+1}(M) \neq 1 \text{ (since } d \leqq c - 1) ,$$

and

$$(\Gamma_d(M), \Gamma_d(M)) \leqq \Gamma_{2d}(M) = 1 .$$

Thus $\Gamma_d(M)$ is abelian and, of course, normal in G but is not centralized by M. From Lemma 3.2 and the definition of H we see that this is impossible, and so $c = 2$.

$M = M_1 \times M_2 \times \cdots$ where M_i is the Sylow q_i-subgroup of M and q_i is an odd prime. Each M_i is of class at most 2 and so M_i is a regular q_i-group [3, p. 183]. Then the elements of order at most q_i form a characteristic subgroup K_i of M_i. Let $K = K_1 \times K_2 \times \cdots$ An automorphism of M_i of order prime to q_i centralizes K_i only if it is the identity automorphism [7, Hilfssatz 1.5]. Therefore no 2-element of H, except for the identity, centralizes K. Hence K cannot be abelian (since h^2 is a nonidentity 2-element of H) and so K must be of class 2.

We now are prepared to define the subgroup H_1. For this purpose decompose V for each K_i into the sum

$$V = V_{i1} \oplus V_{i2} \oplus \cdots$$

where the V_{ij} are the minimal characteristic $F - K_i$ submodules. Let

$C_{ij} = \{x \mid x \in H \text{ and } (K_i, x) = 1 \text{ on } V_{ij}\}$. C_{ij} is a normal subgroup of H although not necessarily normal in G.

Take H_1 to be the intersection of all the C_{ij} which contain h^2. If h^2 is not in any C_{ij} then set H_1 equal to H. In any event $H_1 \lhd H$ and H_1 is normalized by g. As was the case with H, H_1 has no normal 2-subgroup greater than the identity and the greatest normal 2′-subgroup is $H_1 \cap N$.

Now let P be a 2-subgroup of H_1 such that P and g belong to the same Sylow 2-subgroup of G and $P(H_1 \cap N)/(H_1 \cap N)$ is the largest normal 2-subgroup of $H_1/(H_1 \cap N)$. Since, modulo N, P is normalized by g, it follows that g normalizes P.

LEMMA 3.4. *If* $x \in P$, *then* $(h^2, x) = (h, x)^2$.

Proof. First we show that this lemma finishes the proof of Theorem B: h normalizes P so that $(h, x)^2 \in \Phi(P)$ where $\Phi(P)$ is the Frattini subgroup of P. Thus the lemma implies that h^2 centralizes $P/\Phi(P)$. Therefore from [4] we conclude that $h^2 \in P$. Since h^2 is in the greatest normal 2-subgroup of $H_1/(H_1 \cap N)$, it follows that h^2 is in the greatest normal 2-subgroup of $H/(H \cap N)$ from which the conclusion of Theorem B follows.

To prove the lemma, let $k = (h^2, x)(h, x)^{-2}$ and suppose $k \neq 1$. Since k cannot centralize K, (K_i, k) is not the identity on some V_{ij}. Since $k \in H_1$, we must have (K_i, h^2) also not the identity on V_{ij}. (This last statement is the motivation for our choice of H_1).

In what follows let $V' = V_{ij}$, $q = q_i$, and Q, x_1, k_1 the restrictions of K_i, x, k, respectively, to V'. Let $g^{2^{m-n}}$ be the first power of g fixing V' and let g_1 be the restriction of $g^{2^{m-n}}$ to V'. Now h^2 is not the identity on V' and [4, p. 13] g_1 must be exceptional

$$\text{(i.e.,}\quad (g_1 - 1)^{2^{n-1}} = 0)\ ,$$

and thus n must be at least 2. Let $h_1 = g_1^{2^{n-2}}$. $k_1 = (h_1^2, x_1)(h_1, x_1)^{-2}$ and both (Q, h_1^2) and (Q, k_1) are not the identity.

Since g_1 is exceptional and $(Q, h_1^2) \neq 1$, Q cannot be abelian. Thus Q must be of class 2. V' is the sum of absolutely irreducible $F - Q$ submodules all of which are operator isomorphic to each other. Hence $Z(Q)$, the center of Q, is cyclic and is generated by a scalar matrix. Since Q is of exponent q and $Q' \neq 1$, we see that

$$Z(Q) = Q' = \Phi(Q)$$

and so Q is an extra-special q-group [4, p. 15]. We note also that if S is the 2-group generated by x_1 and g_1, then $(Z(Q), S) = 1$ since $Z(Q)$ is generated by a scalar matrix.

Now let V'' be an irreducible $F-QS$ submodule of V'. V'' is an irreducible $F-Q$ module [4, Lemma 2.2.3], and V' is the sum of $F-Q$ modules operator isomorphic to V''. Thus $(Q, h_1^2) \neq 1$ on V'' and g_1 is exceptional on V''. From [4, Theorem 2.5.4] we have the following:

(1) $2^n - 1$ is a power of q, and

(2) if g_1 is faithfully and irreducibly represented on Q_1/Q' (such a Q_1 can always be found since h^2 is not the identity on Q/Q'), then Q can be written as the central product of Q_1 and a group Q_2 and g_1 transforms Q_2 trivially. It now follows [6] that $2^n - 1 = q$ and $|Q_1/Q'| = q^2$.

The representation of Q on V'' is isomorphic to the representation of Q on V' so that $(g_1, Q_2) = 1$ on V'' implies that $(g_1, Q_2) = 1$. Thus the centralizer of g_1 in the space Q/Q' has co-dimension 2 over $GF(q)$. The minimal equation of h_1 on Q_1/Q' must be $t^2 + 1 = 0$ so that h_1^2 must have the representation

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

on Q_1/Q'. We now can conclude that for every power of g_1 (except for the identity, of course), the co-dimension of its centralizer in Q/Q' is 2. Also, since $q \equiv 3 \pmod 4$, $GF(q)$ contains no primitive 4th root of unity. Thus if $n \neq 2$ then in the completely reduced representation of g_1^2 on Q/Q' there is only one nontrivial block. If $n = 2$, there are two nontrivial blocks.

Now if c is a generator of Q', define $\rho(a, b)$ for $a, b \in Q$ by the equation

$$(a, b) = c^{\rho(a,b)} .$$

$\rho(a, b)$ is bilinear and skew symmetric and gives Q/Q' the structure of a symplectic space over $GF(q)$ [4].

ρ is of maximum rank since $Q' = Z(Q)$ so Q/Q' must have dimension $2r$. Since $(S, Q') = 1$, S preserves the symplectic structure of Q/Q'. Thus the representation of S on Q/Q' may be considered as a subgroup of a Sylow 2-subgroup of the symplectic group on Q/Q'.

Q/Q' is of dimension $2r$ over $GF(q)$ so that Q/Q' can be provided with the structure of a vector space U of dimension r over $GF(q^2)$. If $u_1, \cdots, u_r$ is a basis for U, the expression [4]

$$\rho(\Sigma \alpha_i u_i, \Sigma \beta_i u_i) = \Sigma(\alpha_i \beta'_i - \alpha'_i \beta_i)/\gamma ,$$

where $\alpha' = \alpha^q$ and γ is a primitive 4th root of unity, is a skew symmetric bilinear form on U of rank $2r$ with values in $GF(q)$.

Let θ be a primitive 2^{n+1}-th root of unity in $GF(q^2)$ and let T be the group of transformations of $GF(q^2)$ generated by the two transformations $\alpha \to \theta^2\alpha$ and $\alpha \to \theta\alpha'$. All transformations $\boldsymbol{y}$ of U of the form

$$\boldsymbol{y}(\Sigma\alpha_i u_i) = \Sigma(T_i\alpha_i)u_{\sigma(i)} ,$$

where the T_i are taken from T and σ is a permutation taken from a Sylow 2-subgroup of the symmetric group on the numbers $1, 2, \cdots, r$, form a Sylow 2-subgroup of the Symplectic group on Q/Q' [4].

Thus we may assume that $\boldsymbol{x}_1, \boldsymbol{g}_1, \boldsymbol{h}_1$, the representations of x_1, g_1, h_1, respectively, on Q/Q', are of this form. Since $(Q, h_1^2) \neq 1$ and $(Q, k_1) \neq 1$, we have $\boldsymbol{h}_1^2 \neq 1$ and $(\boldsymbol{h}_1^2, \boldsymbol{x}_1) \neq (\boldsymbol{h}_1, \boldsymbol{x}_1)^2$. We now need more information on $\boldsymbol{g}_1$.

LEMMA 3.5. *The permutation σ associated with $\boldsymbol{g}_1$ is the identity permutation.*

Proof. σ is of order less than the order of g_1 from [4, p. 23]. First suppose σ is of order > 2. Then $n > 2$ and so the representation of g_1^2 on Q/Q' has only one nontrivial irreducible block. But the permutation associated with $\boldsymbol{g}_1^2$ is σ^2 which has at least 2 disjoint nontrivial cycles. Clearly this is a contradiction. Thus $\sigma^2 = 1$.

Now suppose $\sigma \neq 1$. Assume, say, $\sigma(1) = 2$, $\sigma(2) = 1$. The representation of g_1 on Q/Q' has only one nontrivial irreducble block so $\boldsymbol{g}_1$ must be the identity on

$$\sum_{i \neq 1,2} \alpha_i u_i .$$

Now $\boldsymbol{g}_1^2(\alpha_1 u_1 + \alpha_2 u_2) = T_2 T_1 \alpha_1 u_1 + T_1 T_2 \alpha_2 u_2$ and so one of $T_2 T_1$ or $T_1 T_2$ must not be the identity of T. But then neither one can be the identity. Therefore the representation of Q/Q' would have 2 nontrivial irreducible blocks. This can happen only if $n = 2$. This implies that $T_2 T_1$ and $T_1 T_2$ are of order 2 and thus must equal the transformation $\alpha \to -\alpha$. (This is the only element of order 2 in T.) Thus the centralizer of g_1^2 in Q/Q' has co-dimension 4 over $GF(q)$ whereas it should have co-dimension 2. This proves that $\sigma = 1$.

Hence $\boldsymbol{g}_1$ fixes each u_i and must act trivially on α_i for all but one value of i, $i = 1$, say. Therefore

$$\boldsymbol{g}_1(\Sigma\alpha_i u_i) = A\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i$$

where A is an element of order 2^n in T. Then

$$\boldsymbol{h}_1(\Sigma\alpha_i u_i) = A^{2^{n-2}}\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i ,$$

and

$$\boldsymbol{h}_1^2(\Sigma\alpha_i u_i) = -\alpha_1 u_1 + \sum_{i\neq 1}\alpha_i u_i \ .$$

We may assume that

$$\boldsymbol{x}_1(\Sigma\alpha_i u_i) = \Sigma T_i \alpha_i u_{\pi(i)} \ .$$

Case 1. $\pi(1) \neq 1$. Assume, say, that $\pi^{-1}(1) = 2$. Straight forward calculation yields

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)(\Sigma\alpha_i u_i) = A^{-2^{n-2}}\alpha_1 u_1 + T_2^{-1}A^{2^{n-2}}T_2\alpha_2 u_2 + \sum_{i\neq 1,2}\alpha_i u_i \ .$$

But $A^{2^{n-1}}$ is the unique element of order 2 in T. Thus

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)^2(\Sigma\alpha_i u_i) = -\alpha_1 u_1 - \alpha_2 u_2 + \sum_{i\neq 1,2}\alpha_i u_i$$

and it is easily verified that this is the same result as $(\boldsymbol{h}_1^2, \boldsymbol{x}_1)$.

Case 2: $\pi(1) = 1$. In this case we easily find that $(\boldsymbol{h}_1^2, \boldsymbol{x}_1)$ is the identity while

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)^2(\Sigma\alpha_i u_i) = (A^{2^{n-2}}, T_1)^2\alpha_1 u_1 + \sum_{i\neq 1}\alpha_i u_i \ .$$

Now the group T easily is seen to be a generalized quaternion group of order 2^{n+1} so that the only conjugates of A in T are A and A^{-1}. Thus

$$(A^{2^{n-1}}, T_1)^2 = A^{2^{n-1}}T_1^{-1}(A^{2^{n-1}})T_1 = 1 \ .$$

Thus $(\boldsymbol{h}_1, \boldsymbol{x}_1)$ is also the identity.

Therefore it has been shown that

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)^2 = (\boldsymbol{h}_1^2, \boldsymbol{x}_1)$$

in all cases. This completes the proof of lemma 3.4, and, by a previous argument, Theorem B now is proved.

4. Groups with $e_2 = 2$. If G is a solvable group whose Sylow 2-groups are of exponent 4, then we know from Theorem A that $l_2(G) \leqq 3$. We now investigate conditions for $l_2(G) \leqq 2$ to hold. The argument is similar to that used in proving Theorem B, but a more restrictive hypothesis is needed. That no loss of generality is involved in assuming the stronger hypothesis is insured by the following reduction theorem, which is stated in a slightly more general form than needed.

A proposition R will be said to be of type 4.1 if it is of the following form:

If G is a finite p-solvable group satisfying condition C, then

$l_p(G) \leqq f(e_p(G))$, where f is a monotonically increasing function defined for nonnegative integral arguments, $f(0) = 0$, and condition C either is vacuous or states that $e_{p_i}(G) \leqq a_i$ for some set, possibly infinite, of primes p_i and nonnegative integers a_i.

Note that the proposition that $l_2(G) \leqq e_2(G)$ if G is a finite solvable group satisfying $e_2(G) \leqq 2$ is of type 4.1. One of the results of this section is that $l_2(G) \leqq e_2(G)$ if G is a finite group of exponent 12. This statement is also of type 4.1 since the condition that G be of exponent 12 is equivalent to stating that $e_2(G) \leqq 2$, $e_3(G) \leqq 1$, and $e_p(G) \leqq 0$ for all other primes.

THEOREM 4.1. *To prove a proposition R of type* 4.1 *it is sufficient to prove the proposition for the following special case*:

(1) *G is the normal product of V by G_1 where V is a vector space over F, a finite field of characteristic p, and G_1 is a p-solvable linear group on V having no normal p-subgroup other than the identity.*

(2) *Any irreducible representation of any p'-subgroup of G_1 over F is in fact absolutely irreducible.*

(3) *All groups of order at most $|G_1|$ satisfy R.*

(4) *V is an irreducible $F - G_1$ module.*

Proof. In proving this theorem we assume R is valid for the special case and then prove it is valid for the general case.

Now suppose G is the group of smallest order which satisfies the hypothesis but not the conclusion of R, and let

$$1 = P_0 \leqq N_0 < P_1 < \cdots < P_l \leqq N_l = G$$

be the upper p-series of G. Since $f(0) = 0$ we must have $l_p(G) > 0$. If F_1/N_0 is the Frattini subgroup of P_1/N_0, then, as is shown in [4], $l_p(G/F_1) = l_p(G)$ so that if $F_1 \neq 1$ we would have a proper factor group of G satisfying the hypothesis but not the conclusion of R.

Hence assume $F_1 = 1$. Thus P_1 is an elementary abelian p-group which we identify with a vector space V_1 over $GF(p)$. G/P_1 is faithfully represented as a linear group G_1 on V_1 and G_1 has no normal p-group greater than the identity.

From [4, p. 4] we may assume that G has only one minimal normal subgroup. This subgroup must be contained in V_1 and we denote it with M. If $M \neq V_1$ and G_1 is faithfully represented on V_1/M then we have $l_p(G/M) = l_p(G)$ so that we would have a contradiction to the minimality of G.

Now suppose $M \neq V_1$ and G_1 is not faithfully represented on V_1/M.

Then the elements of G_1 centralizing V_1/M form a normal subgroup of G_1 greater than the identity. If Q is a minimal normal subgroup of G_1 centralizing V_1/M, then Q must be a p'-group so that V as a Q-module is completely reducible. Thus there exists a Q-module M_1 such that $V_1 = M \oplus M_1$. Q is the identity on M_1 but not on M since Q is faithfully represented on V_1. Now if M_2 is the centralizer of Q in V_1 then M_2 is normal in G, M_2 is not the identity, and M_2 does not contain M. This contradicts the minimality of M.

Thus we see that $M = V_1$ which implies that G_1 is irreducibly represented on V_1. A consequence of this is that if H is any normal subgroup greater than the identity in G_1 then H can have no nonzero fixed vector in V_1 Otherwise all the vectors fixed by H would form a nontrivial submodule of V_1.

Now pick F to be a large enough finite extension of $GF(q)$ such that any irreducible representation of any p'-subgroup of G_1 over F is absolutely irreducible. Let $1 = \theta_0, \theta_1, \cdots, \theta_r$ be a basis for F over $GF(p)$ and let $v_1, v_2, \cdots, v_s$ be a basis for V_1 over $GF(p)$. Finally let V be the vector space over F with basis $v_1, \cdots, v_s$, i.e., the vectors of V are the formal sums

$$\sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij}\theta_i v_j$$

where $c_{ij} \in GF(p)$. G_1 acts on V in the obvious way.

Consider the group $G^* = G_1 V$, i.e., the normal product of V by G_1. If g^* is of order p^m in G^* then either the image g of g^* in G_1 is of order p^m or g is of order p^{m-1} and g is not exceptional on V. In the latter case $(g - 1)^{p^{m-1}} v_i \neq 0$ for some v_i from which it follows that g is not exceptional on V_1. Thus $e_p(G) \geqq (m - 1) + 1 = m$.

Therefore in any event $e_p(G) \geqq e_p(G^*)$. Since $e_q(G^*) = e_q(G)$ for $q \neq p$, G^* satisfies condition C. Furthermore $l_p(G) = l_p(G^*)$ so that if G^* satisfies R so does G.

Now suppose H is any normal p'-subgroup other than the identity in G_1 and suppose

$$v = \sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij}\theta_i v_j$$

is a nonzero vector fixed by H. Since $v \neq 0$ the coefficient of v_j is not zero for some j, $j = 1$ say. Then there exists $\alpha \in F$ such that $\alpha(\sum_{i=0}^{r} c_{i1}\theta_1) = 1$. H must fix αv which can be written in the form $\alpha v = v' + v''$ where

$$v' = v_1 + \sum_{j=2}^{s} c'_{0j} v_j,\ v'' = \sum_{j=2}^{s} \sum_{i=1}^{r} c'_{ij}\theta_i v_j\ .$$

For H to fix αv it must also fix v' which contradicts the fact that

H has no nonzero fixed vector in V_1. Thus H has no nonzero fixed vector in V.

If V is an irreducible $F-G_1$ module then we have arrived at the special case of the theorem. Therefore assume U is a proper submodule.

If G_1 is not faithfully represented on V/U, then let Q be a minimal normal subgroup of G_1 centralizing V/U. Q must be a p'-group so that V is completely reducible as an $F-Q$ module. Thus there exists a nontrivial $F-Q$ submodule on which Q is the identity. This is impossible since Q can have no nonzero fixed vector.

Hence G_1 is faithfully represented on V/U. Thus $l_p(G^*) = l_p(G^*/U)$ and, of course, $e_p(G^*) \geqq e_p(G^*/U)$ so that if G^*/U satisfies R so does G^* and then so does G.

We still have that any normal nonidentity p'-subgroup H of G_1 has no nonzero fixed vector in V/U since V is completely reducible as an $F-H$ module. Therefore if G_1 is not irreducibly represented on V/U then the same argument as before yields that G_1 is faithfully represented on a nontrivial factor module of V/U. Continuing in this way we ultimately arrive at the special case where G_1 is faithfully and irreducibly represented on some vector space over the field F. This finishes the proof of Theorem 4.1.

Among the results we now shall prove is that if G is of exponent 12 then $l_2(G) \leqq e_2(G)$. Before doing this it might be well to justify this work. For in a group of order $2^a 3^b$ the 2-length and the 3-length can vary at most by one. Thus if it were true that the 3-length of a group of exponent 12 was one, then it would be trivial to state that the 2-length was at most two. However in [5, p. 5] is found an example of a group of exponent 12 but with 3-length two.

For the rest of this paper we make the following standing assumptions.

(1) $G = G_1V$, the normal product of V by G_1, where V is a vector space over a finite field F of characteristic 2 and G_1 is a finite, solvable linear group having no normal 2-subgroup other than the identity.

(2) V is an irreducible $F-G_1$ module.

(3) Any representation over F of any p'-subgroup of G_1 is absolutely irreducible.

(4) $e_2(G) \leqq 2$.

We are interested in seeing under what conditions can $l_2(G)$ exceed $e_2(G)$. But if $e_2(G_1) = 0$ then both $e_2(G)$ and $l_2(G)$ are 1, and if $e_2(G_1) = 1$ then $l_2(G_1) = 1$ so that $l_2(G) = e_2(G) = 2$. Thus we may as well assume

(5) $e_2(G_1) = 2$.

Later we shall add to these assumptions the further one that G' is of exponent 12. Actually, until we restrict ourselves to groups of exponent 12, we will make no use of the fact that G_1 is irreducibly represented on V.

Now let N be the largest normal $2'$-subgroup of G_1. We shall show that a certain 2-subgroup, to be described later, must be contained in the greatest normal 2-subgroup of G_1/N. In particular if $l_2(G) > 2$ (which is the same as $l_2(G_1) > 1$), we shall see that there must exist an element of order 4 of a special type in G_1.

First let H be the following normal subgroup of G_1: $x \in H$ if, and only if, for every normal nilpotent subgroup Q of class at most 2 in G_1, x fixes every minimal characteristic $F - Q$ submodule of V. A normal nilpotent subgroup of G_1 must be a $2'$-group so that V splits into the sum of minimal characteristic $F - Q$ modules.

From (5) there are elements of order 4 in G_1, and from (4) all such elements must be exceptional. Thus if g is of order 4 in G_1 then g^2 must be in H by lemma 3.1. Hence H is greater than the identity. H has no normal 2-subgroup except for the identity and the largest normal $2'$-subgroup is $H \cap N$.

Let D be the greatest normal nilpotent subgroup of H. $D = D_1 \times D_2 \times \cdots$ where D_i is a Sylow q_i-subgroup of D for an odd prime q_i. H centralizes any normal abelian subgroup of G_1 so that, by the proof of Lemma 3.3, we obtain $c(D) = 2$. Now, as before, let K_i be the subgroup of D_i consisting of all elements of order at most q_i and let $K = K_1 \times K_2 \times \cdots$ We again have that no non-identity 2-element of H centralizes K.

Now take H_1 to be the subgroup of G_1 consisting of all elements which fix every minimal characteristic $F - K_i$ module for all i. $H_1 \lhd G_1$, and, since $c(K_i) \leqq 2$, $H \leqq H_1$. H_1 has no normal 2-subgroup except for the identity and its greatest normal $2'$-subgroup is $H_1 \cap N$.

Let P be a Sylow 2-subgroup of H_1. $P \neq 1$ since if g is any element of order 4 in G_1 then $g^2 \in H$. Now the square of any element of P must be in H. Thus $P/(P \cap H)$ is of exponent 2 and thus abelian. Therefore $P' < H$. We now prove two lemmas which enable us to show directly that PN/N is normal in G_1/N.

LEMMA 4.2. *Suppose that g and h are two elements of P and V' is a minimal characteristic $F - K_i$ submodule of V. Let Q, g_1, and h_1 be the restrictions of K_i, g, and h, respectively, to V'. Then if $(Q, h_1^2) = 1$ it follows that $(Q, (g_1, h_1)) = 1$.*

Proof. Assume $(Q, (g_1, h_1)) \neq 1$. Therefore neither g_1 nor h_1 centralizes Q. If $(Q, g_1^2) = 1$, then straight forward calculation yields

$$(Q, (g_1h_1)^2) = (Q, (g_1, h_1)) \neq 1 ,$$
$$(Q, (g_1h_1, h_1)) = (Q, (g_1h_1)^{-1}) \neq 1 .$$

Thus, replacing g_1 by g_1h_1 if $(Q, g_1^2) = 1$, we may assume that $(Q, g_1^2) \neq 1$ along with $(Q, h_1^2) = 1$ and $(Q, (g_1, h_1)) \neq 1$.

Now exactly as in the proof of Lemma 3.4 we obtain that Q is an extra special q-group (actually $q = 3$ since g_1 is of order 4 and thus exceptional so that $4 - 1$ must be a power of q), Q/Q' is a symplectic space, g_1 and h_1 preserve the symplectic structure of Q/Q', and we may assume that g_1 and h_1 operate on Q/Q' as follows:

$$g_1(\Sigma\alpha_iu_i) = A\alpha_1u_1 + \sum_{i\neq1} \alpha_iu_i ,$$
$$h_1(\Sigma\alpha_iu_i) = \Sigma T_i\alpha_iu_{\sigma(i)} ,$$

where σ is a permutation of order $\leqq 2$ (since $(Q, h_1^2) = 1$), and A and the T_i are chosen from a group isomorphic to the quaternion group of order 8 (since $q = 3$). In addition A must be of order 4 since $(Q, g_1^2) \neq 1$.

If σ does not fix 1 then (g_1, h_1) would be of order 4 but its centralizer in Q/Q' would have co-dimension 4 over $GF(3)$. Thus (g_1, h_1) would be of order 4 but not exceptional which is impossible.

Hence σ fixes 1 and, since $(Q, h_1^2) = 1$, we must have

$$h_1(\Sigma\alpha_iu_i) = \pm\alpha_1n_1 + \sum_{i\neq1} T_i\alpha_iu_{\sigma(i)} .$$

It is now an easy matter to verify that $(g_1, h_1) = 1$ and the lemma is proved.

COROLLARY. *If $g, h \in P$ and $h^2 = 1$, then $(g, h) = 1$.*

Proof. (g, h) is in P' and thus in H. So if $(g, h) \neq 1$ then $(K_i, (g, h)) \neq 1$ for some K_i. Then lemma states that this cannot happen.

LEMMA 4.3. *If $g, h \in P$, then $(g, h)^2 = 1$.*

Proof. Suppose that $(g, h)^2 \neq 1$. Then for some K_i, $(K_i, (g, h)^2) \neq 1$. Choose V' to be a minimal characteristic $F - K_i$ submodule of V such that $(K_i, (g, h)^2)$ is not the identity on V'. If Q, g_1, and h_1 are defined as in the previous lemma, then, if either (Q, g_1^2) or (Q, h_1^2) is the identity, $(g_1, h_1) = 1$. Therefore assume neither g_1^2 nor h_1^2 centralize Q. Thus g_1 and h_1 are both exceptional of order 4. Q is an extra-special

3-group and we may assume g_1 and h_1 operate on Q/Q' as follows:

$$g_1(\Sigma\alpha_i u_i) = A\alpha_1 u_1 + \sum_{i\neq 1} \alpha_i u_i \, ,$$
$$h_1(\Sigma\alpha_i u_i) = B\alpha_j u_j + \sum_{i\neq j} \alpha_i u_i \, .$$

Now if $j \neq 1$ then $(g_1, h_1) = 1$ and if $j = 1$ then

$$(g_1, h_1)^2(\Sigma\alpha_i u_i) = (A, B)^2\alpha_1 u_1 + \sum_{i\neq 1} \alpha_i u_i \, .$$

But A and B are elements of a quaternion group so that $(A, B)^2$ is the identity and the lemma is proved.

THEOREM 4.4. *$PN/N \lhd G_1/N$.*

Proof. We shall prove that $P(H_1 \cap N)/(H_1 \cap N) \lhd H_1/(H_1 \cap N)$ which is equivalent to the theorem since $H_1 \lhd G_1$.

Let P_1 be the subgroup of P such that $P_1(H_1 \cap N)/(H_1 \cap N)$ is the largest normal 2-subgroup of $H_1/(H_1 \cap N)$. $P_1 \lhd P$ and P_1 contains the center of P [4, Lemma 1.2.3]. Thus by the corollary to Lemma 4.2, P_1 contains all elements of order 2 in P. The elements of order 2 in P form an elementary abelian group P_2 which is normal, modulo $H_1 \cap N$, in H_1. The elements of $H_1/(H_1 \cap N)$ which centralize both P_2 and P_1/P_2 form a normal subgroup of $H_1/(H_1 \cap N)$. But if any $2'$-element centralized both P_2 and P_1/P_2, then, as easily may be seen, this element would centralize P_1 contrary to the fact [4, Lemma 1.2.3] that P_1 contains its centralizer in $H_1/(H_1 \cap N)$. Thus the elements centralizing both P_2 and P_1/P_2 form a normal 2-subgroup of $H_1/(H_1 \cap N)$, and from the corollary to Lemma 4.2 and from Lemma 4.3, P must be contained in this normal 2-subgroup. But P is a Sylow 2-subgroup of H_1 and thus it follows that, modulo $H_1 \cap N$, P is normal in H_1.

COROLLARY. $l_2(H_1) = 1$.

Now let S be a Sylow 2-subgroup of G_1 which contains P. From the theorem it follows that P is normal in S.

LEMMA 4.5. *If P contains all elements of order 4 in S, then $l_2(G_1) = 1$.*

Proof. If $S = P$ we are done. Therefore assume $S \neq P$. Then if $x \in S - P$ we must have $x^2 = 1$. Also $x \in S - P$, $y \in P$ imply that $xy \in S - P$ so that $(xy)^2 = 1$ which implies that $x^{-1}yx = y^{-1}$. Thus x induces the automorphism $y \rightarrow y^{-1}$ of P. This can be an automorphism only if P is abelian. Now if both x_1 and x_2 are in $S - P$ then x_1x_2

centralizes P. But $e_2(G_1)=2$ so that P does contain elements of order 4. Hence x_1x_2 cannot be in $S-P$.

Therefore $|S/P|=2$ and P is abelian. Now if $x\in S-P$, $y\in P$, then $(x,y)=x^{-1}y^{-1}xy=y^2\in\Phi(P)$ and thus x centralizes $P/\Phi(P)$. Hence [4, Lemma 1.2.5] PN/N connot be the largest normal 2-subgroup of G_1/N. But P is maximal in S so that SN/N must be the largest normal 2-subgroup of G_1/N. This implies that $l_2(G_1)=1$.

To our assumptions (1)∼(5) we now add

(6) G is of exponent 12.

This implies that K must be a group of exponent 3 and class at most 2. We prove that $l_2(G_1)=1$ in this case by showing that the hypothesis of Lemma 4.5 are satisfied.

For this purpose assume that g is an element of order 4 in $S-P$. g^2 is in H so $(K,g^2)\neq 1$. Let $V=V_1\oplus V_2\oplus\cdots$ be the decomposition of V into minimal characteristic $F-K$ modules. Since $g\in S-P$, g does not fix some V_i. g^2 does fix each V_i and if g^2 is not the identity on a V_i then g must fix that V_i for otherwise g could not be exceptional [4, p. 13]. We now need the following result:

LEMMA 4.6. *There exist x and y in K such that* $((x,g^2),(y,g^2))\neq 1$.

Proof. Let $C=\{x\mid x\in K,(x,g^2)\in Z(K)\}$. Clearly $C\geqq Z(K)$ but $C\neq K$ since then g^2 would centralize $Z(K)$ and $K/Z(K)$ which would imply that $(K,g^2)=1$. (g^2 centralizes $Z(K)$ by Lemma 3.1 and 3.2.) $K/Z(K)$ is an elementary abelian 3-group so that there must be a $GF(3)-g$ module of $K/Z(K)$ complementary to $C/Z(K)$. Thus $K/Z(K)=L/Z(K)\oplus C/Z(K)$ and g normalizes L. For all $x\in L-Z(K)$, (x,g^2) is not in $Z(K)$.

Now suppose $x,y\in L-Z(K)$ and $(x,g^2)(y,g^2)^{-1}\in Z(K)$. Since $K/Z(K)$ is abelian, straight forward calculation yields

$$(xy^{-1},g^2)\equiv(x,g^2)(y^{-1},g^2)\pmod{Z(K)},$$
$$1=(yy^{-1},g^2)\equiv(y,g^2)(y^{-1},g^2)\pmod{Z(K)}.$$

Thus $(xy^{-1},g^2)\equiv(x,g^2)(y,g^2)^{-1}\equiv 1$ (mod $Z(K)$. This implies that $xy^{-1}\in Z(K)$. Therefore we have shown that $(x,g^2)\equiv(y,g^2)(\text{mod } Z(K))$ if, and only if, $x\equiv y$ (mod $Z(K)$) for $x,y\in L$.

It immediately follows from this that for any $x\in L$, there exists a y such that $x\equiv(y,g^2)(\text{mod } Z(K))$. Now L cannot be abelian since g normalizes L and g^2 does not centralize it. From all this we see that there exist $x,y\in L$ such that $((x,g^2),(y,g^2))\neq 1$.

Now taking x and y to satisfy the lemma, we may assume without loss of generality that $((x, g^2), (y, g^2))$ is not the identity on V_1. This implies that g^2 is not the identity on V_1 so g must fix V_1.

Since g does not fix every V_i, assume g does not fix V_2. Therefore g^2 is the identity on V_2 which then also must be the case for (x, g^2) and (y, g^2).

V is an irreducible $F - G_1$ module so that there must be an element taking V_1 into V_2. Such an element must be of the form zh where $h \in S$ and z is from a Sylow 3-subgroup of G_1 which necessarily must contain K. We shall derive a contradiction by showing that z and K generate elements of order 9 which is impossible in a group of exponent 12.

If $hV_1 = V_m$ then $zV_m = V_2$. Set $g_1 = hgh^{-1}$. Then

$$((x^{h^{-1}}, g_1^2), (y^{h^{-1}}, g_1^2))$$

is not the identity on V_m. Now suppose $g_1 V_2 = V_2$. Then $gh^{-1}V_2 = h^{-1}V_2$, and, since $gV_2 \neq V_2$, this implies that $h^{-1}V_2 = V_j, j \neq 2$. Then we would have $gV_j = V_j$. But $gh^{-1} \in S$ so that $(gh^{-1})^2 \in H$. Thus $(gh^{-1})^2$ fixes V_2 and, therefore, $gh^{-1}V_j = V_2$. $(h^{-1})^2$ also must fix V_2 so we have $h^{-1}V_j = V_2$. From this we conclude that $V_2 = gh^{-1}V_j = gV_2$ which is a contradiction. Hence $g_1V_2 \neq V_2$. A consequence of this is that $V_m \neq V_2$ for $V_m = V_2$ would imply that $h^{-1}V_2 = V_1$ which would imply that $g_1V_2 = hgV_1 = V_2$. Since $V_m \neq V_2$ it follows that z is not the identity and so is of order 3.

If we replace V_1, g, x, and y by $V_m, g_1, x^{h^{-1}}$, and $y^{h^{-1}}$, respectively, we may assume that $zV_1 = V_2, gV_2 \neq V_2$, and $((x, g^2), (y, g^2))$ is not the identity on V_1. Let $x_1 = (x, g^2)$ and $y_1 = (y, g^2)$. x_1 and y_1 must be the identity on V_2 since g_2 is. Since z is of order 3, we have $zV_1 = V_2$, $zV_2 = V_n (n \neq 1, 2)$, and $zV_n = V_1$.

Let $V' = V_1 \oplus V_2 \oplus V_n$. V' is fixed by z and the restrictions of x_1, y_1, and z to V' are

$$z = \begin{pmatrix} 0 & 0 & A \\ B & 0 & 0 \\ 0 & C & 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_1 \end{pmatrix}, \quad y_1 = \begin{pmatrix} N & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & N_1 \end{pmatrix},$$

where I is the identity and 0 the zero matrix. Now (x_1, y_1) is not the identity on V_1 but $(x_1, y_1) \in Z(K)$ and $Z(K)$ is represented on V_1 as a cyclic group generated by a scalar matrix. Thus $(M, N) = \omega I$ where ω is a primitive third root of unity. From $z^3 = 1$ we obtain $C = A^{-1}B^{-1}$.

Now z, x_1, and y_1 all belong to the same Sylow 3-subgroup of G_1. Thus $(zx_1)^3 = (zy_1)^3 = 1$. From this direct computation yields that $M_1 = A^{-1}M^{-1}A, N_1 = A^{-1}N^{-1}A$. Thus $(M_1, N_1) = A^{-1}(M^{-1}, N^{-1})A$. But M and N generate a group of exponent 3 and class 2. It follows easily that $(M^{-1}, N^{-1}) = (M, N) = \omega I$. Thus

$$(x_1, y_1) = \begin{pmatrix} \omega I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \omega I \end{pmatrix}.$$

It is now a simple matter to verify that $(z(x_1, y_1))^3 \neq 1$. Hence $z(x_1, y_1)$ is a 3-element of order greater than 3 which is impossible in a group of exponent 12. This contradiction proves that the hypothesis of Lemma 4.5 is satisfied and thus:

THEOREM 4.7. *If G is a finite group of exponent* 12, *then* $l_2(G) \leqq e_2(G)$.

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ALGEBRAS OF BOUNDED SEQUENCES

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Let l^∞ be the algebra of all bounded sequences of complex numbers. The primary purpose of this paper is to settle, by means of a counter-example, a conjecture about subalgebras of l^∞: If A is a subalgebra of l^∞ which is closed under uniform convergence and separates the points of βN, then $A = l^\infty$.

The main tool used in the construction of the example is a positive result about Boolean algebras (Proposition 1), which seems interesting in its own right.

Our interest in subalgebras of l^∞ stems from recent work on interpolation problems. In 1957, R. C. Buck inquired as to which sequences $\{z_n\}$ in the unit disc of the plane have this property: If $\{w_n\}$ is an arbitrary bounded sequence of complex sequence of complex numbers, there exists a function f, bounded and analytic in the unit disc, such that $f(z_n) = w_n$ for each n. This question was answered very effectively by L. Carleson [2]. It was also answered in a slightly weaker form by D. J. Newman [7]; and, it was partially answered by W. Hayman [5].

Badé and Curtis [1] have obtained strong results on subalgebras of $C(X)$, the algebra of continuous complexvalued functions on a compact Hausdorff space X, in case X has the property that any two disjoint open F_σ subsets have disjoint closures. The smallest nonfinite space with that property is βN, the Čech compactification of the integers. Since l^∞ is isomorphic to $C(\beta N)$, the work of Badé and Curtis has shed some light on a general class of interpolation problems of the type which we previously mentioned [see 6; page 205].

Technical aspects of the work of Badé and Curtis, as well as similar aspects of the interpolation problem for bounded analytic functions, have led to the aforementioned conjecture. Were this conjecture true, it would be a powerful tool in the study of interpolation problems.

We shall settle the conjecture in the negative. We show that, if the continuum hypothesis is valid, there exists a subalgebra A of l^∞ such that

(i) A is closed under uniform convergence;

(ii) every nonnegative sequence in l^∞ is the modulus of a sequence in A;

(iii) $A \neq l^\infty$.

The example is constructed as follows. Let $\mathscr{B}$ be the Boolean

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algebra of measurable subsets of the unit circle, modulo null sets. Let B be the Boolean algebra of subsets of the integers modulo finite sets. As a lattice, B has an unusual property (property (ii) of Lemma 4). We exploit that property and the continuum hypothesis, to produce an isomorphism of $\mathscr{B}$ into B. We extend the inverse of the isomorphism to a homomorphism from B onto $\mathscr{B}$. The adjoint of such a homomorphism is a homeomorphism of the Stone space $S(\mathscr{B})$ into the Stone space $S(B)$. The space $S(\mathscr{B})$ is the maximal ideal space of L^∞, the Banach algebra of essentially bounded measurable functions on the unit circle. The space $S(B)$ is $\beta N - N$. From the algebra of bounded analytic functions in the unit disc, we construct a subalgebra of $L^\infty = C(S(\mathscr{B}))$, and carry that algebra with the imbedding of $S(\mathscr{B})$ in $\beta N - N$, to obtain the desired subalgebra of l^∞.

Homomorphisms of Boolean algebras. In a Boolean algebra B, denote least upper bounds by $\vee$, greatest lower bounds by $\wedge$, and complements by $'$. Denote the corresponding set operations by $\cup$, $\cap$, and $'$. If $a \leqq b$, let $[a, b]$ be the set of all x in B such that $a \leqq x \leqq b$. If E is a subalgebra of B and x is an element of B, let $E[x]$ be the subalgebra generated by E and x:

$$E[x] = \{(e \wedge x) \vee (f \wedge x'); e, f \in E\} .$$

We shall be concerned with the problem of extending Boolean algebra homomorphisms. The basic extension lemma is the following, implicitly given by Sikorski in [8].

LEMMA 1. *Let E be a subalgebra of the Boolean algebra B, and let φ be a homomorphism of E into a Boolean algebra C. If $x \in B$, then φ can be extended to a homomorphism of $E[x]$ into C if and only if there exists an element y in C such that*

(i) $\varphi([0, x] \cap E) \subseteqq [0, y] \cap \varphi(E)$

(ii) $\varphi([x, 1] \cap E) \subseteqq [y, 1] \cap \varphi(E)$.

For each such y, there is a unique extension φ^+ which takes x into y; and then $\varphi^+(E[x]) = \varphi(E)[y]$. If φ is an imbedding and y is such that equality holds in (i) *and* (ii), *then the corresponding extension φ^+ is also an imbedding.*

Proof. We content ourselves with the remark that, if y satisfies (i) and (ii), the extension φ^+ is given by

$$\varphi^+((e_1 \wedge x) \vee (e_2 \wedge x')) = (\varphi(e_1) \wedge y) \vee (\varphi(e_2) \wedge y') .$$

LEMMA 2. (*Sikorski*) *If B is a Boolean algebra and φ is a homomorphism of a subalgebra of B into a complete Boolean algebra C, then φ can be extended to a homomorphism of B into C.*

Proof. In the context of Lemma 1, we have

$$\sup \varphi([0, x] \cap E) \leqq \inf \varphi([x, 1) \cap E) .$$

Therefore, the element y can always be found. Apply Zorn's lemma.

Of course, Lemma 2 says that complete Boolean algebras are injective in the category of Boolean algebras and Boolean homomorphisms.

An *F-space* is a topological space in which any two disjoint open F_σ subsets have disjoint closures. A Boolean algebra is an *F-algebra* if its Stone space is an F-space. An F-algebra can also be described as one which satisfies property (I) of Badé and Curtis [1]: If $x_1, y_1, x_2, y_2, \cdots$ is an orthogonal sequence in B, then there exist orthogonal elements x and y in B, such that x is an upper bound of $\{x_1, x_2, \cdots\}$ and y is an upper bound of $\{y_1, y_2, \cdots\}$.

LEMMA 3. *A homomorphic image of an F-algebra is an F-algebra. If B is an F-algebra in which orthogonal sets are (at most) countable, then B is complete.*

Proof. Let B and C be Boolean algebras, and let φ be a homomorphism from B onto C. If $c_1, c_2, \cdots$ is an orthogonal sequence in C, there can be defined (by induction) an orthogonal sequence of pre-images in B. Choose any b_1 such that $\varphi(b_1) = c_1$. If $b_1, \cdots, b_k$ are orthogonal with $\varphi(b_j) = c_j$, $1 \leqq j \leqq k$, choose any x in B with $\varphi(x) = c_{k+1}$ and set $b_{k+1} = (b_1 \vee \cdots \vee b_k)' \wedge x$. From the alternative description of F-algebras, it is then clear that C is an F-algebra if B is.

Now, suppose that B is an F-algebra and that orthogonal sets in B are at most countable. To prove that B is complete, it suffices to show that every orthogonal set in B has an upper bound. Evidently, the only case of interest is that of an orthogonal sequence $x_1, x_2, \cdots$ which is part of a maximal orthogonal set $\{x_1, y_1, x_2, y_2, \cdots\}$. Since B is an F-algebra, there exist orthogonal elements x, y in B such that x is an upper bound of $\{x_1, x_2, \cdots\}$ and y is an upper bound of $\{y_1, y_2, \cdots\}$. Then x must be the least upper bound of $\{x_1, x_2, \cdots\}$. For, if u is another upper bound of $\{x_1, x_2, \cdots\}$, then $z = x \wedge u$ is also such an upper bound. If z were less than x, we could adjoin $z' \wedge x$ to $\{x_1, y_1, x_2, y_2, \cdots\}$ and obtain a properly larger orthogonal set.

Examples of F-algebras abound. Evidently, any Boolean σ-algebra is an F-algebra. Consequently, any homomorphic image of a σ-algebra is an F-algebra. In particular, if K is a closed subset of the Stone space of a σ-algebra, then the algebra of all (relatively) open-closed subsets of K is an F-algebra. Of course, the same is true of any closed subset of the Stone space of an F-algebra. In other words, a closed subset of a (compact and totally disconnected) F-space is an F-space.

One can use Lemma 3 to show that certain homomorphic images of σ-algebras are complete. We are interested in some homomorphic images of σ-algebras which might be called anti-complete, because, they contain no strictly increasing sequence which has a least upper bound. One such algebra is the algebra of subsets of the integers modulo finite sets, i.e., the algebra of open-closed subsets of the space $\beta N - N$. The following lemma provides us with a class of algebras of the type which we have in mind.

LEMMA 4. *Let $\mathscr{B}$ be an infinite Boolean σ-algebra, with Stone space S. Let K be a nonempty closed G_δ in S such that the interior of K is empty. Let B be the Boolean algebra of (relatively) open-closed subsets of K.*

(i) *Each maximal orthogonal set in B is either finite or uncountable.*

(ii) *If $a_1 \leqq a_2 \leqq \cdots$ and $b_1 \geqq b_2 \geqq \cdots$ in B, and if $a_n < b_n$ for every n, then there is an element c in B such that $a_n < c < b_n$ for every n.*

Proof. If x is in $\mathscr{B}$, let $[x]$ be the corresponding open-closed subset of the Stone space S. The fact that K is a nonempty closed G_δ without interior means that

$$K = \bigcap_n [e_n]$$

where $e_1 > e_2 > \cdots$ is a strictly decreasing sequence in $\mathscr{B}$ and $\inf_n e_n = 0$. The map

$$\gamma(x) = [x] \cap K$$

is a homomorphism from $\mathscr{B}$ onto B, and, $\gamma(x) = \gamma(y)$ if and only if there exists an n such that $x \wedge e_n = y \wedge e_n$.

For the proof of (i), suppose that $\{a_1, a_2, \cdots\}$ is a countable orthogonal set in B. We shall show that there exists a nonzero element of B which is orthogonal to each a_n. We may assume that each a_n is nonzero. Let $x_1, x_2, \cdots$ be an orthogonal sequence in B with $\gamma(x_n) = a_n$. If k is a positive integer, then $\gamma(x_k) = a_k \neq 0$; hence, $x_k \wedge e_n \neq 0$ for all n. Thus, we must have $x_k \wedge (e_n - e_{n+1}) \neq 0$ for infinitely many values of n. From this it is clear that there is a sequence of integers $n_1 < n_2 < \cdots$ such that the elements

$$y_k = x_k \wedge (e_{n_k} - e_{n_k+1})$$

are nonzero, $k = 1, 2, \cdots$. Let $y = \sup_k y_k$. For each k, the element $y \wedge e_{n_k+1}$ is orthogonal to x_k. Therefore, $\gamma(y)$ is orthogonal to each a_k. Since

$$y \wedge e_k \geqq y \wedge e_{n_k} \geqq y_k > 0 ,$$

we see that $\gamma(y) \neq 0$.

The proof of (ii) is essentially the same as the proof that every nonempty closed G_δ in the space $\beta N - N$ has a nonempty interior [4; pages 98, 99]. First note that B has no atoms. Such an atom would correspond to a point of K which is isolated in K. The point would then be a G_δ in S. Any such point is isolated in S and must be in the interior of K. Since B has no atoms, (ii) is trivial if both sequences are eventually constant. Otherwise, the sequence $a_1, b_1, a_2 - a_1, b_1 - b_2, \cdots$ has infinitely many nonzero terms. By (i), there exist nonzero elements a, b in B such that the sequence $a, b, a_1, b'_1, a_2 - a_1, b_1 - b_2, \cdots$ is orthogonal. Since B is an F-algebra, there is an element c in B which is $\{b, b'_1, b_1 - b_2, \cdots\}$ and which is an upper bound of $\{a, a_1, a_2 - a_1, \cdots\}$. Then, for each n

$$a_n < a_n \vee a \leqq c \leqq b_n - b < b_n .$$

That completes the proof.

We now focus our attention on a Boolean algebra B, which has property (ii) of Lemma 4. As we have indicated, the algebra of open-closed subsets of $\beta N - N$ is one such algebra. It is easy to see that a Boolean algebra B has property (ii) if and only if its Stone space $S(B)$ has these three topological properties:

(a) $S(B)$ is an F-space;

(b) every nonempty (closed) G_δ in $S(B)$ has a nonempty interior;

(c) $S(B)$ has no isolated points.

We might remark that, if $S(B)$ satisfies (a), (*b*), and (c), then the Stone space for the algebra of Borel sets in $S(B)$ modulo first category sets is a Stonian space in which every first category set is nowhere dense and on which every measure has nowhere dense support. See Dixmier [3].

LEMMA 5. *Let B be a Boolean algebra which has property* (ii) *of Lemma* 4. *Let I be a countably generated ideal in B, and let F be a countable subset of B which is disjoint from I. If y is a strict upper bound of I, there exists an element z in B such that*

(a) $z < y$

(b) *z is a strict upper bound of I*

(c) *$F \cap [0, z]$ is empty.*

Proof. First let us show that, if u is an element of B which is not in I, there is an $x \leqq y$ such that $u \nleqq x$ and x is a strict upper bound of I. We may suppose that $u \leqq y$; otherwise we may take $x = y$. Since u is not in I, the ideal generated by I and u' is proper. Let J be that ideal. Since J is countably generated, we can find $t < 1$ such that t is a strict upper bound of J. Choose t_1 with $0 < t_1 < t'$, and let $x = y - t_1$. It is easy to see that x is a strict upper bound of I and that $u \nleqq x$.

If $F = \{f_1, f_2, \cdots\}$, the preceding paragraph shows us how to construct a sequence $y_1 \geqq y_2 \geqq \cdots$ such that each y_n is a strict upper bound of I and such that, for each n, we have $f_n \nleqq y_n$. Since B has property (ii) of Lemma 4, there exists an element z in B which is a strict upper bound of I and is a strict lower bound of the dual ideal generated by $y_1, y_2, \cdots$. Any such z has properties (a), (b), and (c).

PROPOSITION 1. Let B be a Boolean algebra which has property (ii) of Lemma 4. Any Boolean algebra of cardinality at most $\aleph_1$ can be imbedded in B.

Proof. Let C be a Boolean algebra, the cardinality of which does not exceed $\aleph_1$. Then C is the union of a well-ordered chain of countable subalgebras. We can arrange that the chain starts with the subalgebra $\{0, 1\}$ and that each member of the chain is generated by the preceding subalgebras and one additional element. By Lemma 1 and induction, it suffices to prove the following. If E is a countable subalgebra of C, if φ is an isomorphism of E onto the subalgebra F of B, and if x is in $C - E$, then there exists an element y in B such that

(a) $\varphi([0, x] \cap E) = [0, y] \cap F$

(b) $\varphi([x, 1] \cap E) = [y, 1] \cap F$.

Let I be the ideal in B which is generated by the left-hand member of (a), and let D be the dual ideal in B which is generated by the left-hand member of (b). Then, both I and D are countably generated, and each element of D is a strict upper bound of I. If we apply property (ii) of Lemma 4 and then Lemma 5, we obtain an element u_0 of B which is a strict upper bound of I and is a strict lower bound of D, while $[0, u_0]$ and $F - I$ are disjoint.

Now, we show that, if v is any element of $F - D$, then there exists u such that $u \geqq u_0$; $v \ngeqq u$; u is a strict lower bound of D; and $[0, u] \cap (F - I)$ is empty. If $v \ngeqq u_0$, we may take $u = u_0$. If $v \geqq u_0$, observe that the dual ideal D_1, which is generated by $D \cup \{v'\}$, is both countably generated and proper. Hence, there exists a nonzero element w in B which is a strict lower bound of D_1. Then $0 < w < v'$, and $u_0 \vee w$ is a lower bound of D, with $u_0 < u_0 \vee w$. We apply Lemma 5 to the ideal $[0, u_0]$ and the subset $F - I$. We obtain an element u such that $u_0 < u < u_0 \vee w$ and $(F - I) \cap [0, u]$ is empty. Then u is a strict lower bound of D, and $v \ngeqq u$, because $u \wedge v' \geqq u \wedge w > 0$.

Suppose $F - D = \{v_1, v_2, \cdots\}$. We employ the argument which we have just concluded, to construct an increasing sequence $u_0 \leqq u_1 \leqq u_2 \leqq \cdots$ such that, for each n, we have the following: $v_n \ngeqq u_n$; $[0, u_n] \cap (F - I)$

is empty; u_n is a strict lower bound of D. Once again we apply property (ii) of Lemma 4 and then Lemma 5, to conclude that there exists an element y in B such that $[0, y] \cap (F - I)$ is empty, y is a strict lower bound of D, and y is a strict upper bound of the ideal generated by $u_1, u_2, \cdots$. This is the y which we seek.

COROLLARY. *Let B be an infinite Boolean F-algebra. Let C be a Boolean algebra of cardinality at most $\aleph_1$ in which orthogonal sets are at most countable. If C^+ is the completion of C by cuts, then C^+ is a homomorphic image of B. Hence, the Stone space $S(C^+)$ can be imbedded in the Stone space $S(B)$.*

Proof. Select any (infinite) discrete sequence in the Stone space $S(B)$. Its closure is homeomorphic to the space βN. Hence, the algebra of open-closed subsets on $\beta N - N$ is a homomorphic image of B.

We see that we need only prove the Corollary when B is the Boolean algebra of open-closed subsets of the space $\beta N - N$. In that case, Proposition 1 provides us with an isomorphism φ, from a subalgebra of B onto the algebra C. We can extend φ to a homomorphism ψ, from B into the completion C^+. Since orthogonal sets in C are at most countable, the same is true in the algebra C^+. By Lemma 3, the range of the homomorphism ψ is complete. Hence, ψ maps B onto C^+.

Suppose C_0 and C_1 are the free Boolean algebras on $\aleph_0$ and $\aleph_1$ generators, respectively. Each of these algebras satisfies the conditions on the algebra C in the Corollary. Therefore, the Stone spaces $S(C_0^+)$ and $S(C_1^+)$ can be imbedded in $\beta N - N$. In particular, $\beta N - N$ contains closed subsets which are Stonian but not hyper-Stonian [3].

PROPOSITION 2. Assume the continuum hypothesis. Let B be the Boolean algebra of subsets of the integers, modulo finite sets. If C is a complete Boolean algebra of cardinality at most $2^{\aleph_0}$, then C is a homomorphic image of B. Hence, the Stone space $S(C)$ can be imbedded in the space $\beta N - N$.

Proof. Since $2^{\aleph_0} = \aleph_1$, Proposition 1 tells us that C can be imbedded in B. Then, we have an isomorphism φ, from a subalgebra of B onto the algebra C. Since C is complete, φ can be extended to a homomorphism of B onto C.

The example. Let L^∞ be the algebra of essentially bounded measurable functions on the unit circle in the plane. Identify functions which agree almost everywhere. Then, with the essential supremum as norm, L^∞ is a commutative Banach algebra which is isometrically isomorphic to $C(X)$, the algebra of continuous functions on its maximal ideal space X. The isomorphism is the Gelfand representation $f \to \hat{f}$. We let H^∞ be the subalgebra of L^∞ which consists of the functions h whose negative Fourier coefficients vanish. If $h \in H^\infty$ then the Poisson formula

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i\theta}) Re \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta$$

extends h to a bounded analytic function in the unit disc; also, each bounded analytic function arises as the Poisson integral of its boundary values

$$h(e^{i\theta}) = \lim_{r \to 1} h(re^{i\theta})$$

which exist almost everywhere. For a summary of these basic facts, see [6].

LEMMA 6. *Let K be a closed subset of X, the maximal ideal space of L^∞, and suppose the interior of K is empty. If u is any nonnegative continuous function on K, there exists a function h in H^∞ such that the modulus of $\hat{h}$ agrees with u on K.*

Proof. Let m be the measure on X which corresponds to normalized Lebesgue measure on the unit circle. In other words, if ψ is the linear functional

$$\psi(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta$$

on L^∞, the Riesz representation theorem, together with the isomorphism between L^∞ and $C(X)$, provides us with a unique probability measure m on X such that

$$\psi(f) = \int_X \hat{f}\, dm , \qquad f \in L^\infty .$$

Then X is a hyperstonian space and m is a normal measure on X, i.e., if S is a measurable subset of X, m assigns the same measure to S and the interior of S:

$$m(S) = m(\text{int}\, S) .$$

Consider the given closed set K in X. Since K has no interior, $m(K) = 0$. Hence, we can construct a nonnegative continuous function on X which vanishes on K and has its logarithm integrable with respect to m. That is, there exists an f in L^∞ such that

(a) $f \geqq 0$
(b) $\hat{f} = 0$ on K
(c) $\int \log f(e^{i\theta})d\theta > -\infty$.

Let u be a nonnegative continuous function on K. We can extend u to a nonnegative continuous function on X. Thus, we have a function g in L^∞ such that $g \geqq 0$ and $\hat{g} = u$ on K. With the function f of the previous paragraph define

$$h(z) = \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log [f(e^{i\theta}) + g(e^{i\theta})]d\theta \ .$$

Since $\log (f + g)$ is integrable and bounded above, h is a bounded analytic function in the unit disc. The boundary values

$$h(e^{i\theta}) = \lim_{r \to 1} h(re^{i\theta})$$

then satisfy

$$|h| = f + g, \text{ almost everywhere} .$$

So $|\hat{h}| = \hat{f} + \hat{g}$. In particular, since $\hat{f} = 0$ on K, we have

$$|\hat{h}|_K = |\hat{g}|_K = u \ .$$

The closed set K of the last lemma may be sufficiently small so that the restriction of $\hat{H}^\infty$ to K is all of $C(K)$. If K is a nonempty G_δ, that does not happen; however, we shall not stop to prove that here. We shall choose a particular K on which it does not happen.

Let K be the set of maximal ideals of L^∞ which contain the function $(1 - z)$. Then K is a closed G_δ without interior in X, and the restriction of $\hat{H}^\infty$ to K is a proper closed subalgebra of $C(K)$. See [6; page 187].

Let $\mathscr{B}$ be the Boolean algebra of measurable sets on the circle, modulo null sets. The Stone space of $\mathscr{B}$ is the space X. The Stone representation for $\mathscr{B}$ is simply the restriction of the Gelfand representation of L^∞ to the collection of characteristic functions of measurable sets. According to Proposition 2, there exists a homeomorphism τ, of X into the space $\beta N - N$. We use τ and the set K to define an algebra of continuous functions on βN. Let A be the algebra of continuous functions f on βN such that, on the set K, the composition $f \circ \tau$ agrees with the restriction of a function in $\hat{H}^\infty$. We assert that

(i) A is a uniformly closed subalgebra of $C(\beta N)$;

(ii) every nonnegative continuous function on βN is the modulus of a function in A;

(iii) $A \neq C(\beta N)$.

It is clear that (i) and (iii) are satisfied. Property (ii) follows from this observation. Suppose g is a nonnegative continuous function on a totally disconnected compact Hausdorff space Y. Let M be a closed subset of Y, and let h be a continuous complex-valued function on Y such that the modulus of h agrees with g on M. Then, there exists f in $C(Y)$ such that $f|_M = h|_M$ and $|f| = g$ on all of Y.

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SOME ASPECTS OF TORSION

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Using S. E. Dickson's characterization of a torsion class, a class of modules closed under taking factors extensions and arbitrary direct sum, we study torsion classes closed under taking submodules and arbitrary direct products. We show that these classes are in one-to-one correspondence with idempotent two sided ideals of the ring. Finally we investigate the structure of rings R for which the torsion class $\mathscr{T}_0 = \{M \mid \mathrm{Hom}_R(M, Q(R)) = 0,\ Q(R)$ the minimal injective for $R\}$ is closed under taking products.

The original purpose of this paper was to show that for certain rings the direct product of torsion modules is again torsion, and by torsion we mean a particular kind of torsion defined in § 3. However, we find that S. E. Dickson [3] has given a set of axioms for torsion theories in Abelian categories and we thought it best to work within his axiomatic system.

In § 1 we summarize the work of Dickson and study, within the context of modules over a ring, torsion theories closed under taking submodules. In § 2, we give a complete characterization of all torsion theories closed under taking submodules and direct products. Finally, in § 3 we show that a fairly wide class of rings enjoy the property that a particular kind of torsion is closed under taking submodules and direct products.

1. Sets of torsion theories.

Dickson [3] has introduced a set of axioms for torsion theories in certain abelian categories sufficiently general to include the category ${}_R\mathscr{M}$ of left modules and homomorphisms over a ring R with identity. His axioms (stated here for ${}_R\mathscr{M}$) are as follows.

A torsion theory for ${}_R\mathscr{M}$ is a pair $(\mathscr{T}, \mathscr{F})$ of classes of modules such that:

I. $\mathscr{T}$ and $\mathscr{F}$ have only 0 in common.

II. $\mathscr{T}$ is closed under taking factors and $\mathscr{F}$ is closed under taking submodules.

III. For each M in ${}_R\mathscr{M}$ there exists a unique submodule $T(M)$ of M such that

$$0 \to T(M) \to M \to M/T(M) \to 0$$

is exact with $T(M) \in \mathscr{T}$ and $M/T(M) \in \mathscr{F}$.

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The modules in $\mathscr{T}$ are torsion modules and those in $\mathscr{F}$ are torsion free. $T(M)$ is the unique largest submodule of M in $\mathscr{T}$.

Dickson [3] proved a number of things about torsion theories (in a context more general than the category ${}_R\mathscr{M}$); we state a few of his results. In what follows $(\mathscr{T}, \mathscr{F})$ is a torsion theory for ${}_R\mathscr{M}$.

A. $\mathscr{T}$ is closed under direct sum and $\mathscr{F}$ is closed under direct product.

B. Both $\mathscr{T}$ and $\mathscr{F}$ are closed under extension.

C. Each class $\mathscr{T}, \mathscr{F}$ uniquely determines the other.

D. Given $\mathscr{T}$ closed under taking factors extension and direct sum, there exists a unique $\mathscr{F} = \{F \mid \mathrm{Hom}_R(T, F) = 0 \text{ all } T \in \mathscr{T}\}$ such that $(\mathscr{T}, \mathscr{F})$ is a torsion theory.

D′. Given $\mathscr{F}$ closed under taking submodules extensions and direct product there exists a unique class $\mathscr{T} = \{T \mid \mathrm{Hom}_R(T, F) = 0$ all $F \in \mathscr{F}\}$ such that $(\mathscr{T}, \mathscr{F})$ is a torsion theory.

Thus in considering torsion theories it is enough to consider classes $\mathscr{T}$ closed under taking factors extensions and direct sums (or classes $\mathscr{F}$ closed under taking submodules extensions and direct products).

For the rest of this paper we shall only consider torsion theories $(\mathscr{T}, \mathscr{F})$ for ${}_R\mathscr{M}$ where $\mathscr{T}$ is closed under taking submodules as well. With respect to this property Dickson [3] proved the following

E. $\mathscr{T}$ is closed under taking submodules if and only if $\mathscr{F}$ is closed under taking minimal injectives.

We shall use the phrase "a torsion theory closed under taking submodules" to describe this situation.

It should be noted that Dickson was working in a context more general than modules over a ring and that we have specialized his results to this case.

In a torsion theory $(\mathscr{T}, \mathscr{F})$ closed under taking submodules, the torsion class $\mathscr{T}$ forms a "strongly complete additive" class in the sense of Walker and Walker [10]; that is, $\mathscr{T}$ is a class closed under taking factors submodules and arbitrary direct sums. Gabriel [6] has shown that there is a one-to-one correspondence between such classes and filters of left ideals of the ring R. Consequently, the torsion theories closed under taking submodules form a set.

In the following theorem we connect such a torsion theory with injective modules.

THEOREM 1.1. *If $(\mathscr{T}, \mathscr{F})$ is a torsion theory closed under taking submodules then*

(1) *There exists modules $X_0 \in \mathscr{T}$ and $Y_0 \in \mathscr{F}$ such that $\mathscr{F} = \{Z \mid \mathrm{Hom}_R(X_0, Z) = 0\}$ and $\mathscr{T} = \{W \mid \mathrm{Hom}_R(W, Y_0) = 0\}$ (X_0 and Y_0 are not unique).*

(2) *The module Y_0 can be selected to be injective.*

Proof. Let $\{X_\alpha\}$ be the *set* of finitely generated modules in $\mathscr{T}$. We say *the set* we mean, of course, one module from each equivalence class (equivalence under isomorphism) of finitely generated modules. Now since $\mathscr{T}$ is closed under taking submodules, factors and direct sums, it is therefore closed under direct limit. Thus a module is in $\mathscr{T}$ if and only if its finitely generated submodules are in $\mathscr{T}$. Now let $X_0 = \Sigma_\alpha X_\alpha$ be the direct sum of the finitely generated modules in $\mathscr{T}$.

Now form the class $\mathscr{F}' = \{M \mid \mathrm{Hom}_R(X_0, M) = 0\}$. It is clear that $\mathrm{Hom}_R(X_0, M) = 0$ if and only if $\mathrm{Hom}_R(X_\alpha, M) = 0$ for all the modules X_α. Therefore the class $\mathscr{F}'$ contains the class $\mathscr{F}$. We show them equal. Let M be in $\mathscr{F}'$ but not in $\mathscr{F}$, then $0 \neq T(M) \leqq M$. But $T(M)$ contains a finitely generated module ($\neq 0$) isomorphic to one of the X_α. It follows that $\mathrm{Hom}_R(X_\alpha, T(M))$, $\mathrm{Hom}_R(X_\alpha, M)$ and $\mathrm{Hom}_R(X_0, M)$ are all nonzero contradicting the assumption that M was not in $\mathscr{F}$. Thus we have shown that $\mathscr{F} = \{M \mid \mathrm{Hom}_R(X_0, M) = 0.\}$

Now choose a *set* of finitely generated modules $\{X'_\alpha\}$ one from each isomorphism class not in T. Factoring out the $\mathscr{T}$ torsion of each of these we obtain a set of modules F_α all in $\mathscr{F}$, where $F_\alpha = X'_\alpha / T(X'_\alpha)$.

Let $Y_0 = Q(\pi F_\alpha)$, the minimal injective containing the product πF_α. Recall that for a module M, $Q(M)$ is injective containing M as a submodule, and for $X \neq 0$ in $Q(M)$ $X \cap M \neq 0$ also. See [4]. The module πF_α is in $\mathscr{F}$ because $\mathscr{F}$ is closed under products. Also from property E cited above $\mathscr{F}$ is closed under taking minimal injectives, consequently Y_0 is in $\mathscr{F}$.

Now form the class $\mathscr{T}' = \{X \mid \mathrm{Hom}_R(X, Y_0) = 0\}$. From the properties of Hom, [9], $\mathscr{T}'$ is closed under taking factors, extensions, and direct sums. But the fact that Y_0 is injective implies that $\mathscr{T}'$ is also closed under taking submodules, for if $0 \to X' \to X$ is exact, then $\mathrm{Hom}_R(X, Y_0) \to \mathrm{Hom}_R(X', Y_0) \to 0$ is also exact. That is, if X is in $\mathscr{T}'$ so is X'. Thus we see that $\mathscr{T}'$ is a torsion class for some torsion theory closed under taking submodules. Moreover, by property D' cited above $\mathscr{T}$ is contained in $\mathscr{T}'$.

From the first part of the proof $\mathscr{T}$ and $\mathscr{T}'$ are determined by their finitely generated modules. If X' is a finitely generated module not in $\mathscr{T}$ then $X' = X'_\alpha$ for some α and there exists a nonzero homomorphism $X' \to F_\alpha = X'_\alpha / T(X'_\alpha)$. It follows that $\mathrm{Hom}_R(X', \pi F_\alpha) \neq 0$ and so $\mathrm{Hom}_R(X', Y_0) \neq 0$. Consequently X' is not in $\mathscr{T}'$. Since $\mathscr{T}$ and $\mathscr{T}'$ coincide on finitely generated modules, $\mathscr{T} = \mathscr{T}'$.

From the properties D and D' cited above the modules X_0 and Y_0 need not be unique because $X_0 \oplus Y'$ and $Y_0 \oplus Y'$ would do as well where $X' \in \mathscr{T}$ and $Y' \in \mathscr{F}$.

Before we develop a more specialized torsion theory in the next section, we should make a few remarks about torsion. In the first place a torsion theory $(\mathscr{T}, \mathscr{F})$ gives an idempotent subfunctor of the identity T, where T applied to $M \xrightarrow{f} S$ gives $T(M) \xrightarrow{T(f)} T(S)$ and $T(f)$ is f restricted to $T(M)$. Two torsion theories are the same if their torsion functors are equivalent.

If we apply this remark to the ring ${}_RR$ considered as a left module over itself we see that the left ideal $T({}_RR)$ is carried into itself by every right multiplication. That is, $T({}_RR)$ is a *two sided* ideal. At first we thought that torsion theories could be indexed by these two sided ideals; that is, T and T' are the same torsion functor if and only if $T({}_RR) = T'({}_RR)$. However, the following example shows this is not correct: Let $\mathscr{T}'$ be the torsion class containing only the zero module and let $\mathscr{T}_0 = \{X \mid \mathrm{Hom}_R(X, Q(R)) = 0\}$ where $Q(R)$ is the minimal injective for R. These clearly give torsion theories; let T_0 and T' be the associated torsion functors. Then one sees that $T_0({}_RR) = 0 = T'({}_RR)$.

In § 3 we shall investigate the torsion theory $\mathscr{T}_0$ defined above in more detail. Also we shall show in the next section that certain torsion theories *can* be indexed by ideals in the ring.

2. Torsion-torsion free classes.

Let $\mathscr{T}$ be a class of modules in ${}_R\mathscr{M}$ which is closed under taking submodules factors, extensions, direct products and direct sums. Then $\mathscr{T}$ is the torsion class for a torsion theory $(\mathscr{T}, \mathscr{F})$ closed under taking submodules and it is *also* the *torsion free* class for another torsion theory $(\mathscr{C}, \mathscr{T})$. Such a pair of torsion theories $(\mathscr{C}, \mathscr{T})$ and $(\mathscr{T}, \mathscr{F})$ will be called a torsion-torsion free theory (TTF theory) and the class $\mathscr{T}$, a TTF class.

Consider for a moment a class $\mathscr{T}$ closed under taking submodules, factors, extension and direct sum. Classes somewhat more general were considered by Gabriel [6, p. 395] as "closed" subcategories of ${}_R\mathscr{M}$. Such a class $\mathscr{T}$ is completely determined by the cyclic modules in it and therefore determines a filter F_T of left ideals of R [6, p. 411] where $L \in F_T$ if and only if $R/L \in T$. The filter F_T is idempotent in the sense that the product of two left ideals in it is again in it [6, p. 412]. Conversely, any such idempotent filter determines such a class $\mathscr{T}$ [6, p. 411-412].

The following theorem was mentioned by R. S. Pierce one day during a conversation.

THEOREM 2.1. *The class $\mathscr{T}$ closed under taking submodules, factors, extension and direct sums is also closed under direct products if and only if its associated filter F_T has a smallest element I. In that case $\mathscr{T}$ is a TTF class and I is the two sided ideal $C(R)$ where C is the torsion functor for the torsion theory $(\mathscr{C}, \mathscr{T})$.*

Proof. If $\mathscr{T}$ is closed under products the natural maps $R \to R/L$ for R/L in $\mathscr{T}$ induce a map $R \xrightarrow{\theta} \prod_{R/L \in T} R/L$ and $Im\,\theta$ is again in $\mathscr{T}$. But then $\operatorname{Ker}\theta$ is in the associated filter and we have $\operatorname{Ker}\theta \subseteqq L$ for each L in the associated filter and the filter therefore has a smallest element.

Conversely, suppose I is the minimal element in the filter associated with $\mathscr{T}$. We shall show that M is in $\mathscr{T}$ if and only if $IM = 0$. If M in $\mathscr{T}$ and $IM \neq 0$, then $Im \neq 0$ for some $m \in M$. The mapping $I \to Im$ has image in T so there exists $X \underset{\neq}{\subset} I$ such that $I/X = Im$. Now since $\mathscr{T}$ is closed under extension and the following exact sequence has both ends in $\mathscr{T}$

$$0 \to I/X \to R/X \to R/I \to 0\ ,$$

it follows that R/X is in $\mathscr{T}$. This latter statement contradicts the minimality of I.

Conversely, suppose that $IM = 0$ and let ΣR_i be a free R module mapping onto M, $\Sigma R_i \xrightarrow{\theta} M \to 0$. Since $\operatorname{Ker}\theta$ contains ΣI_i, θ induces a map of $\Sigma R_i/I_i$ onto M. But since $\mathscr{T}$ is closed under direct sums and factors we have M in $\mathscr{T}$.

Finally, it follows that $\mathscr{T}$ is closed under products because I annihilates a product if and only if it annihilates each factor.

In the case that the filter has a smallest element I then, $C(R) = I$ for R/I is the "largest" factor of R such that I annihilates it.

We remark that the filter $F_I = \{L \mid L \supseteqq I,\ L$ a left ideal$\}$ will be an idempotent filter if and only if $I = I^2$. This idempotence is necessary for the class of modules annihilated by I to be closed under extension (e.g. $0 \to I/I^2 \to R/I^2 \to R/I \to 0$ is a short exact sequence with I annihilating each end but not the middle unless $I = I^2$).

COROLLARY 2.2. *There is a one-to-one correspondence between TTF classes T_I and idempotent two sided ideals I*

$$I \to \{M \mid IM = 0\} = T_I\ .$$

Proof. This follows from Gabriel's correspondence between "closed" categories of modules and idempotent filters of left ideals [6, p. 412] together with the above theorem which classifies those classes which are also closed under products.

COROLLARY 2.3. *The following are equivalent for an ideal I*

(1) $I = I^2$

(2) $T_I = \{M \mid IM = 0\}$ *is a TTF class*

(3) *the natural map* $0 \to \operatorname{Ext}_{R/I}(A, C) \to \operatorname{Ext}_R(A, C)$ *is an isomorphism for R modules A, C such that $IA = IC = 0$.*

Proof. Only the condition (3) requires attention. For R modules A, C such that $IA = IC = 0$, R extensions can be considered as R/I extensions because the class of modules annihilated by I is closed under extension.

If $I \neq I^2$, by the remark preceding Corollary 2.2,

$$0 \to I/I^2 \to R/I^2 \to R/I \to 0$$

represents an R extension in $\operatorname{Ext}_R(R/I, I/I^2)$ which is not an R/I extension and in this case the map is not an isomorphism.

The following theorem gives information about a special class of TTF classes.

THEOREM 2.4. *Suppose that* $\mathscr{T}$ *is a TTF class and* $(\mathscr{C}, \mathscr{T})$, $(\mathscr{T}, \mathscr{F})$ *the associated torsion theories with torsion functors* C, T*; then the following are equivalent.*

(1) *for all* M, $M = C(M) \oplus T(M)$

(2) $R = C(R) \oplus T(R)$ *ring direct sum*

(3) $\mathscr{F} = \mathscr{C}$

(4) $T(C(M)) = 0$ *and* $C(M/T(M)) = M/T(M)$ *for all* M.

Proof. (1) $\Rightarrow$ (2) is trivial.

The class $\mathscr{T}$ is $\{M \mid C(R)M = 0\}$. Let $\mathscr{A}$ be the class

$$\{M \mid T(R)M = 0\} .$$

Assuming the decomposition $R = C(R) \oplus T(R)$ gives a decomposition $1 = e_1 + e_2$ where e_i are orthogonal central idempotents in R. Then $\mathscr{T}$ and $\mathscr{A}$ can be characterized as $\mathscr{T} = \{M \mid e_1M = 0\}$ and

$$\{M \mid e_2M = 0\} = \mathscr{A}$$

and e_1 acts like identity on modules in $\mathscr{T}$. Using this one sees that A can now be characterized as either

$$\mathscr{A} = \{M \mid \operatorname{Hom}_R(M, T) = 0 \text{ all } T \text{ in } \mathscr{T}\}$$

or

$$\mathscr{A} = \{M \mid \operatorname{Hom}_R(T, M) = 0 \text{ all } T \text{ in } \mathscr{T}\} .$$

It follows that $\mathscr{F} = \mathscr{C}$ and $(2 \Rightarrow 3)$.

Assuming (3), $\mathscr{F} = \mathscr{C}$ is closed under sums, products, submodules, and factors, so $T(C(M)) = 0$ and since $M/T(M)$ in $\mathscr{F}$, $C(M/T(M)) = M/T(M)$ which is condition (4).

Finally, assuming (4), form $C(M) \cap T(M)$ for an R module M. Since this is a submodule of $T(M)$ and $\mathscr{T}$ is closed under submodules, it is in $\mathscr{T}$. But $T(C(M)) = 0$ means that $C(M)$ has no $\mathscr{T}$ torsion submodules so $C(M) \cap T(M) = 0$.

Now form $M/(C(M) + T(M))$, this is a factor of $M/C(M)$ which is in $\mathscr{T}$. It follows that $M/(C(M) + T(M))$ is in $\mathscr{T}$. Since we are assuming that $C(M/T(M)) = M/T(M)$, $M/(C(M) + T(M))$ is also in $\mathscr{C}$ since it is a factor of $M/T(M)$ which is in $\mathscr{C}$. But then

$$M/\big(C(M) + T(M)\big)$$

is in both $\mathscr{C}$ and $\mathscr{T}$ and is therefore zero. Thus we have established that $M = C(M) \oplus T(M)$. This completes the proof of the theorem.

3. A particular torsion theory. In this section we define a torsion theory which coincides with the usual one for integral domains. This torsion has been used before [See Prop. 1 of 7].

Let $Q(R)$ be the injective hull of R considered as a left module over itself [4]. Let $\mathscr{T}_0 = \{M \mid \mathrm{Hom}_R(M, Q(R)) = 0\}$. It is not hard to show that $\mathscr{T}_0 = \{M \mid \mathrm{Hom}_R(M', R) = 0$ for all submodules $M' \subseteqq M.\}$

The methods of § 1 show that the class $\mathscr{T}_0$ is a torsion class closed under taking submodules. It is not, in general, closed under arbitrary direct products (e.g. $R = Z$ the ring of integers). However, in the following theorem we shall show that for a fairly wide class of rings $\mathscr{T}_0$ is a TTF class. Eilenberg [5] introduced the notation of a perfect ring and Bass [2] has given a characterization of them. We use Bass' characterization instead of Eilenberg's original definition.

DEFINITION. R is right perfect if

(1) R/N is semisimple with minimum condition where N is the Jacobson radical

(2) every nonzero left module M has nonzero socle $S(M)$, where the socle is the submodule generated by the simple submodules of M.

Condition (2) is equivalent to "every nonzero module has simple submodules."

THEOREM 3.1. *If R is right perfect then $\mathscr{T}_0$ is a TTF class.*

Proof. We must show that the product of $\mathscr{T}_0$-torsion modules is again $\mathscr{T}_0$-torsion.

Let us begin by defining for each ordinal α a submodule $S^\alpha(M)$ of the R module M as follows:

(1) $S^0(M) = 0$

(2) if β is a limit ordinal $S^\beta(M) = \bigcup_{\alpha<\beta} S^\alpha(M)$

(3) if $\beta = \alpha + 1$, $S^{\beta}(M)$ is the inverse image of $S(M/S^{\alpha}(M))$ under the map $M \to M/S^{\alpha}(M)$. Equivalently, $S^{\beta}(M) = \{m \mid Nm \in S^{\alpha}(M)\}$ where N is the radical of R.

From the assumptions on the ring R we know that for some α_0, $S^{\alpha_0}(M) = M$ (and the cardinal of that α_0 is less than or equal to the cardinal of M). Also for $\alpha < \alpha_0$ we know that $S^{\alpha+1}(M)/S^{\alpha}(M)$ is a direct sum of simple R (hence R/N) modules.

We now define the *composition factors* of the R module M to be those simple modules which appear as summands in $S^{\alpha+1}(M)/S^{\alpha}(M)$ for some ordinal α. Note that since R/N is semi-simple with minimum condition, R has (up to isomorphism) only a finite set of simple modules and for each M the composition factors of M form a subset of that set.

Now we show that an R-module M is $\mathscr{T}_0$-torsion if and only if for each composition factor S of M, $\operatorname{Hom}_R(S, R) = 0$. Suppose first that for each composition factor S of M, $\operatorname{Hom}_R(S, R) = 0$ but that $\operatorname{Hom}_R(M, Q(R)) \neq 0$. Let f be a nonzero homomorphism of M to $Q(R)$, since R is essential in $Q(R)$, $f(M) \cap R \neq 0$. Let $f^{-1}(f(M) \cap R) = M_0$ in M and we have a nonzero homomorphism $f_0 : M_0 \to R$ where $f_0 = f|_{M_0}$. Because $M_0 \cap S^{\alpha_0}(M) = M_0$ we can be assured that there exists a first ordinal α_1 such that f_0 restricted to $S^{\alpha_1}(M) \cap M_0$ is nonzero.

Now we note that α_1 cannot be a limit ordinal, for if $S^{\alpha_1}(M) = \bigcup_{\beta<\alpha_1} S^{\beta}(M)$ with f_0 restricted to $S^{\beta}(M) \cap M_0$, the zero map, then f_0 restricted to $S^{\alpha_1}(M) \cap M_0$ is also the zero map.

Let $\alpha_1 = \gamma + 1$ then f_0 induces a nonzero homomorphism of

$$(S^{\gamma+1}(M) \cap M_0)/(S^{\gamma}(M) \cap M_0)$$

into R. Since

$$(S^{\gamma+1}(M) \cap M_0)/(S^{\gamma}(M) \cap M_0)$$

is isomorphic to

$$(S^{\gamma+1}(M) \cap M_0 + S^{\gamma}(M))/S^{\gamma}(M) \;,$$

we have a nonzero homomorphism of a submodule of $S^{\gamma+1}(M)/S^{\gamma}(M)$ into R. This gives a nonzero homomorphism of one of the composition factors of M into contradicting the assumption that $\operatorname{Hom}_R(S, R) \neq 0$ for composition factors S of M.

Conversely, suppose that $\operatorname{Hom}_R(S, R) \neq 0$ for some composition factor S of M. This gives a nonzero homomorphism $f : M' \to R$ where M' is a submodule of M (actually M' can be taken to be $S^{\alpha+1}(M)$ where S appears as a summand of $S^{\alpha+1}(M)/S^{\alpha}(M)$). Since $Q(R)$ is the injective hull of R we have the solid diagram

$$\begin{array}{ccc} M' & \longrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ R & \longrightarrow & Q(R) \end{array}$$

which can be extended by a dotted arrow f' which gives a nonzero element of $\mathrm{Hom}_R(M, Q(R))$. Thus M is not $\mathscr{T}_0$-torsion.

Finally we complete the proof of the theorem by showing that the set of composition factors of a product $\prod_{i\in I} M_i$ is the union of the composition factors of each of the factors M_i. For each simple R module S_i there is an idempotent $\bar{e}_i$ in R/N such that $\bar{e}_i S_i \neq 0$ and $\bar{e}_i S_j = 0$ for $i \neq j$. Since R is an SBI ring [8, p. 53, 54] these can be raised to idempotents e_i in R. Also we still have the relation $e_i S_i \neq 0$ and $e_i S_j = 0$ for $i \neq j$.

Now we shall show that S_i is a composition factor of M if and only if $e_i M \neq 0$. First if S_i is a composition factor of M then S_i appears as a summand of $S^{\alpha+1}(M)(M)$, for some ordinal α. Thus, $e_i S^{\alpha+1}(M) \not\subseteq S^{\alpha}(M)$, and, a fortiori, $e_i M \neq 0$.

Conversely, if $e_i M \neq 0$, then $e_i m \neq 0$ for a suitable $m \in M$. There is a first ordinal α such that $e_i m \in S^{\alpha}(M)$. We claim that $\alpha = \gamma + 1$ is not a limit ordinal for if $S^{\alpha}(M) = \bigcup_{\beta<\alpha} S^{\beta}(M)$ with $e_i m \notin S^{\beta}(M)$, then $\overline{e_i m} \notin S^{\alpha}(M)$also. It follows that $e_i m$ represents a nonzero coset $\overline{e_i m}$ in $S^{\gamma+1}(M)/S^{\gamma}(M)$ and $e_i \overline{e_i m} = \overline{e_i m} \neq 0$. Therefore, one of the summands of $S^{\gamma+1}(M)/S^{\gamma}(M)$ is S_i for otherwise e_i would annihilate $S^{\gamma+1}(M)/S^{\gamma}(M)$. Thus we have shown that S_i is a composition factor of M.

We now complete the proof of the theorem. It is clear that e_i annihilates a product $\prod_{j\in J} M_j$ if and only if e_i annihilates each factor. So the set of composition factors of a product is the union of the composition factors of each of the factors. Thus for right perfect rings the product of $\mathscr{T}_0$-torsion modules is again $\mathscr{T}_0$-torsion.

The class of right perfect rings is fairly large, containing rings with minimum condition and semi-primary rings [2].

However, we shall show by an example that the rings for which $\mathscr{T}_0$-torsion is closed under products is even bigger.

Let $R = \prod_{i\in N} K_i$ the product of an infinite number of copies of a field. R has an infinite set of orthogonal, indecomposable idempotents $\{e_i\}$. It isn't hard to show that a module M is $\mathscr{T}_0$-torsion if and only if $e_i M = 0$ for all i. It is clear that this property is inherited by products, so for such rings the product of $\mathscr{T}_0$ torsion is again $\mathscr{T}_0$ torsion. Note also that the ring R has some non trivial torsion modules, for instance $\Pi K_i/\Sigma K_i$.

This brings up the question of the structure of rings for which

(0) is the only $\mathscr{T}_0$ torsion module. Clearly, in this case, $\mathscr{T}_0$ torsion will be closed under products. The following theorem gives several equivalent characterizations of this condition. In a ring R we denote by $r(S)$, the right annihilator of S, $r(S) = \{x \mid Sx = 0\}$.

THEOREM 3.2. *For a ring R the following conditions are equivalent:*
(1) $\mathscr{T}_0 = (0)$
(2) *For every simple R-module S,* $\operatorname{Hom}_R(S, R) \neq 0$
(3) *For every maximal left ideal L of R,* $r(L) \neq 0$
(4) *For every left ideal* $L \neq R$, $r(L) \neq 0$

Proof. We prove the implications cyclically in order. If $\mathscr{T}_0 = 0$ then $\operatorname{Hom}_R(S, Q(R)) \neq 0$ for simple modules S. But if $f: S \rightarrow Q(R)$ is nonzero, then $f(S) \cap R \neq 0$ and it follows that $f(S) \subseteqq R$ since $f(S)$ is also simple. Therefore $\operatorname{Hom}_R(S, R) \neq 0$ and (1) implies (2).

Now let L be a maximal left ideal of R and consider the sequence $0 \rightarrow L \rightarrow R \rightarrow R/L \rightarrow 0$. Note that R/L is simple. The sequence induces $0 \rightarrow \operatorname{Hom}_R(R/L, R) \rightarrow \operatorname{Hom}_R(R, R)$ exact. Considering $\operatorname{Hom}_R(R/L, R)$ as a submodule of $\operatorname{Hom}_R(R, R)$, the right multiplications of R we see that $\operatorname{Hom}_R(R/L, R) = r(L)$ the right annihilator of L. It follows that $r(L) \neq 0$ because $\operatorname{Hom}_R(R/L, R) \neq 0$.

Since every left ideal $L \neq R$ can be embedded in a maximal one $\bar{L}$ and the relation $L \subseteqq \bar{L}$ implies $r(\bar{L}) \subseteqq r(L)$, it follows that (3) implies (4).

Finally, if for each $L \neq R$, $r(L) \neq 0$, then for each such L, $\operatorname{Hom}_R(R/L, R) \neq 0$ (a right multiplication by a nonzero element of $r(L)$ gives such a nonzero homomorphism). It follows that

$$\operatorname{Hom}_R(R/L, Q(R)) \neq 0$$

for all $L \neq R$ and the filter of left ideals associated with $\mathscr{T}_0$ in this case consists only of R itself. That is, $C(R) = R$. Then from Theorem 2.1 we know that $\mathscr{T}_0 = (0)$. This completes the proof of the theorem.

REMARK. M. Auslander has shown [1] that a commutative semi-primary ring satisfies condition (2) of the above theorem. Hence such rings have only zero torsion.

The following example shows that left and right torsion theories need not give rise to the same two sided ideal. Let

$$R = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \middle| \, x, y, z \in K, \text{ a field} \right\} .$$

The left $\mathscr{T}_0$ torsion modules for R are those annihilated by

$$C(R) = \left\{ \begin{pmatrix} 0 & 0 \\ y & z \end{pmatrix} \middle| \, y, z \in K \right\} ,$$

whereas the right $\mathscr{T}_0$ torsion modules are those annihilated by the ideal $C_r(R) = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \middle| \, x, y \in K \right\}$.

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BOUNDARY MEASURES OF ANALYTIC DIFFERENTIALS AND UNIFORM APPROXIMATION ON A RIEMANN SURFACE

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A classical theorem of F. and M. Riesz establishes a one-to-one correspondence between analytic differentials of class H_1 on the interior of the unit disc and finite complex-valued Borel measures on the boundary of the disc which are orthogonal to polynomials. The main result of this paper gives a similar correspondence when the unit disc is replaced by a compact subset, satisfying a finite connectivity condition, of any noncompact Riemann surface. The analytic differentials on the interior of the set satisfy a boundedness condition analogous to the classical H_1 differentials and the measures on the boundary of the set are those orthogonal to all meromorphic functions with a finite number of poles in the complement of the set. This result is then used to obtain theorems on uniform approximation on the set by such meromorphic functions.

This paper extends results of Bishop in [2] and [5] where he considers compact subsets of the plane staisfying a simple connectivity condition.[1] He obtained such a one-to-one correspondence between boundary measures and analytic differentials and used his result together with an approximation theorem for nowhere dense sets to give a proof of Mergelyan's approximation theorem [6]. We are able to extend Mergelyan's theorem to our more general sets and also show that "local" approximation implies approximation on the whole set.

I. Boundary measures of analytic differentials.

A. DEFINITIONS AND PRELIMINARIES.

In this section S will denote an open Riemann surface. If K is a compact subset of S, we denote by $C(K)$ the algebra of all continuous complex-valued functions on K with norm $\|f\| = \sup_{x \in K} |f(x)|$, and by $A(K)$ the closed subalgebra of $C(K)$ consisting of those functions which are limits of meromorphic functions on S with finitely many poles in

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[1] The case with smooth boundary is discussed by Royden in [7].

$S \sim K$. By Runge's Theorem when S is the plane, or by the extension of Runge's Theorem due to Behnke and Stein [**1**, p. 445 and p. 456] in the general case, $A(K)$ can also be characterized as all functions of $C(K)$ which are uniform limits on K of functions analytic in a neighborhood of K.

The sets for which our results are obtained are defined as follows.

DEFINITION 1. A compact subset K of S will be called *n-balanced* if there exists a finite family $\{U_i\}_{i=1}^n$ of components of $S \sim K$ such that any point of the boundary of K lies on the boundary of one of the U_i. An open subset of S will be called *n-balanced* if it is the interior of its closure and its closure is a compact n-balanced set.

The following properties are clear.

LEMMA 1. *The interior of a compact n-balanced set is an open m-balanced set for some* $m \leqq n$. *The boundary of a compact n-balanced set is a nowhere dense compact n-balanced set.*

The measures on the boundary of K to be considered are now defined.

DEFINITION 2. If K is a compact subset of S, we denote by $M(K)$ all finite complex-valued Borel measures μ on the boundary of K such that $\int f d\mu = 0$ for all $f \in A(K)$.

Several preliminary definitions will be necessary to describe the boundedness condition on the analytic differentials to be studied.

By an arc we will mean a continuous map $f: [a, b] \rightarrow S$ of a closed interval $a \leqq t \leqq b$ into S. We will identify arcs $f: [a, b] \rightarrow S$ and $g: [c, d] \rightarrow S$ whenever $b - a = d - c$ and $g(t) = f(t + a - c)$. The image of $[a, b]$ under f will be denoted by $|f|$. By a subarc of f we mean the restriction of f to a subinterval $[c, d]$, $a \leqq c < d \leqq b$. If $g: [a_1, b_1] \rightarrow S$ is such that $f(b) = g(a_1)$ then by the product of f and g, written fg, we mean the arc $h: [a, b + b_1 - a_1] \rightarrow S$ defined by

$$h(t) = \begin{cases} f(t) & \text{if } a \leqq t \leqq b \\ g(t + a_1 - b) & \text{if } b \leqq t \leqq b + b_1 - a_1 . \end{cases}$$

An arc $f: [a, b] \rightarrow S$ is an analytic arc if f can be extended to be analytic with nonzero derivative in a neighborhood of $[a, b]$. A piecewise analytic arc is a product of a finite number of analytic arcs. A simple closed curve is an arc $f: [a, b] \rightarrow S$ such that $f(a) = f(b)$, and if $x \neq a$ and $x \neq b$ then $f(x) \neq f(a)$ and f is one-to-one on the open interval (a, b).

DEFINITION 3. If U is an open subset of S we say that a sequence $\{\gamma_i\}$ delimits U if

(i) each γ_i is a finite family of disjoint piecewise analytic simple closed curves α_{ij} such that $|\alpha_{ij}| \subset U$ and $\bigcup_j |\alpha_{ij}|$ is the boundary of an open set $V_i \subset U$ and each α_{ij} is positively oriented with respect to V_i.

(ii) if T is any compact subset of U, then for all sufficiently large i, $T \subset V_i$.

DEFINITION 4. If U is an open subset of S with compact closure K and γ is a finite family of piecewise analytic curves α_j such that $|\alpha_j| \subset U$ and ω is an analytic differential on U, we denote by $\|\omega\|_\gamma$ the norm of the linear functional F on $C(K)$ defined by $F(h) = \int_\gamma h\omega$.

DEFINITION 5. Let U be an open subset of S with compact closure. The class $H(U)$ consists of all analytic differentials ω on U such that there exists a sequence $\{\gamma_i\}$ which delimits U and an $M > 0$ such that $\|\omega\|_{\gamma_i} < M$ for all i.

Our aim is to establish, in case K is an n-balanced set, a one-to-one correspondence between $M(K)$ and $H(U)$, where U is the interior of K. The correspondence will be between a differential and its boundary measure, in the following sense.

DEFINITION 6. Let U be an open subset of S with compact closure and let B be its boundary. A finite complex-valued Borel measure μ on B is said to be a boundary measure of $\omega \in H(U)$ if the sequence of Definition 5 can be chosen so that

$$\int_{\gamma i} h\omega \rightarrow \int h d\mu \quad \text{as} \quad i \rightarrow \infty$$

for all $h \in C(U \cup B)$.

We do not need any restrictions on K other than compactness in order to show the existence of a boundary measure for every differential $\omega \in H(U)$. The following theorem has the same proof as Theorem 1 in [5].

THEOREM 1. *Let U be an open subset of S with compact closure K. Then any $\omega \in H(U)$ has a boundary measure $\mu \in M(K)$.*

In order to "fit together" sequences which delimit two different open sets to obtain a sequence which delimits the union, we will need the following lemma.

LEMMA 2. *Let γ and δ each be a finite family of disjoint piece-*

wise analytic simple closed curves, α_j *and* β_j *respectively, such that* $\bigcup_j |\alpha_j|$ *is the boundary of an open set* Γ *with each* α_j *positively oriented with respect to* Γ *and similarly* $\bigcup_j |\beta_j|$ *is the boundary of an open set* Δ *with each* β_j *positively oriented with respect to* Δ*. Then there exists a finite collection of analytic coordinate functions* h_i *with domain* V_i, V_i *a neighborhood of a point* $p_i \in S$ *(the* p_i *need not be distinct), so that given any neighborhood* U_i *of* $h_i(p_i)$ *such that* $U_i \subset h_i(V_i)$ *and any* $\varepsilon_i > 0$, *there exists* φ, *a finite family of disjoint piecewise analytic simple closed curves* ψ_j, *such that* $\bigcup_j |\psi_j|$ *is the boundary of an open set* Φ *and*

(i) *each* ψ_j *is positively oriented with respect to* Φ

(ii) *each* ψ_j *is the product of a finite number of subarcs, each of which is either a subarc of some* α_j *or* β_j *or is an arc* f *such that for some* i, *the arc* $h_i \circ f$ *has length less than* ε_i *and* $|h_i \circ f| \subset U_i$.

(iii) $\Gamma \cup \Delta \subset \Phi \subset \Gamma \cup \Delta \cup \bigcup_{i=1}^{n} h_i^{-1}(U_i)$

The proof is left as an exercise for the reader.[2]

B. Plane Sets.

In this section we consider the special case where S is the plane.

The proofs of the following lemma and theorem are the same as Lemma 4 and Theorem 1 in [5].

Lemma 3. *If* K *is a compact n-balanced subset of the plane and if* μ *and* ν *are both in* $M(K)$ *and* $\int (t-z)^{-1} d\mu(t) = \int (t-z)^{-1} d\nu(t)$ *for all* z *in the interior of* K, *then* $\mu = \nu$.

Theorem 2. *Let* U *be an n-balanced open subset of the plane and* K *be its closure. Then given* $\omega \in H(U)$, *its boundary measure, which exists by Theorem* 1, *is unique and if* $\omega = f(z)dz$ *then*

$$f(z) = (2\pi i)^{-1} \int (t-z)^{-1} d\mu(t)$$

for all $z \in U$.

The next lemma is a modification of Lemma 6 in [3]. The assumption that ν is orthogonal to all functions analytic in a neighborhood of K rather than just all polynomials enables us to obtain the measure β_{x_0} with support in K. The proof is not given, as the same proof applies with only obvious minor modifications and we prove a general version for any open Riemann surface as Lemma 5 below.

[2] A proof may be found in the author's thesis.

LEMMA 4. *Let K be a compact subset of the complex plane. Let ν be a measure on K orthogonal to $A(K)$. Then for almost all real numbers x_0, there exists a measure β_{x_0} on the set $K \cap \{z : Re\ z = x_0\}$ such that*

$$\int_{R_{x_0}} h d\nu = - \int_{L_{x_0}} h d\nu = \int h d\beta_{x_0}$$

for all $h \in A(K)$, where

$$R_{x_0} = K \cap \{z : Re\, z \geqq x_0\} \quad \text{and} \quad L_{x_0} = K \cap \{z : Re\, z \leqq x_0\} .$$

THEOREM 3. *Let K be a compact n-balanced subset of the complex plane with interior U. Then if $\mu \in M(K)$, there exists an analytic differential $\omega \in H(U)$ such that μ is the boundary measure of ω.*

Proof. The proof is by induction on n. If $n = 1$, K is balanced in the sense of [5] and Theorem 3 of [5] is the required result.

Suppose for $n > 1$ the theorem is true for m-balanced sets for all $m < n$. For $z \in U$, define

$$f(z) = (2\pi i)^{-1}\int (t - z)^{-1} d\mu(t) .$$

Now suppose x_0 is as in Lemma 4 and furthermore that $\{z : Re\, z = x_0\}$ intersects the interior of at least one of the bounded components U_i of Definition 1. Then L_{x_0} and R_{x_0} are both m-balanced for some $m < n$. Thus since $\mu \mid L_{x_0} + \beta_{x_0} \in M(L_{x_0})$ and $\mu \mid R_{x_0} - \beta_{x_0} \in M(R_{x_0})$ by Lemma 4 and Runge's theorem, the induction hypothesis applies and they are boundary measures of analytic differentials $f_1(z)dz$ and $f_2(z)dz$ respectively.

For z in the interior of L_{x_0},

$$\begin{aligned} f_1(z) &= (2\pi i)^{-1} \int (t - z)^{-1} d(\mu \mid L_{x_0} + \beta_{x_0})(t) \\ &= (2\pi i)^{-1} \int (t - z)^{-1} d(\mu \mid L_{x_0} + \beta_{x_0})(t) \\ &+ (2\pi i)^{-1} \int (t - z)^{-1} d(\mu \mid R_{x_0} - \beta_{x_0})(t) \\ &= (2\pi i)^{-1} \int (t - z)^{-1} d\mu(t) = f(z) \end{aligned}$$

and for z in the interior of R_{x_0} we have similary

$$f_2(z) = (2\pi i)^{-1} \int (t - z)^{-1} d\mu(t) = f(z) .$$

Now let $x_0 < x_1$ both restricted as above. Then $\mu \mid R_{x_0} - \beta_{x_0}$ is a boundary measure for $f(z)dz$ on the set R_{x_0}. Denote the delimiting sequence by $\{\gamma_j\}$. Also $\mu \mid L_{x_1} + \beta_{x_1}$ is a boundary measure for $f(z)dz$ on the set L_{x_1}. Denote the delimiting sequence by $\{\delta_j\}$. Suppose Γ_j is the open set bounded by γ_j and Δ_j the open set bounded by δ_j, as required in Definition 3. We apply Lemma 2 to $\gamma_j, \delta_j, \Gamma_j, \Delta_j$ where U_i are chosen so that $h_i^{-1}(U_i) \subset U$ and ε_i chosen so that the length of the arc in U_i which is not from δ_j or γ_j is less than η_i and $\Sigma\eta_i \sup_{z \in U_i} |f(z)| < 1$.

The lemma yields φ_j a finite union of disjoint piecewise analytic simple closed curves in U which form the boundary of the open set Φ_j, and $\Gamma_j \cup \Delta_j \subset \Phi_j \subset U$. If S is a compact subset of U, let $x_0 < x_2 < x_1$. Then $S_1 = S \cap \{z : Re\, z \leqq x_2\}$ is a compact subset of the interior of L_{x_1} and $S_2 = S \cap \{z : Re\, z \geqq x_2\}$ is a compact subset of the interior of R_{x_0}. Thus for all j sufficiently large,

$$S_1 \subset \Delta_j \quad \text{and} \quad S_2 \subset \Gamma_j \quad \text{and} \quad S = S_1 \cup S_2 \subset \Delta_j \cup \Gamma_j \subset \Phi_j \ .$$

Therefore $\{\varphi_j\}$ is a delimiting sequence for U. Furthermore,

$$\begin{aligned} \|\omega\|_{\varphi_j} = \int_{\varphi_j} |f(z)|\,|dz| &\leqq \int_{\gamma_j} |f(z)|\,|dz| + \int_{\delta_j} |f(z)|\,|dz| \\ &+ \Sigma\eta_i \sup_{z \in U_i} |f(z)| \leqq \|\omega\|_{\gamma_j} + \|\omega\|_{\delta_j} + 1 \ . \end{aligned}$$

Thus $\omega \in H(U)$.

By Theorems 1 and 2 there exists a boundary measure ν on the closure of U such that for $z \in U$,

$$f(z) = (2\pi i)^{-1}\int (t - z)^{-1} d\nu(t) \quad \text{and} \quad \nu \in M(\text{clsr}\, U) \subset M(K) \ .$$

Applying Lemma 3 to μ and ν we see that $\mu = \nu$ and thus μ is the boundary measure of ω.

C. Subsets of an Open Riemann Surface.

In this section we consider the general case where S is any open Riemann surface. The function $(t - z)^{-1}$ used in the plane case must be replaced by the elementary differential of Behnke and Stein [1]. The result needed is the following: there exists an elementary differential $\alpha(p)$ which for fixed p is a meromorphic differential on S with exactly one pole, a simple pole at p with residue 1. Furthermore, if h is an analytic coordinate function on an open set $V \subset S$ and $\alpha(p) = A(z, p)dz$ on $h(V)$, then $A(z, p)$ is meromorphic in p on S with exactly one pole, a simple pole at $h^{-1}(z)$. Thus if $h^{-1}(z_0) \notin K$, $A(z_0, p) \in A(K)$.

We prove the following generalization of Lemma 4.

LEMMA 5. *Let K be a compact subset of S. Let ν be a measure on K orthogonal to $A(K)$. Then if f is a nonconstant function analytic on S, for almost all real numbers x_0, there exists a measure β_{x_0} on the set $K \cap \{p : Re\, f(p) = x_0\}$ such that*

$$\int_{R_{x_0}} h d\nu = -\int_{L_{x_0}} h d\nu = \int h d\beta_{x_0}$$

for all $h \in A(K)$ where

$$L_{x_0} = \{p : Re\, f(p) \leqq x_0\} \cap K \text{ and } R_{x_0} = \{p : Re\, f(p) \geqq x_0\} \cap K .$$

Proof. Since f is nonconstant, for all but finitely many real numbers x, the differential of f does not vanish on $K \cap \{p : Re\, f(p) = x\}$. Let x_1 have this property and let $x_2 > x_1$ be such that the differential of f does not vanish on $K \cap \{p : x_1 \leqq Re\, f(p) \leqq x_2\}$. Since the differential of f does not vanish, there exists a neighborhood of any point of $K \cap \{p : x_1 \leqq Re\, f(p) \leqq x_2\}$ on which f is a coordinate function. Cover $K \cap \{p : x_1 \leqq Re\, f(p) \leqq x_2\}$ by finitely many neighborhoods $\{U_i\}_{i=1}^n$ such that the closure of U_i is compact and contained in V_i and f is a coordinate function on V_i. Denote by f_i^{-1} the inverse of f as a coordinate function on V_i.

There exists a nonnegative measure μ on K such that $|\nu(B)| \leqq \mu(B)$ for all Borel sets B. Let ϕ be the nonnegative, nondecreasing function defined by $\phi(x) = \mu(\{p : Re\, f(p) \leqq x\})$. Then $\phi'(x)$ will exist for almost all x. Let x_0 be such that $\phi'(x_0)$ exists and $x_1 \leqq x_0 \leqq x_2$. Thus ν vanishes on all subsets of $L_{x_0} \cap R_{x_0}$ and since $h \in A(K)$ implies $\int h d\nu = 0$ we have

$$\int_{R_{x_0}} h d\nu = -\int_{L_{x_0}} h d\nu \quad \text{for all} \quad h \in A(K) .$$

Suppose now that h is a meromorphic function with finitely many poles outside K. Let W be an open neighborhood of K on which h is analytic. Let $W_i = W \cap U_i$. Choose ε, $0 < \varepsilon < 1$ and let

$$T = \bigcup_{i=1}^{n} \{p \in W_i : Re\, f(p) = x_0 \text{ and } \operatorname{dist}(f(p), f(K \cap W_i)) \leqq \varepsilon\} .$$

Let $\|h\| = \sup_{p \in T} |(h(p)|$.

If $Re\, f(p) > x_0$, define $h_1(p) = (2\pi i)^{-1}\int_T h\alpha(p)$ and if $Re\, f(p) < x_0$, define $h_2(p) = (2\pi i)^{-1}\int_T h\alpha(p)$ where in each case integration is in a positive direction with respect to $\{p : Re\, f(p) \leqq x_0\}$. Suppose p_0 is interior to T relative to $\{z : Re\, z = x_0\}$. Then for some i_0, $f(p_0)$ is interior to $f(W_{i_0} \cap T)$ relative to $\{z: Re\, z = x_0\}$. Let $\tau_{i_0} = f(T \cap W_{i_0})$. Since the W_i cover T, we can choose, for $i \neq i_0$, measurable sets $\tau_i \subset \{z: Re\, z = x_0\} \cap f(W_i)$ so that $f_i^{-1}(\tau_i)$ are pairwise disjoint and each is disjoint from $f_{i_0}^{-1}(\tau_{i_0})$ and so that $T = \bigcup_{i=1}^n f_i^{-1}(\tau_i)$. Then if $p \in U_{i_0}$, $(2\pi i)^{-1}\int_T h\alpha(p)$ becomes

$$(2\pi i)^{-1}\int_{\tau_{i_0}} h(f_{i_0}^{-1}(\zeta))(\zeta - f(p))^{-1}d\zeta + (2\pi i)^{-1}\sum_{i=1}^{n}\int_{\tau_i} h(f_i^{-1}(\zeta))g_i(f(p), \zeta)d\zeta$$

where g_i is analytic in $f(\text{clsr } U_{i_0})$ in the first variable and in $f(\text{clsr } U_i)$ in the second variable. The first term has continuous boundary values both from the right and the left at p_0 with difference $h(p_0)$ and the integrals in the summation are all continuous in p at p_0. Thus h_1 and h_2 have continuous boundary values $h_1(p_0)$ and $h_2(p_0)$ and

$$h_1(p_0) - h_2(p_0) = h(p_0)\ .$$

If we define $h_1(p) = h(p) + h_2(p)$ in $Re\, f(p) < x_0$ and $h_2(p) = h(p) + h_1(p)$ in $Re\, f(p) > x_0$, then h_1 and h_2 are analytic in a neighborhood of K and $\mathrm{h} = h_1 - h_2$. Thus

$$\int_{R_{x_0}} h d\nu = \int_{R_{x_0}} h_1 d\nu - \int_{R_{x_0}} h_2 d\nu = \int_{R_{x_0}} h_1 d\nu + \int_{L_{x_0}} h_2 d\nu$$

and

$$\begin{aligned}\left|\int_{R_{x_0}} h_1 d\nu\right| &= \left|\int_{R_{x_0}}\left[(2\pi i)^{-1}\int_T h\alpha(p)\right]d\nu(p)\right| \\ &\leqq \int_{\{p\,:\,Re\, f(p) > x_2\}}\left|\int_T h\alpha(p)\right| d\mu(p) \\ &\quad + \sum_{i=1}^{n}\int_{R_{x_0}\cap U_i}\left|\int_T h\alpha(p)\right| d\mu(p)\ .\end{aligned}$$

Cover $K \cap [p: Re\, f(p) \geqq x_2\}$ by a finite number of open analytic neighborhoods, which are the domains of analytic coordinate functions ψ_k, each with range the unit circle D. Continuing the inequalities we have

$$\begin{aligned}&\leqq \|h\|\sum_{i=1}^{n}\int_{R_{x_0}\cap U_i}\left(\int_{\tau_i}|\zeta - f(p)|^{-1}d\zeta\right)d\mu(p) \\ &+ \|h\|\sum_{i=1}^{n}\int_{R_{x_0}\cap U_i}\left(\sum_{j=1}^{n}\int_{\tau_j}|g_{ij}(f(p), \zeta)|\, d\zeta\right)d\mu(p) \\ &+ \|h\|\sum_{k}\int_{\psi^{-1}(D)}\left(\sum_{i=1}^{n}\int_{\tau_i}|\gamma_{ki}(\psi_k(p), \zeta)|\, d\zeta\right)d\mu(p)\end{aligned}$$

where g_{ij} is analytic in the first variable in $f(\text{clsr } U_i)$ and in the second variable in $f(\text{clsr } U_j)$ and γ_{ki} is analytic in the first variable in D and the second variable in $f(\text{clsr } U_i)$. The g_{ij} and γ_{ki} are therefore bounded and we have a constant L, independent of ε, so that the above expression is less than or equal to

$$\| h \| \left\{ n \int_{x_0}^{x_2} \int_{-M}^{M} [(x_0 - x)^2 + t^2]^{-1/2} dt\, d\phi(x) + L \right\}$$

where M is chosen, independent of ε, so that if $y = Im\, f(p)$ and $v = Im \zeta$ where $p \in R_{x_0}$ and ζ is in some τ_i, then $|y - v| < M$.

A bound N, independent of ε, is found for

$$\int_{x_0}^{x_2} \int_{-M}^{M} [(x_0 - x)^2 + t^2]^{-1/2} dt\, d\phi(x)$$

as in [3, p. 42]. Thus

$$\left| \int_{R_{x_0}} h_1 d\nu \right| \leqq \| h \| (nN + L)$$

and a similar estimate can be made for $\left| \int_{L_{x_0}} h_2 d\nu \right|$. Combining these we have

$$\left| \int_{R_{x_0}} h d\nu \right| \leqq Q \| h \| .$$

where Q is independent of ε, and thus

$$\left| \int_{R_{x_0}} h d\nu \right| \leqq Q \sup \{ | h(p) | : p \in K \cap \{p : \; Re\, f(p) = x_0\} \} .$$

Therefore $h \longrightarrow \int_{R_{x_0}} h d\nu$ is a bounded linear functional on a dense subset of $A(K) \mid K \cap \{p\colon Re\, f(p) = x_0\}$ and therefore on $A(K) \mid K \cap \{p\colon Re\, f(p) = x_0\}$. By the Hahn-Banach theorem we can extend this bounded linear functional to $C(K \cap \{p\colon Re\, f(p) = x_0\})$ and then apply the Riesz representation theorem to obtain the desired measure β_{x_0}.

LEMMA 6. *Suppose K is an n-balanced compact subset of S. Suppose f is a nonconstant analytic function on S and $K_1 = K \cap \{p\colon Re\, f(p) \geqq x_0\}$. Then K_1 is a compact m-balanced set for some $m \leqq n$.*

Proof. K_1 is clearly compact.

Let $\{U_i\}_{i=1}^n$ be the finite set of components of $S \sim K$ from Definition 1. A point q on the boundary of K_1 is either on the boundary of K or in the intersection of the interior of K with the boundary of $\{p: Re\, f(p) \geqq x_0\}$. In the former case, q is on the boundary of some U_i and therefore on the boundary of the component of $S \sim K$ which contains U_i, which we call V_i. There are $n\, U_i$ and therefore $m\, V_i$ with $m \leqq n$. In the later case, q is on the boundary of some component Q of $\{p: Re\, f(p) < x_0\}$. Suppose $Q \subset K$, then clsr $U \subset K$ and clsr Q is compact. Q is open, so $Re\, f(p) = x_0$ on the boundary of Q. Since clsr Q is compact, $Re\, f(p)$ must assume its minimum on clsr Q is compact, $Re\, f(p)$ must assume its minimum on clsr Q and by the minimum modulus theorem for real parts of analytic functions, the minimum must be assumed on the boundary, but there $Re\, f(p) = x_0$. Thus $Re\, f(p) \geqq x_0$ on Q which is a contradiction. Since Q is not contained in K, it must interset some U_i. Therefore $Q \subset V_i$ and q is on the boundary of V_i. This shows K_1 is m-balanced.

LEMMA 7. *Under the hypotheses of Lemma* 5, *the measure* $\nu \mid R_{x_0} - \beta_{x_0}$ *is orthogonal to* $A(R_{x_0})$ *and the measure* $\nu \mid L_{x_0} + \beta_{x_0}$ *is orthogonal to* $A(L_{x_0})$.

Proof. Let h be a rational function on S with poles at $p_1, p_2, \cdots, p_n$ in $S \sim R_{x_0} = S \sim K \cap \{p: Re\, f(p) < x_0\}$. Let $p_1, \cdots, p_k$ be those poles not in $S \sim K$. Each p_i, $i = 1, \cdots, k$ is in some component Q_i of $\{p: Re\, f(\mathrm{p}) < x_0\}$. By the proof of Lemma 6, such a component cannot be contained in K. Thus we may choose q_i, $i = 1, \cdots, k$, $q_i \in Q_i \sim K$ and let J_i be a curve in Q_i joining p_i and q_i. Let

$$B = S \sim \bigcup_{i=1}^{k} J_i \sim \bigcup_{i=1}^{k} \{p_i\} \quad \text{and} \quad \tilde{B} = S \sim \bigcup_{i=1}^{k} \{q_i\} \sim \bigcup_{i=k+1}^{n} \{p_i\} \,.$$

Then by Theorem 6 in [4], h, which is analytic on B, can be uniformly approximated on R_{x_0}, a compact subset of B, by functions f_j analytic on $\tilde{B}$. Now letting $B_1 = \tilde{B}$ and $\tilde{B}_1 = S$ we apply Theorem 13 in [1, p. 456] and approximate f_j on R_{x_0} by meromorphic functions g_j with poles on the boundary of B_1, i.e., at the points $q_1, \cdots, q_k, p_{k+1}, \cdots, p_n$. But these are all in $S \sim K$. Thus $g_j \in A(K)$. By Lemma 5, $\int g_j d(\nu \mid R_{x_0} - \beta_{x_0}) = 0$. Thus $\int h d(\nu \mid R_{x_0} - \beta_{x_0}) = 0$ and $\nu \mid R_{x_0} + \beta_{x_0}$ is orthogonal to $A(R_{x_0})$. The same argument shows $\nu \mid L_{x_0} + \beta_{x_0}$ is orthogonal to $A(L_{x_0})$.

Given any finite collection of functions $\{g_k\}_{k=1}^{l}$ on S, and a real number x_0, we define an equivalence relation on the points of S as follows. The points p and q are equivalent, if, for all k, $Re\, f_k(p) \leqq x_0$ if and only if $Re\, f_k(q) \leqq x_0$.

LEMMA 8. *Let K be a compact subset of S and $\{U_i\}$ an open* *covering of K. Then there exists a finite collection of nonconstant functions, each analytic on S, such that given any x_0, $1/4 < x_0 < 3/4$, each equivalence class of the relation defined with these functions lies in a single member of the covering.*

Proof. Fix a metric on S. By the Lebesgue covering lemma, there exists $\rho > 0$ such that any set of diameter less than or equal to ρ, containing a point of K, lies in a single member of the covering $\{U_i\}$. Cover K by a finite number of sets of diameter less than $\rho/3$ which are homeomorphic to a closed disc. Call these sets $\{D_i\}_{i=1}^m$. For i, j such that $D_i \cap D_j$ is empty, let f_{ij} be a function analytic on S such that $Re\, f_{ij} < 1/4$ on D_i and $Re\, f_{ij} > 3/4$ on D_j. This is possible since by the Behnke-Stein extension of Runge's theorem [1, p. 445 and p. 456] we can approximate a function which is identically zero on a neighborhood of D_1 and identically one on a neighborhood of D_j by functions analytic on S.

Now if A is an equivalence class of the equivalence relation defined by these functions, we will show diam $A \leqq \rho$.

Let $p_0 \subset A$. Then for some i_0, $p_0 \in D_{i_0}$. Let $i_0, i_1, \cdots, i_k$ be all i such that $D_{i_0} \cap D_i$ is not empty. Let $p \in K \cap \{p : Re\, f_{i_0 j}(p) \leqq 3/4$, all $j \neq i_0, i_1, \cdots, i_k\}$. Suppose $p \notin \bigcup_{i=i_0}^{i_k} D_i$. Then since $p \in K \subset \bigcup_{i=1}^{n} D_i$, $p \in D_{j_0}$, some $j_0 \neq i_0, \cdots, i_k$. Thus $f_{i_0 j_0}(p) > 3/4$ which contradicts the choice of p. We have shown

$$K \cap \left\{p : Re\, f_{i_0 j}(p) \leqq \frac{3}{4} \text{ all } j \neq i_0, \cdots, i_k\right\} \subset \bigcup_{i=i_0}^{i_k} D_i .$$

Now since $p_0 \in D_{i_0}$, $Re\, f_{i_0 j}(p) < 1/4$, all $j \neq i_0, \cdots, i_k$, but $p_0 \in A$, so for all $p \in A$, we have $Re\, f_{i_0 j}(p) < 1/4$ for $i \neq i_0, \cdots, i_k$. Therefore

$$A \subset K \cap \left\{p : Re\, f_{i_0 j}(p) \leqq \frac{3}{4} \text{ all } j \neq i_0, \cdots, i_k\right\} \subset \bigcup_{i=i_0}^{i_k} D_i .$$

Each $D_{i_0}, \cdots, D_{i_k}$ intersects D_{i_0} and diam $D_i < \rho/3$. Therefore diam $\bigcup_{i=i_0}^{i_k} D_i < \rho$ and the proof is complete.

THEOREM 4. *If K is a compact subset of S and $\{U_i\}_{i=1}^n$ is an open covering of K, and if μ is a measure on K which is orthogonal to $A(K)$, then there exist measures ν_i with support contained in a compact set $T_i \subset K \cap U_i$ such that ν_i is orthogonal to $A(T_i)$ and $\mu = \nu_1 + \nu_2 + \cdots + \nu_n$.*

Proof. Let f_k be the functions of Lemma 8, $k = 1, \cdots, l$. The proof will be by induction on l. If $l = 0$, let $T = K \subset U_{i_0}$ and $\nu_{i_0} = \mu$ is orthogonal to $A(K) = A(T)$.

Suppose the theorem is true for $l-1$. Let $1/4 < x_0 < 3/4$ and $R_{x_0} = K \cap \{p : Re\, f_l(p) \geqq x_0\}$ and $L_{x_0} = K \cap \{p : Re\, f_l(p) \leqq x_0\}$. R_{x_0} and L_{x_0} are both compact, and $\{U_i\}_{i=1}^n$ is a covering for each. An equivalence class of points of R_{x_0} of the relation defined by $f_1, \cdots, f_{l-1}$ lies in a single member of $\{U_i\}_{i=1}^n$. Similarly for an equivalence class of points of L_{x_0}. Thus we may apply the induction hypothesis to the measures $\mu_1 = \mu \mid R_{x_0} - \beta_{x_0}$ which is orthogonal to $A(R_{x_0})$ by Lemma 7, and $\mu_2 = \mu \mid L_{x_0} + \beta_{x_0}$ which is orthogonal to $A(L_{x_0})$ by Lemma 7. Thus we have measures ν_{ji} with support contained in a compact set $T_{ji} \subset U_i \cap K$ which is orthogonal to $A(T_{ji})$ $j = 1, 2,$ $i = 1, \cdots, n$ and

$$\mu_1 = \nu_{11} + \nu_{12} + \cdots + \nu_{1n}\,, \qquad \mu_2 = \nu_{21} + \nu_{22} + \cdots + \nu_{2n}\,.$$

Thus $\mu = \mu_1 + \mu_2 = (v_{11} + \nu_{21}) + (\nu_{12} + \nu_{22}) + \cdots + (\nu_{1n} + \nu_{2n})$ and $\nu_{1i} + \nu_{2i}$ has support contained in $T_{1i} \cup T_{2i} \subset U_i \cap K$. If $f \in A_{1i} T_{1i} \cup T_{2i})$, then $f \mid T_{1i} \in A(T_{1i})$ and $f \mid T_{2i} \in A(T_{2i})$ and

$$\int f d(\nu_{1i} + \nu_{2i}) = \int f d\nu_{1i} + \int f d\nu_{2i} = 0\,.$$

Thus $\nu_{1i} + \nu_{2i}$ is orthogonal to $A(T_{1i} \cup T_{2i})$ and the theorem is proved.

THEOREM 5. *If K is a compact subset of S and if for every $p \in K$, there is a closed neighborhood W of p such that $f \mid W \in A(K \cap W)$, then $f \in A(K)$.*

Proof. Suppose $f \notin A(K)$. Then there exists a measure μ on K such that μ is orthogonal to $A(K)$ and $\int f d\mu \neq 0$. Let V be the interior of W. Then $\{V\}$ is an open covering of K. Let $\{V_i\}_{i=1}^n$ be a finite subcovering. Apply the last theorem with this covering to get measures ν_i with support contained in a compact set $T_i \subset V_i \cap K \subset W_i \cap K$ and ν_i is orthogonal to $A(T_i)$ and $\mu = \nu_1 + \nu_2 + \cdots + \nu_n$. $f \mid W \in A(K \cap W_i)$ implies $f \mid T_i \in A(T_i)$. Thus $\int f dv_i = 0$ and $\int f d\mu = 0$ which is contradiction.

COROLLARY 1. *If K is a compact subset of S and for every $p \in K$ there exists an analytic coordinate function h with $h(p) = 0$ and the range of h is $\{z : |z| < 1\}$ and an r, $0 < r < 1$, such that $A(h(K) \cap \{z : |z| \leqq r\}) = C(h(K) \cap \{z : |z| \leqq r\})$, then $A(K) = C(K)$.*

Proof. Let $f \in C(K)$. For every p,

$$\begin{aligned} f \circ h^{-1} \mid h\{z : |z| \leqq r\} &\in C(h(K) \cap \{z : |z| \leqq r\}) \\ &= A(h(K) \cap \{z : |z| \leqq r\}). \end{aligned}$$

Thus $f \mid K \cap h^{-1}\{z: |z| \leqq r\} \in A(K \cap h^{-1}\{z: |z| \leqq r\})$. Applying the theorem, $f \in A(K)$.

Thus a local condition on a compact set in the plane which implies that any continuous function can be uniformly approximated by rational functions, such as Theorems 2.4 and 3.4 in [6], can be applied in coordinate neighborhoods of every point of K to show $A(K) = C(K)$. As a special case, using Theorem 2.4 of [6] we have the next corollary which we will need to prove uniqueness of boundary measures.

COROLLARY 2. *If K is a nowhere dense compact n-balanced subset of S, then $A(K) = C(K)$.*

We also obtain a generalization from the plane to Riemann surface of the approximation theorem of Bishop [4].

COROLLARY 3. *If K is a compact nowhere dense subset of an open Riemann surface and M is the minimal boundary of $A(K)$, then $M = K$ implies $A(K) = C(K)$.*

Proof. Let h be an analytic coordinate function at $p \in K$ such that $h(p) = 0$ and the range of h is $\{z: |z| < 1\}$. Let r be $0 < r < 1$. Let M' be the minimal boundary of $A(h(K) \cap \{z: |z| \leqq r\})$. Let $z \in h(K)$ and $|z| \leqq r$, then $h^{-1}(z) \in K = M$. There exists $f \in A(K)$ such that $f(h^{-1}(z)) = 1$ and $|f(q)| < 1$ if $q \in K$ and $q \neq h^{-1}(z)$.

$$f \circ h^{-1} \in A(h(K) \cap \{z: |z| \leqq r\}), \quad f \circ h^{-1}(z) = 1, \quad |f \circ h^{-1}(z)| < 1$$

if $\zeta \in h(K)$ and $|\zeta| \leqq r$, $\zeta \neq z$. Thus $z \in M'$. Since

$$M' = h(K) \cap \{z: |z| \leqq r\},$$

by Theorem 4 in [4], we have

$$A(h(K) \cap \{z: |z| \leqq r\}) = C(h(K) \cap \{z: |z| \leqq r\}).$$

Now the theorem applies and we have $A(K) = C(K)$.

LEMMA 9. *Suppose K is an n-balanced compact subset of S. If μ is a measure on the boundary B of K which is orthogonal to all rational functions on S with poles in the interior of K* or *in $S \sim K$, then $\mu = 0$.*

Proof. The hypothesis implies μ is orthogonal to $A(B)$. By Lemma 1, B is an n-balanced nowhere dense compact subset of S. Thus by Corollary 2, $A(B) = C(B)$ and $\mu = 0$.

THEOREM 6. *If K is a compact n-balanced subset of S and ω is an analytic differential on the interior of K, then the boundary measure μ of ω which exists by Theorem 1 is unique, and if h is an analytic coordinate function on an open set $V \subset S$ and $\omega = f(z)dz$ on $h(V)$, then*

$$f(z) = (2\pi i)^{-1}\int -A(z, q)\, d\mu(q)$$

where $\alpha(p) = A(z, p)dz$ on $h(V)$.

Proof. Suppose μ and ν are both boundary measures of ω. Let g be a rational function on S with poles in the interior or the complement of K. Then

$$\int gd(\mu - \nu) = \int gd\mu - \int gd\nu = \lim_n \int_{\delta_n} g\omega - \lim_n \int_{\gamma_n} g\omega\ .$$

If n is large enough so both δ_n and γ_n surround all the poles of g which lie in the interior of K, then

$$\int_{\delta_n} g\omega = \Sigma_{p \in \mathrm{int} K} Res_p(g\omega) = \int_{\gamma_n} g\omega\ .$$

Therefore $\int gd(\mu - \nu) = 0$ and by Lemma 9, $\mu = \nu$.

$A(z, q)$ is meromorphic in q with a simple pole of residue -1 at $h^{-1}(z)$. Thus

$$(2\pi i)^{-1}\int - A(z, q)\, d\mu(q) = (2\pi i)^{-1} \lim_n \int_{\gamma_n} -A(z, q)\omega =$$
$$- Res_{h^{-1}(z)}(A(z, q)\omega) = f(z)\ .$$

THEOREM 7. *Let K be a compact n-balanced subset of S with interior U and let $\mu \in M(K)$. Then there exists a differential $\omega \in H(U)$ such that μ is the boundary measure of ω.*

Proof. Let $f_1, \cdots, f_l$ be the finite set of functions analytic on S and satisfying the conditions of Lemma 8 using coordinate neighborhoods for the covering. The proof will be by induction on l. If $l = 0$, K lies in a single coordinate neighborhood and we may consider K as a subset of the plane. In this case we have the result in Theorem 3.

Suppose the theorem is true for $l - 1$. Let $1/4 < x_0 < 3/4$ satisfy the conditions of Lemma 5 for f_l, μ, K. Let L_{x_0}, R_{x_0}, and β_{x_0} be as in Lemma 5. By Lemma 6, R_{x_0} and L_{x_0} are compact m-balanced sets for some m and by Lemma 7, $\mu \mid R_{x_0} - \beta_{x_0} \in M(R_{x_0})$ and $\mu \mid L_{x_0} + \beta_{x_0} \in M(L_{x_0})$. Since $f_1, \cdots, f_{l-1}$ partition R_{x_0} and L_{x_0} in the sense of Lemma 8, the induction hypothesis applies. Thus we have analytic differentials ω_1 and ω_2 on the interiors of R_{x_0} and L_{x_0} for which $\mu \mid R_{x_0} - \beta_{x_0}$ and $\mu \mid L_{x_0} + \beta_{x_0}$ are the boundary measures respectively.

If h_1 is an analytic coordinate function on $V_1 \subset \operatorname{int} R_{x_0}$ and $\omega_1 = f_1(z)dz$ on $\mathrm{h}_1(V_1)$ and $\alpha(q) = A_1(z, q)dz$ on $\mathrm{h}_1(V_1)$ then

$$\begin{aligned} f_1(z) &= (2\pi i)^{-1} \int - A_1(z, q)d(\mu \mid R_{x_0} - \beta_{x_0})(q) \\ &= (2\pi i)^{-1} \int - A_1(z, q)d(\mu \mid R_{x_0} - \beta_{x_0})(q) \\ &+ (2\pi i)^{-1} \int - A_1(z, q)d(\mu \mid L_{x_0} + \beta_{x_0})(q) \\ &= (2\pi i)^{-1} \int - A_1(z, q)d\mu(q) \; . \end{aligned}$$

Similarly, if h_2 is an analytic coordinate function on $V_2 \subset \operatorname{int} L_{x_0}$ and $\omega_2 = f_2(z)dz$ on $h_2(V_2)$ and $\alpha(q) = A_2(z, q)dz$ on $h_2(V_2)$ then

$$f_2(z) = (2\pi i)^{-1} \int - A_2(z, q)d\mu(q) \; .$$

Since we have this for almost all x_0 between 1/4 and 3/4 we can define, for any coordinate function h on $V \subset U$, a differential $\omega = f(z)dz$ on $h(V)$ with

$$f(z) = (2\pi i)^{-1} \int - A(z, q)d\mu(q)$$

where $\alpha(q) = A(z, q)dz$ on $h(V)$, $\omega = \omega_1$ on $\operatorname{int} R_{x_0}$, and $\omega = \omega_2$ on $\operatorname{int} L_{x_0}$.

Let $1/4 < x_1 < x_2 < 3/4$ and both x_1 and x_2 satisfy the conditions of Lemma 5. Let the delimiting sequence of Definition 6 for the boundary measures $\mu \mid L_{x_2} + \beta_{x_2}$ and $\mu \mid R_{x_1} - \beta_{x_1}$ be $\{\delta_i\}$ and $\{\gamma_i\}$ respectively. Let Δ_i and Γ_i be the open sets of which δ_i and γ_i are the boundaries. Let p_j, V_j be the finite collection of points and coordinate neighborhoods obtained in Lemma 2 with h_j the analytic coordinate function on V_j. Let U_j be a closed neighborhood of p_j so that $U_j \subset V_j \cap U$. Let k_j be the maximum of $|f(h_j(p))|$ for $p \in U$ where $\omega = f(z)dz$ on V_j. Let $\varepsilon_j = (k_j^{-1}2^{-j})$. Using these U_j and ε_j we apply Lemma 2 to get φ_i a finite union of disjoint piecewise analytic simple closed curves forming the boundary of Φ_i and $|\varphi_i| \cup \Phi_i \subset U$. Furthermore, since $\{\delta_i\}$ and $(\gamma_i\}$ delimit the interiors of L_{x_2} and R_{x_1}, respectively, and $\Gamma_i \cup \Delta_i \subset \Phi_i$, $\{\varphi_i\}$ delimits U.

Finally we see that

$$\| \omega \|_{\varphi_i} \leqq \| \omega \|_{\gamma_i} + \| \omega \|_{\delta_i} + \Sigma_j k_j \varepsilon_j \leqq \| \omega \|_{\gamma_i} + \| \omega \|_{\delta_i} + 1 \; .$$

Therefore $\omega \in H(U)$ and by Theorem 1, ω has a boundary measure ν on the boundary of clsr U.

Now let g be a rational function on S with poles in U or $S \sim K$. Choose x, $1/4 < x_0 < 3/4$, as in Lemma 5 and so that no pole of g lies on $\{p: Re\, f_1(p) = x_0\}$. Let $\{\sigma_i\}$ and $\{\tau_i\}$ be the delimiting sequence of Definition 6 for the boundary measures $\mu \mid L_{x_0} + \beta_{x_0}$ and $\mu \mid R_{x_0} - \beta_{x_0}$ respectively. Then

$$\int g d(\mu - \nu) = \int g d(\mu \mid R_{x_0} - \beta_{x_0}) + \int g d(\mu \mid L_{x_0} + \beta_{x_0}) - \int g d\nu$$
$$= \lim_i \int_{\tau_i} g\omega + \lim_i \int_{\sigma_i} g\omega - \lim_i \int_{\varphi_{n_i}} g\omega .$$

Letting i be large enough so that all the poles of g in U are surrounded by φ_{n_i} and by either τ_i or σ_i and using the residue theorem we have

$$\int g d(\mu - \nu) = \int_{\tau_i} g\omega + \int_{\sigma_i} g\omega - \int_{\varphi_{n_i}} g\omega = 0 .$$

Thus by Lemma 9, $\mu - \nu = 0$ and μ is the boundary measure of ω.

COROLLARY 4. *If K is a compact n-balanced subset of S with interior U, then $A(K)$ consists of all functions in $C(K)$ which are analytic on U.*

Proof. Clearly every function in $A(K)$ is analytic on U. Suppose $A(K)$ does not contain all such functions in $C(K)$. Then there exists a continuous linear functional L orthogonal to $A(K)$ with $L(f) \neq 0$ for some $f \in C(K)$, f analytic on U. The boundary of K is the Silov boundary of the algebra of functions in $C(K)$ analytic on U, so there exists a measure μ on the boundary of K so that $\int g d\mu = L(g)$, all $g \in C(K)$, analytic on U. Thus $\mu \in M(K)$ and there exists $\omega \in H(U)$, so that

$$0 \neq L(f) = \int f d\mu = \lim_i \int_{\gamma_j} f\omega = 0$$

since f is analytic on U.

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FUNCTIONS WHICH OPERATE ON CHARACTERISTIC FUNCTIONS

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Let G be a locally compact abelian group and $B^{+}(G)$ the family of continuous, complex-valued non-negative definite functions on G. Set

$$B_1^{+}(G) = \{f \in B^{+}(G): f(0) < 1\}$$
$$\Phi(G) = \{f \in B^{+}(G): f(0) = 1\}$$

A complex-valued function defined on the open unit disk is said to *operate* on $\{B_1^{+}(G), B^{+}(G)\}$ if $f \in B_1^{+}(G)$ implies $F(f) \in B^{+}(G)$, similarly for $\{\Phi(G), \Phi(G)\}$. Recently C. S. Herz has given a proof of a conjecture of W. Rudin that F operates on $\{B_1^{+}(G), B^{+}(G)\}$ if and only if

$$(*) \qquad F(z) = \sum_{m,n=0}^{\infty} c_{mn} z^m \bar{z}^n,\ c_{mn} \geqq 0,\ |z| < 1\ .$$

for a certain class of G. We shall show by independent methods that F operates on $\Phi(R^1)$ if F is given by (*) for $|z| \leqq 1$ and $F(1) = 1$. This answers a question posed by E. Lukacs and provides in addition an alternate proof of Herz's theorem.

Let $\mathfrak{A}, \mathfrak{B}$ denote two familes of functions $a, b: X \to Y$. A function $F: Z \subseteqq Y \to Y$ is said *to operate on* $(\mathfrak{A}, \mathfrak{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteqq Z$ we have $F(a) \in \mathfrak{B}$. If $\mathfrak{A} = \mathfrak{B}$ we say simply that *F operates on* $\mathfrak{A}$. Recently there has been considerable interest in determining, for particular families $(\mathfrak{A}, \mathfrak{B})$ the class of functions which operate.

If $\mathfrak{A}$ is the family of complex-valued 2π-periodic functions on R^1 which have absolutely convergent Fourier series

$$\mathfrak{A} = \left\{a : a(\theta) \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \text{ with } \sum_{k=-\infty}^{\infty} |a_k| < \infty\right\}$$

then a classic result of N. Wiener [10] states that $1/a \in \mathfrak{A}$ provided that $a(\theta) \neq 0\ (0 \leqq \theta < 2\pi)$. P. Lévy [3] generalized Wiener's theorem by proving that analytic functions operate on $\mathfrak{A}$.

If $\mathfrak{A}$ is the family of all non-negative-definite matrices $(a_{i,j})$ with $-1 < a_{i,j} < 1$ then I. J. Schoenberg [8] proved that any continuous function F which operates on $\mathfrak{A}$, $F: (a_{i,j}) \to (F(a_{i,j}))$ must be of the form

$$F(x) = \sum_{n=0}^{\infty} c_n x^n$$
$$(c_n \geqq 0 \quad -1 < x < 1)$$

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The theorem of Wiener-Lévy can be obtained in a more general setting. Let G be a locally compact abelian group and $\hat{G}$ its dual group, i.e. the set of continuous homomorphisms of G into the multiplicative group of complex numbers of modulus one, endowed with the weak topology. For μ a complex-valued, regular measure on G with finite total variation we define its Fourier-Stieltjes transform by

$$\hat{\mu}(\hat{x}) = \int_G \hat{x}(x)\mu(dx) \quad (\hat{x} \in \hat{G})$$

and denote by $B(\hat{G})$ the family of such transforms. Then

THEOREM. *Real entire functions operate on* $B(\hat{G})$ (*see* [7] *for definition*).

In particular by taking $G = Z$ (the group of integers) we obtain the Wiener-Lévy theorem.

A few years ago a converse to this theorem was obtained by H. Helson, J. P. Kahane, Y. Katznelson and W. Rudin [1]. They proved that if F operates on $B(\hat{G})$ then F is a real-entire function.

In probability theory the elements of $B(\hat{G})$ which are of most direct interest are those $\hat{\mu}$ which arise from nonnegative measures μ, i.e. according to Bochner's theorem the $\hat{\mu}$ which are nonnegative-definite on $\hat{G}$. Let $B^+(\hat{G})$ denote this family. Rudin has conjectured [6] that the functions which operate on $(B_1^+(Z), B^+(Z))$[1] must have the form

$$F(z) = \sum_{\substack{n,m=0 \\ (c_{m,n} \geq 0)}}^{\infty} c_{n,m} z^n \bar{z}^m .$$

Recently C. S. Herz [2] published a proof of Rudin's conjecture for $(B_1^+(G), B^+(G))$ under certain restrictions on G. His proof consists of (1) showing that if F, defined on the unit disk, operates on $(B_1^+(G), B^+(G))$ then F operates on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$ where Γ_0 is the discrete multiplicative group of complex numbers of modulus one, and (2) characterizing the functions which operate on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$.

Lukacs posed in [5] the question of determining the class of functions which operate on the set of characteric functions $\Phi(R^1)$, where $\Phi(G) = \{f \in B^+(G) : f(0) = 1\}$.

We shall answer here the question posed by Lukacs, directly and by quite independent methods. This will actually yield an alternate proof of Herz's more general result by making use of some of his preliminary propositions. In § 1 we state the main theorem and outline the proof. The details occupy us in § 2–§ 4. In § 5 we show how to obtain the more general result.

[1] $Z =$ the additive group of integers with discrete topology, $B_1^+(G) = \{f \in B^+(G) : f(0) < 1\}$

1. Statement of the main theorem and outline of the proof.

THEOREM 1. *If F operates on $\Phi(R^1)$ then F is given by*

$$(*) \qquad F(z) = \sum_{\substack{n,m=0 \\ (c_{n,m} \geqq 0)}}^{\infty} c_{n,m} z^n \bar{z}^m \qquad (|z| \leqq 1) .$$

with $\sum_{n,m=0}^{\infty} c_{m,n} = 1$.

Assuming that F is continuous it is first shown that F operates on $B_1^+(R^1)$. It then follows that

$$F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k(r) \exp(ik\theta)$$

$(0 \leqq r \leqq 1)$ where $a_k(r) \geqq 0$ $(k = 0, \pm 1, \pm 2, \cdots)$. Having obtained this representation we prove that not only is $a_k(r)$ nonnegative, but also absolutely monotonic. Thus

$$(1) \qquad F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} a_{k,n} r^n \exp(ik\theta)$$

with $a_{k,n} \geqq 0$. On the other hand, if the theorem is to be true, then

$$F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \left\{ \sum_{\substack{n,m \geqq 0 \\ n-m=k}} c_{n,m} r^{n+m} \right\} \exp(ik\theta) .$$

In order to pass from (1) to (*) $a_k(r)$ must actually be of the form

$$a_k(r) = r^{|k|} \sum_{n=0}^{\infty} b_{k,n} r^{2n}$$

with $b_{k,n} \geqq 0$. To prove that the exponents of r in $a_k(r)$ increase by two can be done directly (Lemma 5). To prove that $a_k(r) = O(r^{|k|})$ (near $r = 0$) we introduce the more general representation of F

$$F(r_1 \exp(i\lambda_1 t) + r_2 \exp(i\lambda_2 t) + \cdots + r_n \exp(i\lambda_n t))$$
$$= \sum_{\substack{k_i=-\infty \\ 1 \leqq i \leqq n}}^{\infty} \alpha_{k_1,k_2,\cdots,k_n}(r_1, r_2, \cdots, r_n) \exp\left\{ i \sum_{j=1}^{n} k_j \lambda_j t \right\}$$

where $(r_1, r_2, \cdots, r_n)$ varies in a suitable cube of R^n. The vanishing of $a_k(r)$ to the correct order is then deduced from the simple observation that $\alpha_{k_1,k_2,\cdots,k_n}(r_1, r_2, \cdots, r_n) = O(r_1 r_2 \cdots r_n)$ if all $k_j \neq 0$ (Lemma 4).

Finally we turn to the question of continuity. Since $F(\phi)$ is a continuous function for every $\phi \in \Phi(R^1)$, the natural approach would be to prove directly that $z_n \to z_0$ implies $F(z_n) \to F(z_0)$ by constructing a ch.f. ϕ together with a bounded sequence $\{t_n\}$ such that $\phi(t_n) = z_n$.[2] However, as the referee has observed it suffices to prove a slightly weaker interpolation property; namely that some $\phi \in \Phi(R^1)$ exists which interpolates, on a bounded sequence, some subsequence of the $\{z_n\}$. His lemma and proof are given in § 4.

2. Several lemmata. In this section we assume that F is continuous on $\Delta = \{z: |z| \leqq 1\}$ and operates on $\Phi(R^1)$.

LEMMA 1. *If* $p \in B_1^+(R^1)$ *then* $F(p) \in B_1^+(R^1)$.

Proof. It suffices by Cramey's criterion [5, p. 65] to show that

$$\int_0^A\int_0^A F(p(t-u)) \exp(ix(t-u))dtdu \geqq 0$$

for all real x and $A > 0$. If the lemma were false there would exist therefore and $A_0 > 0$ and x_0 such that

$$\int_0^{A_0}\int_0^{A_0} F(p(t-u)) \exp(ix_0(t-u))dtdu = -d < 0^3 . \tag{2}$$

The function

$$p_\varepsilon(t) = \begin{cases} (1-p(0))\left(1 - \dfrac{|t|}{\varepsilon}\right) & \text{if } |t| \leqq \varepsilon \\ 0 & \text{if } |t| > \varepsilon \end{cases}$$

is in $B_1^+(R^1)$ for every $\varepsilon > 0$, [5, p. 70] and thus $\phi_\varepsilon = p_\varepsilon + p \in B^+(R^1)$. It is, in fact, in $\Phi(R^1)$ since $\phi_\varepsilon(0) = 1$. Because F operates on $\Phi(R^1)$.

$$\int_0^{A_0}\int_0^{A_0} F(\phi_\varepsilon(t-u)) \exp(ix_0(t-u))dtdu \geqq 0 . \tag{3}$$

On the other hand

$$\left|\int_0^{A_0}\int_0^{A_0} \{F(p(t-u)) - F(\phi_\varepsilon(t-u))\} \exp(ix_0(t-u))dtdu\right|$$
$$= \left|\int\int_{G_\varepsilon} \{F(p(t-u)) - F(\phi_\varepsilon(t-u))\} \exp(ix_0(t-u))dtdu\right| \leqq 4A_0\varepsilon$$
$$G_\varepsilon = \{(t, u): 0 \leqq t \leqq A_0,\ 0 \leqq u \leqq A_0,\ |t-u| \leqq \varepsilon\}$$

since $|F(z)| \leqq 1$ on Δ. If we take $\varepsilon < d/4A_0$ then (3) contradicts (2).

Let n be a positive integer and $2\pi, \lambda_1, \lambda_2, \cdots \lambda_n$ be rationally independent real numbers. For each vector $\boldsymbol{m} = (m_1, m_2, \cdots, m_n)$ with

[2] We were not able to deduce this strong interpolation property for $\Phi(R^1)$ and this necessitated a somewhat round about argument in the original version of this paper.

[3] That the integral in (2) is real follows from the easily verified identity $F(\bar{z}) = \overline{F(z)}$.

integral components and each vector $r = (r_1, r_2, \cdots, r_n)$ with $0 \leqq r_i < 1/n\ (1 \leqq i \leqq n)$ we formally define $a_m(r)$ by

$$(4) \qquad a_m(r) = \operatorname*{limit}_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F\Big(\sum_{k=1}^{n} r_k \exp(i\lambda_k t)\Big) \exp\Big\{-it\sum_{k=1}^{n} m_k\lambda_k\Big\}\, dt\ .$$

LEMMA 2. *The limit in* (4) *exists and is independent of* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *(provided that* $2\pi, \lambda_1, \lambda_2, \cdots, \lambda_n$ *are rationally independent real numbers).*

Proof. Combining Lemma 1 with the observation that

$$\sum_{k=1}^{n} r_k \exp(i\lambda_k \cdot) \in B_1^+(R^1)$$

we see

$$F\Big(\sum_{k=1}^{n} r_k \exp(i\lambda_k \cdot)\Big) \in B_1^+(R^1)$$

and hence the limit in (4) exists [5, p. 43].

The Kronecker-Weyl theorem [9] next shows that

$$(5) \qquad \begin{aligned} a_m(r) = \Big(\frac{1}{2\pi}\Big)^n \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} F\Big(\sum_{k=1}^{n} r_k \exp(i\phi_k)\Big) \\ \times \exp -i\sum_{k=1}^{n} m_k\phi_k \quad d\phi_1 d\phi_2 \cdots d\phi_n \end{aligned}$$

and hence $a_m(r)$ is independent of the particular $\{\lambda_j\}$ chosen.

A function f defined on the cube $0 \leqq x_i < a\ (1 \leqq i \leqq n)$ is called *absolutely monotonic function* if

$$\frac{\partial^{j_1+j_2+\cdots+j_n}}{\partial x_1^{j_1}\partial x_2^{j_2}\cdots\partial x_n^{j_n}} f(x_1, x_2, \cdots, x_n) \geqq 0$$

throughout the cube for $j_1, j_2, \cdots, j_n = 0, 1, 2, \cdots$ Just as in the case of one variable, an absolutely monotonic function admits a power series expansion with nonnegative coefficients.

LEMMA 3. *The pointwise limit of absolutely monotonic functions is absolutely monotonic.*

Proof. For $n = 1$ the lemma is well known. We then proceed by induction to $n + 1$. Suppose

$$\operatorname*{limit}_{k\to\infty} f_k(r_1, r_2, \cdots, r_{n+1}) = f(r_1, r_2, \cdots, r_{n+1})\ .$$

For fixed $r_1, r_2, \cdots, r_n$ we have

$$f_k(r_1, r_2, \cdots, r_{n+1}) = \sum_{j=0}^{\infty} a_{k,j}(r_1, r_2, \cdots, r_n) r_{n+1}^j \longrightarrow f(r_1, r_2, \cdots, r_{n+1})$$

and hence

$$f(r_1, r_2, \cdots, r_{n+1}) = \sum_{j=0}^{\infty} a_j(r_1, r_2, \cdots, r_n) r_{n+1}^j$$

with

$$a_j(r_1, r_2, \cdots, r_n) = \operatorname*{limit}_{k\to\infty} a_{k,j}(r_1, r_2, \cdots, r_n) .$$

Since $a_{k,j}(r_1, r_2, \cdots, r_n)$ is an absolutely monotonic function the induction hypothesis implies $a_j(r_1, r_2, \cdots, r_n)$ is likewise so and lemma is proved.

LEMMA 4. *In the cube* $0 \leqq r_i < 1/n \, (1 \leqq i \leqq n)$

(4i) $a_{\boldsymbol{m}}(r)$ *is an absolutely monotonic function*

$$(6) \qquad a_{\boldsymbol{m}}(r) = \sum_{\substack{0 \leqq i_j < \infty \\ 1 \leqq j \leqq n}} \alpha_{i_1, i_2, \cdots i_n}(\boldsymbol{m}) r_1^{i_1} r_2^{i_2} \cdots r_n^{i_n}$$

and

(4ii) *If* $m_i \neq 0$ *for every* $i \, (1 \leqq i \leqq n)$ *then* $\alpha_{i_1, i_2, \cdots, i_n}(\boldsymbol{m}) = 0$ *if* $i_j = 0$ *for some* $j \, (1 \leqq j \leqq n)$.

Proof. 1. Generalizing a result of Rudin [6, p. 618] we will show that if f is continuous in the cube $0 \leqq x_i < a \, (1 \leqq i \leqq n)$ and satisfies

$$(7) \qquad \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} f(a_1 + b_1 \cos\theta_1, a_2 + b_2 \cos\theta_2, \cdots, a_n + b_n \cos\theta_n) \times \prod_{k=1}^{n} \cos j_k\theta_k d\theta_k \geqq 0$$

for all integers $j_1, j_2, \cdots, j_n = 0, 1, 2, \cdots$ whenever $0 \leqq b_j \leqq a_j$, $a_j + b_j < a$, then f is absolutely monotonic in the cube $0 \leqq x_i < a \, (1 \leqq i \leqq n)$.

2. To see that $a_{\boldsymbol{m}}(r)$ satisfies (7) (with $a = 1/n$) we observe that

$$I = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} a_{\boldsymbol{m}}(a_1 + b_1 \cos\theta_1, \cdots, a_n + b_n \cos\theta_n) \times \prod_{k=1}^{n} \cos j_k\theta_k \, d\theta_k$$

$$= \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} a_{\boldsymbol{m}}(a_1 + b_1 \cos\theta_1, \cdots, a_n + b_n \cos\theta_n) \times \exp -i \sum_{k=1}^{n} j_k\theta_k \, d\theta_1 d\theta_2 \cdots d\theta_n$$

since the integrand in I is an even function of each of the $\{\theta_k\}$. Next, the integral representation of $a_{\boldsymbol{m}}(r)$ and the Kronecker-Weyl theorem yields

$$I = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi}$$
$$\times F((a_1 + b_1 \cos\theta_1)\exp(i\phi_1) + \cdots + (a_n + b_n \cos\theta_n)\exp(i\phi_n))$$
$$\times \exp -i \sum_{k=1}^{n} (j_k\theta_k + m_k\phi_k)\, d\theta_1 \cdots d\theta_n d\phi_1 \cdots d\phi_n \,.$$

A final application of the Kronecker-Weyl theorem shows

$$I = \operatorname*{limit}_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F\Big(\sum_{k=1}^{n} (a_k + b_k \cos\zeta_k t)\exp(i\lambda_k t)\Big)$$
$$\times \exp -i \sum_{k=1}^{n} (j_k\zeta_k + m_k\lambda_k)t\, dt^4$$

and this limit is nonnegative because

$$\sum_{k=1}^{n} (a_k + b_k \cos\zeta_k\cdot)\exp(i\lambda_k\cdot) \in B_1^+(R^1)\,,$$

Lemma 1 and [5, p. 43].

3. Suppose first that f satisfies (7) and is of class C^∞. To show that

$$\frac{\partial^{j_1+j_2++\cdots+j_n}}{\partial x_1^{j_1}\partial x_2^{j_2}\cdots\partial x_n^{j_n}} f(x_1, x_2, \cdots, x_n) \geqq 0 \tag{8}$$

in the cube $0 \leqq x_i < a\,(1 \leqq i \leqq n)$ we let $N = j_1 + j_2 + \cdots + j_n$ and write, by Taylor's theorem,

$$f(a_1 + b_1\cos\theta_1, \cdots, a_n + b_n\cos\theta_n)$$
$$= \sum_{k=0}^{N} \frac{1}{k!}\Big(b_1\cos\theta_1\frac{\partial}{\partial x_1} + \cdots + b_n \cos\frac{\partial}{\partial x_n}\Big)^k f\,\Big|_{\substack{x_i=a_i\\1\leqq i\leqq n}} \tag{9}$$
$$+ \frac{1}{(N+1)!}\Big(b_1\cos\theta_1\frac{\partial}{\partial x_1} + \cdots + b_n\cos\theta_n\frac{\partial}{\partial x_n}\Big)^{N+1} f\,\Big|_{\substack{x_i=a_i+\eta_i b_i\cos\theta_i\\1\leqq i\leqq n}}\,.$$

Multiply (9) by $\prod_{k=1}^{n}\cos j_k\theta_k\, d\theta_k$ and integrate from 0 to 2π. Set $b_i = b < \min_k a_k$ and let $b \downarrow 0$ to obtain (8).

4. If f is *a priori* only continuous, we proceed as follows: let $g\colon R^1 \to R^1$ satisfy

(i) $g \in C^\infty$

(ii) $g(t) > 0$ if $0 < t < 1$; $g(t) = 0$ otherwise

(iii) $\int_0^1 g(t)dt = 1$.

If f satisfies (7), then so does

$$f_\delta(x_1, x_2, \cdots, x_n) = \int_0^1\int_0^1 \cdots \int_0^1$$
$$\times f(x_1 + \delta y_1, \cdots, x_n + \delta y_n)\prod_{k=1}^{n} g(y_k)dy_k$$

[4] The numbers $2\pi, \lambda_1, \cdots, \lambda_n, \zeta_1, \cdots, \zeta_n$ are taken to be rationally independent real numbers.

on the cube $0 \leqq x_i < a - \delta\,(1 \leqq i \leqq n)$. Now $f_\delta \in C^\infty$ and the argument in 3. applies to show that f_δ is absolutely monotonic. But $f_\delta \to f$ (pointwise) in the cube $0 \leqq x_i < a\,(1 \leqq i \leqq n)$ and Lemma 3 permits us to complete the proof of 4(i).

5. If $m_k \neq 0\,(1 \leqq k \leqq n)$ then from (5) we see

$$a_{\boldsymbol{m}}(0, r_2, \cdots, r_n) = a_{\boldsymbol{m}}(r_1, 0, r_3, \cdots, r_n) = \cdots$$
$$= a_{\boldsymbol{m}}(r_1, r_2, \cdots, r_{n-1}, 0) = 0$$

and this yields (4)ii.

LEMMA 5. *If*

$$a_k(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r\exp(i\phi))\exp(-ik\phi)\,d\phi \tag{10}$$
$$k = 0, \pm 1, \pm 2, \cdots$$

then

5(i) $a_k(-r) = (-1)^k a_k(r)$

and

5(ii) $a_k(r) = \sum_{j=0}^{\infty} a_{k,j} r^j \quad -1 \leqq r \leqq 1$

with

$$a_{k,j} \geqq 0 \quad \sum_{j=0}^{\infty} a_{k,j} < \infty .$$

Thus

$$a_k(r) = \begin{cases} \sum_{j=0}^{\infty} a_{k,2j} r^{2j} & \textit{if } k \textit{ is an even integer} \\ \sum_{j=0}^{\infty} a_{k,2j+1} r^{2i+1} & \textit{if } k \textit{ is an odd integer}. \end{cases}$$

Proof. For 5(i) note

$$a_k(-r) = \frac{1}{2\pi}\int_0^{2\pi} F(r\exp i(\phi + \pi))\exp(-ik\phi)\,d\phi = (-1)^k a_k(r) .$$

Proceeding as in the proof of Lemma 4, we show that

$$\int_0^{2\pi} a_k(\cos\theta)\exp -i\nu\theta\,d\theta \geqq 0$$
$$\nu = 0, \pm 1, \pm 2, \cdots$$

so that $a_k(\cos\cdot) \in B^+(R^1)$. It follows from [4, p. 202] that

$$a_k(\cos\theta) = \sum_{j=0}^{\infty} b_{k,j}\cos j\theta$$

with

$$b_{k,j} \geqq 0 \sum_{j=0}^{\infty} b_{k,j} < \infty .$$

If T_j denotes the jth Tchebychev polynomial then

$$a_k(x) = \sum_{j=0}^{\infty} b_{k,j} T_j(x) \qquad -1 \leqq x \leqq 1 . \tag{11}$$

But for $0 \leqq x \leqq 1$, Lemma 4 yields the representation

$$a_k(x) = \sum_{j=0}^{\infty} a_{k,j} x^j$$

with

$$a_{k,j} \geqq 0 \sum_{j=0}^{\infty} a_{k,j} < \infty .$$

Using elementary properties of the Tchebychev polynomials and the fact that the Fourier series of a C^∞ function may be differentiated term-by-term, 5(i) and (11) imply that the equality

$$\sum_{j=0}^{\infty} a_{k,j} x^j = \sum_{j=0}^{\infty} b_{k,j} T_j(x)$$

extends to $-1 \leqq x \leqq 1$, and this proves 5(ii).

3. Proof of Theorem 1 with hypothesis of continuity. $F(r \exp(i\phi))$ is a continuous, periodic, nonnegative definite function. We can therefore write

$$F(r \exp(i\phi)) = \sum_{k=-\infty}^{\infty} a_k(r) \exp(ik\phi) \tag{12}$$

$$0 \leqq r \leqq 1 \qquad 0 \leqq \phi \leqq 2\pi$$

with

$$a_k(r) \geqq 0 \; (k = 0, \pm 1, \pm 2, \cdots) \sum_{k=-\infty}^{\infty} a_k(r) = F(r) .$$

In (12) we set $z = r \exp(i\phi)$ and use Lemma 5 to conclude that

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m + \sum_{1 \leqq m \leqq n < \infty} (d_{n,m} z^n / \bar{z}^m + e_{n,m} \bar{z}^n / z^m) \tag{13}$$

with

$$c_{n,m} \geqq 0 \; (n, m = 0, 1, 2, \cdots)$$
$$d_{n,m} \geqq 0 \; e_{n,m} \geqq 0 \; (1 \leqq m \leqq n < \infty)$$
$$\sum_{n,m=0}^{\infty} c_{n,m} + \sum_{1 \leqq m \leqq n < \infty} (d_{n,m} + e_{n,m}) = 1 .$$

We will now show that $d_{n_0,m_0} = 0$. Let $2\pi, \lambda_1, \cdots, \lambda_{n_0}, \lambda$ be rationally independent real numbers and set

(14) $$z = r \exp(i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(i\lambda_k t)$$

in (13) where

$$0 \leqq r < 2/3 \quad r_k = r/2n_0 \quad (1 \leqq k \leqq n_0).$$

Let $\boldsymbol{m} = (m_0, \underbrace{1, 1, \cdots, 1}_{n_0})$ and note by Lemma 4

(15) $$\begin{aligned} a_{\boldsymbol{m}}(r, r_1, r_2, \cdots, r_{n_0}) &= C_{\boldsymbol{m}} r r_1 r_2 \cdots r_{n_0} + o(r r_1 r_2 \cdots r_{n_0}) \\ &= C_{\boldsymbol{m}} \left(\frac{1}{2n_0}\right)^{n_0} r^{n_0+1} + o(r^{n_0+1}). \end{aligned}$$

Examing the term $z^\alpha/\bar{z}^\beta$ with z as in (14) we obtain

(16) $$\begin{aligned} &\frac{\left(r \exp(i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(i\lambda_k t)\right)^\alpha}{\left(r \exp(-i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(-i\lambda_k t)\right)^\beta} \\ &= r^{\alpha-\beta}\left(\exp(i\lambda t) + \frac{1}{2n_0}\sum_{k=1}^{n_0} \exp(i\lambda_k t)\right)^\alpha \exp(i\beta\lambda t) \\ &\quad \times \sum_{p=0}^{\infty} b_p \left\{\frac{1}{2n_0}\sum_{k=1}^{n_0} \exp(-i(\lambda_k - \lambda)t)\right\}^p \qquad (b = 1) \end{aligned}$$

so that only the terms $z^\alpha/\bar{z}^\beta$ with $\beta = m_0 - j$, $\alpha = n_0 + j$ $(0 \leqq j \leqq m_0 - 1)$ yield a contribution to $a_{\boldsymbol{m}}(r, r_1, r_2, \cdots, r_{n_0})$. But with z as in (14)

$$\begin{aligned} &\operatorname*{limit}_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} z^{n_0+j}/\bar{z}^{m_0-j} \exp(-i(m_0\lambda + \lambda_1 + \cdots + \lambda_{n_0})t)dt \\ &= D_j r^{n_0-m_0+2j} \end{aligned}$$

with $D_j \neq 0$ for $j = 0$. Thus (15) implies that $d_{n_0,m_0} = 0$. A similar argument shows $e_{n_0,m_0} = 0$ and the theorem is proved with the hypothesis of continuity.

4. The continuity of F[5]. We begin with an interpolation lemma.

LEMMA 6. *Let $z_n \to z_0$ $(|z_n| < 1, n = 0, 1, 2, \cdots)$. There exists a ch.f. ϕ, a sequence (of real numbers) $t_k \to 1$ and a sequence (of integers) $\{n_k\}$ such that $\phi(t_k) = z_{n_k}$.*

Proof. Let $\tau_n = 1 - (2/3)9^{-n}$; then $(9^n/2)\tau_n \equiv (1/6) \pmod 1$ while $(9^{n+m}/2)\tau_n \equiv (1/2) \pmod 1$ for $m > 0$. Hence

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[5] We wish to acknowledge our thanks to the referee for the statement and proof of Lemma 6.

$$\cos\frac{\pi}{2}9^n\tau_n = \frac{\sqrt{3}}{2},\ \cos\frac{\pi}{2}9^{n+m}\tau_n = 0\ (m > 0)$$

and $\cos(\pi/2)9^n = 0$. Let $\{\eta_n\}$ be a sequence of positive numbers such that

$$|z_0| + \sum_{n=1}^{\infty}\eta_n < 1 .$$

We define inductively a sequence $\{\phi_n\}$ of positive-definite functions as follows; let

$$\phi_0(t) = |z_0| e^{i(\arg z_0)t} .$$

Assume that $\phi_0, \phi_1, \cdots, \phi_p$ have been defined such that $\phi_j(1) = 0$ for $j > 0$. Choose integers m_{p+1} and n_{p+1} such that

$$r_{p+1} = \left|\sum_{j=0}^{p}\phi_j(\tau_{m_{p+1}}) - z_{n_{p+1}}\right| < \frac{\eta_{p+1}}{2} ,$$

and define

$$\phi_{p+1}(t) = 2r_{p+1}(\cos\varepsilon_{p+1}t)\left(\cos\frac{\pi}{2}9^{m_{p+1}}t\right)e^{i\lambda_{p+1}t}$$

where ε_{p+1} and λ_{p+1} are chosen such that

$$\phi_{p+1}(\tau_{m_{p+1}}) = z_{n_{p+1}} - \sum_{j=0}^{p}\phi_j(\tau_{m_{p+1}}) .$$

We shall assume that the sequence $\{m_k\}$ is strictly increasing. If we set $t_k = \tau_{m_k}$ and

$$\phi(t) = \sum_{j=0}^{\infty}\phi_j(t) + \varepsilon\Delta(t)$$

where $\Delta(x) = \max(0, 1 - 2|x|)$ and $\varepsilon > 0$ is such that $\phi(0) = 1$ then $\phi(t_k) = z_{n_k}$ $(k = 1, 2, \cdots)$ and $\phi \in \Phi(R^1)$.

LEMMA 7. *F is continuous in the open unit disk $\{z: |z| < 1\}$.*

Proof. Suppose not; then there would exist a $z_0, |z_0| < 1$ and a sequence $\{z_n\}$ ($|z_n| < 1$) such that $z_n \to z_0$ and $F(z_n) \not\to F(z_0)$. By passing to a subsequence if necessary we can assume that $\{F(z_n)\}$ converges. By Lemma 6 there is a ch.f. ϕ and a sequence (of real numbers) $\{t_k\}$ with limit one such that $\phi(t_k) = z_{n_k}$. But then

$$F(z_0) = F(\phi(1)) = \lim_{k\to\infty} F(\phi(t_k)) = \lim_{k\to\infty} F(z_{n_k})$$

which is a contradiction.

REMARK. For future reference let us note that Lemma 1 now shows that F operates on $B_1^+(R^1) \cup \Phi(R^1)$.

LEMMA 8. *F is continuous on* $-1 \leqq x \leqq 1$.

Proof. By observing that $F(\cos\cdot) \in \Phi(R^1)$, we obtain, just as in Lemma 5

$$F(x) = \sum_{n=0}^{\infty} p_n T_n(x)$$

where $p_n \geqq 0$ and

$$\sum_{n=0}^{\infty} p_n = 1 .$$

Since $| T_n(x) | \leqq 1$ on $-1 \leqq x \leqq 1$, F is continuous there.

THEOREM 2. *F is continuous on Δ.*

Proof. As we have already remarked, F operates on $B_1^+(R^1) \cup \Phi(R^1)$. Now Lemmata 2–5 carry over mutatis mutandis to prove that

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m \qquad |z| < 1 \tag{20}$$

where $c_{n,m} \geqq 0$. Setting $z = x$ in (20) and using Lemma 8 we see that

$$\lim_{x \uparrow 1} \sum_{k=0}^{\infty} \sum_{\substack{n,m \geqq 0 \\ n+m=k}}^{\infty} c_{n,m} x^k = F(1) = 1 .$$

But the $\{c_{n,m}\}$ are nonnegative and hence

$$\sum_{n,m=0}^{\infty} c_{n,m} = 1 .$$

Thus our series in (20) extends to a continuous function on Δ. We assert that F is equal to this extension. For let $\phi \in \Phi(R^1)$ $t_k \to t_0$ with $0 < |\phi(t_k)| < 1$, $|\phi(t_0)| = 1$. Then $F(\phi)$ is a continuous function and thus $\lim F(\phi(t_k)) = F(\phi(t_0))$. But

$$\begin{aligned} \lim F(\phi(t_k)) &= \lim \sum_{n,m=0}^{\infty} c_{n,m}(\phi(t_k))^n(\overline{\phi(t_k)})^m \\ &= \sum_{n,m=0}^{\infty} c_{n,m}(\phi(t_0))^n(\overline{\phi(t_0)})^m \end{aligned}$$

and thus

$$F(\phi(t_0)) = \sum_{n,m=0}^{\infty} c_{n,m}(\phi(t_0))^n(\overline{\phi(t_0)})^m .$$

5. Concluding remarks. In order to obtain the general theorem we require two propositions due to Herz [2 p. 165, p. 167].

PROPOSITION 1. If a locally compact abelian group H has elements of arbitrarily high order then every F which operates on $(B_1^+(H), B^+(H))$ is continuous.

PROPOSITION 2. If a locally compact abelian group H has elements of arbitrarily high order, then every F which operates on $(B_1^+(H), B^+(H))$ operates on $(B_1^+(Z), B^+(Z))$.

REMARKS. 1. In Propositions 1 and 2 it is assumed that F is defined on $\{z: |z| < 1\}$.

2. Proposition 1 does not include our Lemma 7 since we assume merely that F operates on $\Phi(R^1)$, not on $(B_1^+(R^1), B^+(R^1))$.

THEOREM 2. *If a locally compact abelian group H has elements of arbitrarily high order, then F operates on $(B_1^+(H), B^+(H))$ if and only if*

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m , \qquad (|z| < 1)$$

where $c_{n,m} \geqq 0$.

Proof. By Propositions 1 and 2 we may assume that $H = Z$ and that F is continuous. It suffices, by the proof of Theorem 1, to show that F operates on $(B_1^+(R^1), B^+(R^1))$. Suppose $\lambda \in B_1^+(R^1)$ and set $\phi = F(\lambda)$. Since ϕ is continuous all that must be verified is that ϕ is a nonnegative-definite function. For any $\delta > 0$, the sequence $\{\lambda_n = \lambda(n\delta)\}$ is nonnegative definite and therefore by the hypothesis $\{\phi(n\delta)\}$ is a nonnegative definite sequence for any $\delta > 0$. Since ϕ is continuous

$$\int_0^A \int_0^A \phi(u - v) \exp(ix(u - v)) du dv$$
$$= \operatorname*{limit}_{\delta \downarrow 0} \sum_{n,m=1}^{A/\delta} \phi((n - m)\delta) \exp ix\delta(n - m)\, \delta^2 .$$

But since $\{\phi(n\delta)\}$ is a nonnegative-definite sequence for each $\delta > 0$

$$\sum_{n,m=1}^{A/\delta} \phi((n - m)\delta) \exp ix\delta(n - m) \quad \delta^2 \geqq 0$$

and hence by Cramer's criterion ϕ is nonnegative definite.

We conclude with a few remarks.

1. There is a formal relation between the result of [1] and our Theorem 1. Every real-entire function F can be written in the form

$$F = (F_1 - F_2) + i(F_3 - F_4)$$

where F_1, F_2, F_3 and F_4 satisfy (*). On the other hand every $\hat{\mu} \in B(\hat{G})$ is of the form

$$\hat{\mu} = (\hat{\mu}_1 - \hat{\mu}_2) + i(\hat{\mu}_3 - \hat{\mu}_4)$$

where $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$ are in $B^+(\hat{G})$. A direct proof of our theorem starting from this observation would be desirable.

2. The proof given here of Theorem 1 demonstrates in one stroke that F is real-analytic in Δ and if it is expressed as a power series in z and $\bar{z}$ it has nonnegative coefficients. If one could prove directly that F operates on all Fourier transforms assuming values in Δ then proof of the theorem could be completed in two steps:

(A) F is real-analytic [7, Chapter VI] and thus

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m$$

(B) $c_{n,m} \geqq 0$ $(n, m = 0, 1, 2, \cdots)$ The second step is a consequence of the explicit representation

$$c_{n,m} = \underset{r \downarrow 0}{\text{limit}} \underset{T \to \infty}{\text{limit}} \frac{1}{r^{n+m}} \frac{1}{2T} \int_{-T}^{T} F\left(\sum_{k=1}^{n+m} r_k \exp(i\lambda_k t)\right) \times \exp\left(\sum_{k=1}^{n} \lambda_k t - i \sum_{k=1}^{m} \lambda_{n+k} t\right) dt$$ [6]

where the inner limit exists and is positive by virtue of Lemma 1 and [5, p. 43] and the outer limit exists by (A) above.

3. For nondiscrete G with elements of arbitrarily high order one can show by using the methods used in the proof of Theorem 1, that F operates on $\Phi(G)$ if and only if F satisfies (*). If G is discrete this needn't be the case, and F needn't even be continuous as, $F(z) = 0$ $(|Z| < 1)$, $=1$ $(|z| = 1)$, which operates on $\Phi(Z)$ already shows. For such discrete groups we don't know if it is true that F operates on $\Phi(G)$ implies that F must operate on $B_1^+(G)$. If it were true then at least the structure of F for $|z| < 1$ could be determined.

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ALMOST INVARIANT MEASURES

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Let μ be a regular complex-valued Borel measure on a locally compact topological (LC) group G which is finite on compact sets; and for each $s \in G$ define the measure $T_s\mu$ by $T_s\mu(E) = \mu(E + s)$, $E \in B_c(G)$ the collection of all Borel subsets of G with compact closure. If f is a function on G then for each $s \in G$ we set $T_sf(t) = f(t + s)$, $t \in G$. Let X be a translation invariant subspace of $C_0(G)$, the space of continuous complex-valued functions on G which vanish at infinity, i.e., a subspace such that $f \in X$ implies $T_{-s}f \in X$, $s \in G$; and let U be an open symmetric neighborhood of zero in G. Then we shall say μ *acts U-almost invariantly* on X if $\int_G |h(t)| d|\mu|(t) < \infty$, $h \in X$, and

$$\int_G h(t)dT_s\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G h(t)dT_{s_i}\mu(t) \qquad (s \in U,\ h \in X),$$

where $s_1, s_2, \cdots, s_n$ are fixed elements of U. We shall say μ is a *U-almost invariant measure* on G if $\{T_s\mu \mid s \in U\}$ spans a finite dimensional space of measures. When $U = G$ we shall say μ *acts almost invariantly* and μ is an *almost invariant measure*, respectively. The main results of this paper show that if μ acts U-almost invariantly on X then there exists some continuous function f such that

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t)\,, \qquad h \in X,$$

where dm is right invariant Haar measure on G; and that μ is a U-almost invariant measure if and only if there exists a continuous f such that $d\mu(t) = f(t)dm(t)$ and $\{T_sf \mid s \in U\}$ spans a finite dimensional space of functions.

We shall also establish the equivalence for connected groups of the two notions of acting almost invariantly and of the two notions of almost invariance, and shall say something about the uniqueness of measures which act U-almost invariantly.

We shall denote by $V(G)$ the linear space of all regular complex valued Borel measures on a LC group G, and by $C_c(G)$ the subspace of $C_0(G)$ consisting of those functions with compact support. Throughout the paper we shall use m and dm to denote right invariant Haar measure on the LC group G, i.e., $m(E + s) = m(E)$.

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REMARKS. (a) The concept of a measure which acts U-almost invariantly is o generalization of the notion of a measure acting invariantly, i.e. a measure μ such that $\int_G h(t)dT_s\mu(t) = \int_G h(t)d\mu(t)$, $s \in G$, $h \in X$. For abelian LC groups measures which act invariantly were considered in [1].

(b) Since, in general, we shall consider nonabelian groups it would perhaps be better to speak of measures which "act right U-almost invariantly" or are "right U-almost invariant". However, in the interests of notational simplicity we choose the terminology given above. It is easy to see that a similar development can be made using left invariant Haar measure, $T_s(E) = \mu(s + E)$ and $T_sf(t) = f(s + t)$.

(c) The restriction in the definitions that U be an open symmetric neighborhood of zero in G is mainly one of convenience. Indeed, it is not difficult to see that if W is a Borel subset of G with finite positive Haar measure for which

$$\int_G h(t)dT_s\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G h(t)dT_{s_i}\mu(t) \qquad (s \in W,\ h \in X)$$

then some left translate of $W + W$ contains an open symmetric neighborhood U of zero in G for which a similar relation holds. However, in the proofs which follow it is necessary that the Haar measure of U be positive.

2. Measures which act almost invariantly. For G a LC group and X a translation invariant subspace of $C_0(G)$ we shall denote by $L(X)$ the topological linear space of all linear complex-valued functionals on X with the topology given by pointwise convergence; i.e. a net of functionals $< F_\alpha > \subset L(X)$ converges to $F_0 \in L(X)$ if and only if $\lim F_\alpha(h) = F_0(h)$, $h \in X$. If $\mu \in V(G)$ acts U-almost invariantly on X, then for each $s \in G$ we define the functional $F_s \in L(X)$ by

$$F_s(h) = \int_G h(t)dT_s\mu(t) \qquad (h \in X).$$

This notation for the functionals F_s will be used consistently in the remainder of the paper. It should be noted that the functionals F_s need not be continuous.

The main result of this section is the following theorem which is comparable to Theorem 2 in [1].

THEOREM 1. *Let G be a LC group, X a translation invariant subspace of $C_0(G)$, $\mu \in V(G)$ and U an open symmetric neighborhood of zero* *in G. If μ acts U-almost invariantly on X then there exists a continuous function f such that*

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X) .$$

PROOF. Since μ acts U-almost invariantly on X it is clear from the definition 1 that, without loss of generality, we may write

$$F_s = \sum_{i=1}^{n} \alpha_i(s)F_{s_i} \qquad (s \in U)$$

where $F_{s_1}, F_{s_2}, \cdots, F_{s_n}$, are assumed to form a linearly independent subset of $L(X)$. Let C be the subspace of $L(X)$ spanned by $F_{s_1}, F_{s_2}, \cdots, F_{s_n}$.

It is easy to verify that the mapping $\varphi: G \rightarrow L(X)$ defined by $\varphi(s) = F_s$, $s \in G$, is continuous; and hence the mapping $\psi = \varphi \mid_U$ is a continuous mapping on U to C in the relative topology inherited from $L(X)$. Thus, since C is a finite dimensional subspace of $L(X)$, the mapping ψ is also continuous if we put on C the topology given by the norm, $\| \sum_{i=1}^{n} b_i F_{s_i} \| = \sum_{i=1}^{n} | b_i |$.

Furthermore, in this norm topology it is clear that the projection mappings $P_k : C \rightarrow C$ defined by

$$P_k\Big(\sum_{i=1}^{n} b_i F_{s_i}\Big) = b_k F_{s_k} , \qquad k = 1, 2, \cdots, n$$

are also continuous.

But then from the continuity of the composite mappings $P_k \circ \psi$, $k = 1, 2, \cdots, n$, it is immediate that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are continuous functions on U.

Let A be the set of all functions in $C_c(G)$ with support contained in U. For each $g \in A$ we define the linear functional $F_g \in L(X)$ by

$$F_g = \int_G g(s)F_s dm(s) .$$

This vector valued integral makes sense since the support of g lies in U and so

$$(1) \qquad F_g = \int_G g(s)\Big[\sum_{i=1}^{n} \alpha_i(s)F_{s_i}\Big]dm(s) = \sum_{i=1}^{n} \int_G g(s)\alpha_i(s)dm(s)F_{s_i} ,$$

where the coefficients in the last expression exist since $\alpha_1, \alpha_2, \cdots, \alpha_n$ are continuous on U.

Set $B = \{F_g \mid g \in A\}$. From (1) it is clear that $B \subset C$, and simple verification shows that B is linear space. Hence B is a closed subspace of C.

Let $< g_\beta > \subset A$ be a net of functions such that

(i) $g_\beta \geqq 0$, all β;

(ii) $\int_G g_\beta(t)dm(t) = 1$, all β;

(iii) for any open symmetric neighborhood W of zero in G there is a β_0 such that for $\beta \succ \beta_0$ the support of g_β is contained in W. (We shall call such a net of functions a *compact approximate identity.*) Then since $\alpha_1, \alpha_2, \cdots, \alpha_n$ are continuous on U, using (1), we obtain:

$$\begin{aligned}\lim_\beta F_{g_\beta} &= \lim_\beta \sum_{i=1}^n \int_G g_\beta(s)\alpha_i(s)dm(s)F_{s_i} \\ &= \sum_{i=1}^n \alpha_i(0)F_{s_i} = F_0 .\end{aligned}$$

Therefore $F_0 \in B$, and so there exists some $k \in A$ such that $F_0 = F_k = \int_G k(s)F_s dm(s)$.

But then for each $h \in X$,

$$\begin{aligned}\int_G h(t)d\mu(t) &= F_0(h) \\ &= F_k(h) \\ &= \int_G k(s)F_s(h)dm(s) \\ &= \int_G k(s)\int_G h(t-s)d\mu(t)dm(s) \\ &= \int_G\int_G k(s+t)h(-s)dm(s)d\mu(t) \\ &= \int_G\int_G k(-s+t)h(s)\Delta(-s)dm(s)d\mu(t) \\ &= \int_G h(s)\Delta(-s)\int_G k(-s+t)d\mu(t)dm(s) \\ &= \int_G h(s)f(s)dm(s) ,\end{aligned}$$

where Δ is the modular function of G and f is the continuous function defined by

$$f(s) = \Delta(-s)\int_G k(-s+t)d\mu(t) .$$

The applications of Fubini's theorem are valid since it is clear that

$$\int_G |k(s)| \int_G |h(t-s)| d|\mu|(t)dm(s) < \infty .$$

This completes the proof of the theorem.

REMARKS. (a) Clearly the function f is, in general, not unique.

(b) For Euclidean groups R^m, $m > 0$, it is easy to see that we may choose f to be infinitely differentiable.

(c) One cannot conclude that a measure which acts U-almost invariantly is either a U-almost invariant measure or even absolutely continuous with respect to m. For example let G be any infinite compact abelian group, X the space spanned by any nonzero continuous character $(., \gamma)$, $U = G$ and μ the measure with unit mass concentrated at zero. Then μ is neither almost invariant nor absolutely continuous, but it does act almost invariantly on X.

The next theorem shows that for connected groups the two notions of acting almost invariantly are identical.

THEOREM 2. *Let G be a connected LC group, X a translation invariant subspace of $C_0(G)$, $\mu \in V(G)$ and U an open symmetric neighborhood of zero in G. Then the following are equivalent:*

(i) *μ acts almost invariantly on X.*

(ii) *μ acts U-almost invariantly on X.*

PROOF.[1] Clearly (i) implies (ii)

Now suppose μ acts U-almost invariantly on X. Then the space C spanned by $\{F_s \mid s \in U\}$ is a finite dimensional subspace of $L(X)$. Let $E = \{s \mid s \in G, F_s \in C\}$. Without loss of generality we may write $F_s = \sum_{i=1}^n \alpha_i(s) F_{s_i}$, $s \in E$; where $s_1, s_2, \cdots, s_n$ are fixed elements of U.

Clearly E is not empty as $U \subset E$. We shall show that E is both open and closed, and hence, since G is connected, $E = G$; i.e. μ acts almost invariantly on X.

It is immediate from the finite dimensionality of C and the continuity of the mapping $\varphi: s \to F_s$, cited in the proof of Theorem 1, that E is a closed subset of G.

On the other hand, let $s_0 \in E$. Since U is an open symmetric neighborhood of zero in G there is an open symmetric neighborhood W of zero such that $W + s_i \subset U$, $i = 1, 2, \cdots, n$. Then $W + s_0$ is a neighborhood of s_0, and for each $s + s_0 \in W + s_0$ we have:

$$\begin{aligned} F_{s+s_0}(h) &= \int_G h(t) dT_{s+s_0}\mu(t) \\ &= \int_G h(t-s) dT_{s_0}\mu(t) \\ &= \sum_{i=1}^n \alpha_i(s_0) \int_G h(t-s) dT_{s_i}\mu(t) \\ &= \sum_{i=1}^n \alpha_i(s_0) F_{s+s_i}(h) \qquad (h \in X) . \end{aligned}$$

[1] The author is indebted to J. Lindenstrauss for suggesting the simple proof given here.

But $s + s_i \in W + s_i \subset U$, $i = 1, 2, \cdots, n$; and so $F_{s+s_i} \in C$; $i = 1, 2, \cdots, n$.

Thus for each $s + s_0 \in W + s_0$, we see that $F_{s+s_0} \in C$; and consequently E is open.

3. Almost invariant measures.

Theorem 1 provides us almost immediately with a necessary and sufficient condition for a measure to be U-almost invariant.

THEOREM 3. *Let G be a LC group, $\mu \in V(G)$ and U an open symmetric neighborhood of zero in G. Then the following are equivalent:*

(i) *μ is a U-almost invariant measure on G.*

(ii) *There is a continuous function f on G such that $d\mu(t) = f(t)dm(t)$ and $\{T_s f \mid s \in U\}$ spans a finite dimensional space of functions.*

Proof. Clearly (ii) implies (i). Suppose μ is a U-almost invariant measure. Then evidently μ acts U-almost invariantly on $X = C_c(G)$; and so by Theorem 1 there exists a continuous function f on G such that

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in C_c(G)) .$$

Consequently, from the regularity of μ it is easy to deduce that $d\mu(t) = f(t)dm(t)$ and that $\{T_s f \mid s \in U\}$ spans a finite dimensional space of functions; and this completes the proof.

Given a topological group G, let $FDT(G)$ be the space of all continuous complex-valued functions f on G such that $\{T_s f \mid s \in G\}$ spans a finite dimensional space. As an immediate consequence of Theorem 3 we have the following theorem on almost invariant measures.

THEOREM 4. *Let G be a LC group and $\mu \in V(G)$. Then the following are equivalent:*

(i) *μ is an almost invariant measure on G.*

(ii) *There is an $f \in FDT(G)$ such that $d\mu(t) = f(t)dm(t)$.*

REMARKS. (a) For U-almost invariant measures it is clear that the dimensions of the spaces spanned by $\{T_s\mu \mid s \in U\}$ and $\{T_s f \mid s \in U\}$ must be the same.

(b) If μ is almost invariant and $T_s\mu = \sum_{i=1}^n \alpha_i(s) T_{s_i}\mu$, $s \in G$, it can be shown that $\alpha_1, \alpha_2, \cdots, \alpha_n \subset FDT(G)$; and that we may write $f = \sum_{i=1}^n f(s_i)\alpha_i$.

(c) In general for U-almost invariant measures the function f given by Theorem 3 need not belong to $FDT(G)$. For example, let $G = Z$, the additive group of the integers; $U = \{0\}$, and let μ be the measure with unit mass concentrated at zero. Then $f(0) = 1$, $f(t) = 0$, $t \neq 0$; and $f \notin FDT(Z)$.

(d) For a topological group G, let $D(G)$ be the space of all linear combinations of products of continuous complex-valued functions on G which are either additive or multiplicative; i.e. functions f such that either $f(s + t) = f(s) + f(t)$ or $f(s + t) = f(s)f(t)$. If G is an abelian topological group it is known that $FDT(G) = D(G)$ [2, p. 25]. Thus if G is a LCA group we can conclude that the function f of Theorem 4 belongs to $D(G)$.

(e) If $G = R^m$, $m > 0$, then the preceding remark implies that each almost invariant measure μ must be of the form

$$d\mu(t) = \sum_{j=1}^{l} P_j(t) \exp(b_j, t) dm(t) ,$$

where P_j are arbitrary polynomials with complex coefficients, $j = 1, 2, \cdots, l$; b_j are m-vectors of complex numbers, $j = 1, 2, \cdots, l$ and $t = (x_1, x_2, \cdots, x_m)$.

An immediate corollary to Theorem 4 is the following:

COROLLARY. *Let G be a LC group; $\mu \in V(G)$, $\mu \neq 0$, μ singular with respect to right invariant Haar measure. Then for each Borel set W in G with finite positive Haar measure, $\{T_s\mu \mid s \in W\}$ spans an infinite dimensional subspace of $V(G)$.*

Proof. Suppose the contrary, i.e. there exists a Borel set W of finite positive Haar measure for which $\{T_s\mu \mid s \in W\}$ spans a finite dimensional subspace of $V(G)$. Then from a remark of section one there exists an open symmetric neighborhood U of zero in G such that $\{T_s\mu \mid s \in U\}$ also spans a finite dimensional subspace of $V(G)$.

Thus, by Theorem 3, μ would be absolutely continuous with respect to Haar measure, and hence zero; contrary to the hypotheses of the corollary.

Considering measures $\mu \in V(G)$ as acting on the space $C_c(G)$, Theorem 2 implies that for connected LC groups the notions of almost invariant measures and U-almost invariant measures are equivalent. We state this result as Theorem 5.

THEOREM 5. *Let G be a connected LC group, $\mu \in V(G)$ and U an* *open symmetric neighborhood of zero in G. Then the following are equivalent*:

(i) μ *is an almost invariant measure on G.*

(ii) μ *is a U-almost invariant measure on G.*

4. Uniqueness theorems.

As noted previously, a measure μ may act U-almost invarianty on a subspace X of $C_0(G)$ without being a U-almost invariant measure. The next two theorems provide conditions which insure that a measure which acts U-almost invariantly is a U-almost invariant measure. The first theorem is a generalization of Theorem 1 in [1], and its proof is patterned after that in [1].

THEOREM 6. *Let G be a LC group, X a dense translation invariant subalgebra of* $C_0(G)$, $\mu \in V(G)$ *and U an open symmetric neighborhood of zero in G. If* μ *acts U-almost invariantly on X then* μ *is a U-almost invariant measure.*

Proof. Without loss of generality we may assume that

$$\int_G h(t-s)d\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G h(t-s_i)d\mu(t) \qquad (s \in U, h \in X). \tag{2}$$

For each $f \in C_c(G)$, since X is dense in $C_0(G)$, there is a function $g \in X$ such that g vanishes at no point of the support of f. Let $k = f/g$. Clearly $k \in C_c(G)$. Again by the denseness of X there is a sequence $< g_m > \subset X$ which converges uniformly to k.

Then it is easy to verify that

$$\lim_m \int_G g_m(t-s)g(t-s)d\mu(t) = \int_G f(t-s)d\mu(t) \qquad (s \in U), \tag{3}$$

and that

$$\lim_m \sum_{i=1}^{n} \alpha_i(s) \int_G g_m(t-s_i)g(t-s_i)d\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G f(t-s_i)d\mu(t) \qquad (s \in U). \tag{4}$$

But $< g_m g > \subset X$ as X is a subalgebra, and hence from (2) the left hand sides of (3) and (4), and thus the right hand sides, are equal.

Since this holds for each $f \in C_c(G)$ we conclude from the regularity of μ that μ is U-almost invariant.

If the group G is compact then the functionals F_s are bounded, and it is easy to see that in this case the preceding theorem remains true if we only require that X be a dense translation invariant subspace. This leads us to search for conditions on X other than the ones that it be a dense subalgebra which will insure that a measure which acts U-almost invariantly is a U-almost invariant measure. A result in this direction is given by the following theorem.

THEOREM 7. *Let G be a LC group, X a translation invariant subspace of $C_0(G)$, $\mu \in V(G)$ and U an open symmetric neighborhood of zero in G. If X contains a compact approximate identity and μ acts U-almost invariantly on X then μ is a U-almost invariant measure.*

Proof. Since μ acts U-almost invariantly on X, $\{F_s \mid s \in U\}$ spans a finite dimensional subspace B of $L(X)$.

Let C be the linear subspace of $V(G)$ spanned by $\{T_s\mu \mid s \in U\}$ and define the mapping $\Phi: C \rightarrow B$ by $\Phi(T_s\mu) = F_s$, $s \in U$. Clearly Φ maps C onto B.

Furthermore, we claim Φ is one-to-one. Indeed, let $\nu = \sum_{j=1}^{l} c_j T_{r_j}\mu$ be an element of C such that $\Phi(\nu) = 0$, i.e. $\int_G h(t)d\nu(t) = 0$, $h \in X$. Let $< g_\beta > \subset X$ be a compact approximate identity. Then, since X is translation invariant, for each $f \in C_c(G)$ we have

$$\begin{aligned} 0 &= \lim_\beta \int_G f(r) \int_G g_\beta(t - r)d\nu(t)dm(r) \\ &= \lim_\beta \int_G g_\beta(-r) \int_G f(r + t)d\nu(t)dm(r) \\ &= \int_G f(t)d\nu(t) \end{aligned}$$

since $\int_G f(. + t)d\nu(t)$ is continuous. The applications of Fubini's theorem are valid as both f and $< g_\beta >$ belong to $C_c(G)$.

Thus, by regularity, $\nu = 0$, and hence Φ is one-to-one.

But then Φ is a one-to-one linear mapping of C onto the finite dimensional space B. Therefore C is finite dimensional, i.e. μ is U-almost invariant.

REMARKS. (a) We have not, of course, circumvented the denseness assumption of Theorem 6; as any translation invariant subspace of $C_0(G)$ which contains a compact approximate identity is necessarily dense in $C_0(G)$.

(b) Let $\mu \in V(G)$ act U-almost invariantly on a translation invariant subspace X of $C_0(G)$, and let f be any function given by Theorem 1 such that

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X) .$$

In general the precise nature of f is not clear. A plausible conjecture, in the light of the structure of U-almost invariant measures, might be that one could always find some f as above for which $\{T_s f \mid s \in U\}$ spans a finite dimensional space of functions. Some support for this conjecture can be found in the fact that for compact groups and measures μ which act almost invariantly one can construct such a function f as a linear combination of the characters common to the space X and the support of the Fourier-Stieltjes transform of μ.

5. Additional comments. In a previous version of this paper a seemingly more general problem was considered. A subsemi-group S of a LC group G will be called *admissible* if

(i) S is an open subset of G and

(ii) the zero of G is a point of closure of S.

For such subsemigroups (with the obvious changes in the previous notation) one can consider translation invariant subspaces X of $C_0(S)$, measures $\mu \in V(S)$ and open symmetric neighborhoods U of zero in G such that

$$\int_S h(t)dT_s\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_S h(t)dT_{s_i}\mu(t) \qquad (s \in U \cap S, h \in X) ,$$

i.e. one can consider measures μ on S which act U-almost invariantly on translation invariant subspaces X of $C_0(S)$. Similarly one can consider $\mu \in V(S)$ which are U-almost invariant measures on S, i.e. $\{T_s\mu \mid s \in U \cap S\}$ spans a finite dimensional subspace of $V(S)$.

It is now seen that the most appropriate way to investigate such measures is to reduce the problem to the context of groups which was discussed in the preceding four sections. Let us indicate how this reduction takes place. We shall restrict ourselves to the case where $\mu \in V(S)$ acts U-almost invariantly; the situation for U-almost invariant measures is similar.

Suppose X is a translation invariant subspace of $C_0(S)$, $\mu \in V(S)$ and U is an open symmetric neighborhood of zero in G; and assume that μ acts U-almost invariantly on X. Define a new measure $\bar{\mu} \in V(G)$ by $\bar{\mu}(E) = \mu(E \cap S)$, $E \in B_c(G)$. This clearly defines a measure in $V(G)$ as S is an open subset of G. Also, since S is open the functions in $C_0(S)$ must vanish on the boundary of S, and hence we may consider $C_0(S)$ as a subspace of $C_0(G)$ by defining for each $f \in C_0(S)$, $f(t) \equiv 0$, $t \notin S$. Let Y be the subset of $C_0(G)$ which consists of X, considered as a subspace in $C_0(G)$, and all its translates by elements of G. Clearly Y is a translation invariant subspace of $C_0(G)$. Moreover, it is easy to check that $\int_G |h(t)| \, d|\bar{\mu}|(t) < \infty$, $h \in Y$, and that $\{F_s \mid s \in U \cap S\}$ spans a finite dimensional subspace of $L(Y)$.

But $U \cap S$ has finite positive Haar measure since S is open; and so by a remark in the first section there must exist some open symmetric neighborhood W of zero in G such that $\{F_s \mid s \in W\}$ spans a finite dimensional subspace of $L(Y)$, i.e. $\bar{\mu}$ acts W-almost invariantly on Y.

We now can employ the developement of the preceding sections to investigate $\bar{\mu}$, and then restricting the functions and measures so obtained to the admissible subsemi-group S we get the analogous information about the measure μ. In particular, one can in this fashion establish theorems for $\mu \in V(S)$, S an admissible subsemi-group of G, which are analogs of Theorems 1–7 above.

The reduction just obtained makes it clear that nothing really new is to be gained by a separate consideration of admissible subsemi-groups. Therefore a detailed exposition of this situation has been omitted.

REMARKS. It should be noted that a similar development for arbitrary subsemi-groups of G is not possible. Indeed, let $G = R$, the additive group of the real line; S_1 and S_2 the subsemi-groups of G defined by $S_1 = \{s \mid s \geqq 0\}$ and $S_2 = \{s \mid s > 1\}$; and define the measure $\mu_1 \in V(S_1)$ by $\mu_1(E) = 1$ if $0 \in E$, $\mu_1(E) = 0$ if $0 \notin E$; and the measure $\mu_2 \in V(S_2)$ by $\mu_2(E) = 1$ if $3/2 \in E$, $\mu_2(E) = 0$ if $3/2 \notin E$. Clearly neither S_1 nor S_2 is an admissible subsemi-group, as each violates one of the conditions for admissibility. Furthermore it is easy to check that $\{T_s\mu_i \mid s \in S_i\}$ $i = 1, 2$, span finite dimensional spaces, but that there exist no continuous functions f_i on S_i, $i = 1, 2$, for which $d\mu_i(t) = f_i(t)dm(t)$ and $\{T_s f_i \mid s \in Si\}$ span finite dimensional spaces of functions, $i = 1, 2$; i.e. the analog of Theorem 4 fails.

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GENERALIZED CHARACTER SEMIGROUPS: THE SCHWARZ DECOMPOSITION

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The author's résumé: A structure theorem due to Š. Schwarz asserts that if S is a finite abelian or a compact abelian semigroup admitting relative inverses, than the character semigroup of S is decomposed into a disjoint union of character groups of certain maximal subgroups of S. In this note, among other things, we generalize this Schwarz Decomposition Theorem to a broader class of semigroups, the so-called pseudo-invertible semigroups. We also relax the range of the characters from the semigroup of complex numbers to a more general semigroup.

For notations and terms not defined here see A. D. Wallace [11].

Throughout this paper, let S be always a compact commutative semigroup, unless otherwise stated. By a *character* of S is meant a continuous homomorphism of S into the multiplicative semigroup C of the complex numbers endowed with the usual Euclidean topology. The collection of all characters of S, with the value-wise multiplication of functions, endowed with the compact-open topology, forms a semigroup which will be denoted by $(S, C)^\wedge$ or simply $S^\wedge$, and will be called the *character semigroup* of S. Hewitt and Zuckerman [4] use the term *semicharacter*, in the discrete case, for not identically zero characters. Here we use $(\hat{S}, C)$ or simply $\hat{S}$, as distinguished from $S^\wedge$, to denote the collection of semicharacters of S. We note that $\hat{S}$, in general, need not be a semigroup. We first draw attention to the fact that if χ is a character of S, then $|\chi(x)| \leqq 1$ for every x in S. For, otherwise $\chi(S)$ would not be compact. Thus, in the study of characters, only the unit disc $\{z: |z| \leqq 1\}$ of the complex numbers is used. Let us write D for this unit disc. The set D itself forms an important semigroup which is compact, connected, commutative, cancellable,[1] has zero 0 and unit 1; moreover the circumference $\{z: |z| = 1\}$ of D is the maximal subgroup $H(1)$ and $D\backslash H(1)$ is an ideal. However, only some of these are needed as we shall see below.

Throughout the rest of this paper, let T be an arbitrary, but fixed, compact commutative cancellable semigroup with zero z and unit u[2] such that $T\backslash H(u)$ is a subsemigroup of T. By a *generalized character*

[1] A semigroup S is cancellable if and only if for any nonzero elements a, b, c in S such that $ab = ac$ or $ba = ca$, then $b = c$.

[2] It is to be understood that $z \neq u$.

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of S is meant a continuous homomorphism of S to T. As in the case of character semigroup, the collection $(S, T)^\wedge$ of all generalized characters of S, with value-wise multiplication of mappings and the compact-open topology, forms a commutative (topological) semigroup which will be called the *generalized character semigroup* of S. We write $(\hat{S}, T)$ for the collection of all not identically zero elements in $(S, T)^\wedge$. It is quite easy to see that if S is a group, then $(\hat{S}, T) = (\hat{S}, H(u))$ and $(\hat{S}, T)$ is a group.

THEOREM 1. *If S is discrete, then $(S, T)^\wedge$ is compact.*

Proof. Since S is discrete, the compact-open topology on $(S, T)^\wedge$ is the relative topology, on the set (not topologized) $(S, T)^\wedge$, of the Tychonoff product topology on the product $P\{T : s \in S\}$, which is compact by the Tychonoff theorem. The compactness of $(S, T)^\wedge$ now follows from the fact that $(S, T)^\wedge$ is a closed subset of $P\{T : s \in S\}$.

DEFINITION 1. For any χ in $(S, T)^\wedge$, the *support* of χ, $sp(\chi)$, is the set $\{x : x \in S, \chi(s) \neq z\}$.

We have immediately $sp(\chi_1 \cdot \chi_2) = sp(\chi_1) \cap sp(\chi_2)$ for any χ_1, χ_2 in $(S, T)^\wedge$. It is clear that if χ in $(S, T)^\wedge$ is not identically zero, $sp(\chi)$ is an open subsemigroup of S; such an open subsemigroup will be called a *supporting subsemigroup*. Since the support of the zero generalized character is the void set $\square$, as a convenience we also call $\square$ a supporting subsemigroup. *We write henceforth, $\mathscr{P}(S)$ or $\mathscr{P}$ for the collection of all supporting subsemigronps of S.*

DEFINITION 2 [8]. The *Rees partial-ordering* $\leqq$ on the set E of idempotents in S is the subset $\{(e_1, e_2) : (e_1, e_2) \in E \times E, e_1e_2 = e_1\}$. If (e_1, e_2) is in $\leqq$ we write, equivalently, $e_1 \leqq e_2$.

LEMMA 0 [13]. *Let $(X, \leqq)$ be a nonvoid compact topological space endowed with a quasi-ordering $\leqq$ such that for each t in X the set $\{x : x \in X, x \leqq t\}$ is closed in X. Then $(X, \leqq)$ has a minimal element.*

Proof. See Ward [13, Theorem 1].

LEMMA 1. *If S_0 is a compact subsemigroup of S, then $E(S_0)$ has a unique minimal element with respect to the Rees partial-ordering $\leqq$.*

Proof. Since S_0 is a compact semigroup, the set $E(S_0)$ of all idempotents in S_0 is a nonvoid closed subset of S_0. It is fairly easy to see that $\{x : x \in E(S_0), x \leqq t\}$ is closed for each t in $E(S_0)$. It then follows from Lemma 0 that $(E(S_0), \leqq)$ has a minimal element, say e_0. If there were another minimal element e_1 in $E(S_0)$, then one concludes $e_1 = e_0 e_1 = e_0$. This proves the uniqueness.

DEFINITION 3. If $P \in \mathscr{P}(S)$, then $\sigma(P)$ is the set

$$\{\chi : \chi \in (S, T)^\frown, sp(\chi) = P\} .$$

For each P in $\mathscr{P}(S)$, $\sigma(P)$ is easily seen to be a cancellable subsemigroup of $(S, T)^\frown$. In general, $(S, T)^\frown$ may be decomposed into the union of the disjoint family $\{\sigma(P) : P \in \mathscr{P}(S)\}$ of cancellable subsemigroups of $(S, T)^\frown$.

DEFINITION 4 [1]. *A semigroup S admitting relative inverses* is a semigroup such that to each x in S there is a pair (e, x') in $E \times S$ such that $xe = x = ex$ and $xx' = e = x'x$.

A well-known result of A. H. Clifford [1] says that a semigroup is a semigroup admitting relative inverses if and only if it is the disjoint union of its maximal subgroups. The more general class of semigroups that we are interested in is the following.

DEFINITION 5 [3]. A semigroup S is *pseudo-invertible* if and only if, to each element x in S there is an $\bar{x}$ in S such that

(i) $x\bar{x} = \bar{x}$;

(ii) $\bar{x}x^{n+1} = x^n$ for some positive integer n, and

(iii) $\bar{x}^2 x = \bar{x}$.

The element $\bar{x}$ satisfying conditions (i), (ii) and (iii) above turns out to be unique if it exists [3], in which case it is called the *pseudo-inverse* of x. A semigroup S is pseudo-invertible if and only if, to every x in S there is an integer $n > 0$ such that x^n is in some subgroup of S [3], [5], [6]. From this, one sees that the class of pseudo-invertible semigroups includes all semigroups admitting relative inverses, all periodic semigroups; all semigroups of matrices; all finite dimensional affine semigroups (for definition of an affine semigroup, see [2]) and many others.

LEMMA 2. *Let S be a compact commutative pseudo-invertible semigroup. Then each supporting subsemigroup P of S is open and closed. Therefore, if $P \neq \square$ then P has a unique minimal idempotent e_P with respect to the Rees partial-ordering on $E(P)$.*

Proof. To show each P in $\mathscr{P}$ is open and closed, we may consider only $P \neq \square$. For any nonvoid P in $\mathscr{P}$, there is a χ in $(S, T)^\wedge$ such that $P = \{x: x \in S, \chi(x) \neq z\}$. We show, for pseudo-invertible S, also, $P = \{x: x \in S, \chi(x) \in H(u)\}$ and consequently P is open and closed. To this end, let x be an arbitrary element of the nonvoid set $P = \{x: x \in S, \chi(x) \neq z\}$. Then since S is pseudo-invertible, there is a positive integer n such that $x^n \in H(e)$ for some e in E; thus, since $E(T) = \{z, u\}$, we must have $\chi(x)^n = \chi(x^n) \in H(u)$. Consequently, since $T \backslash H(u)$ is a subsemigroup, we obtain $\chi(x) \in H(u)$. This proves

$$\{x : x \in S, \chi(x) \neq z\} = P = \{x : x \in S, \chi(x) \in H(u)\} ,$$

so that P is open and closed

The set P being a closed subsemigroup of the compact semigroup S, by Lemma 1, $E(P)$ has a unique minimal element e_P with respect to the Rees partial ordering.

In the following, for $Q \subset S, e_Q$ will be the least idempotent in Q if it exists.

THEOREM 2. *Let S be a compact commutative pseudo-invertible semigroup. Then the generalized character semigroup $(S, T)^\wedge$ of S may be decomposed into the union of the disjoint family $\{(\widehat{H(e_P)}, T) : P \in \mathscr{P}\}$ of groups, where we agree that $(\hat{H}_\square, T) = \{0\}$.*

The proof of this theorem is contained in the following two lemmas. It should be noted that when $T \subset C$, $(\widehat{H(e)}, T)$ is the familiar character group $\widehat{H(e)}$.

LEMMA 3. *Under the hypothesis of Theorem 2, for any nonvoid P in $\mathscr{P}(S)$, the maximal subgroup $H(e_P)$ of S is the kernel of P; and the mapping $r_P: P \rightarrow H(e_P)$ which takes every x in P to xe_P is a (continuous) retraction of P onto $H(e_P)$.*

Proof. Let x be an arbitrary element of S. Let $\Gamma(x) = \{x^n : n \geqq 1\}^-$ and $N(x) = \cap \{x^n\Gamma(x) : n \geqq 1\}$, then $N(x)$ is the kernel of $\Gamma(x)$ as well as, since S is compact, a closed subgroup of S. Thus $N(x)$ contains a unique idempotent which is designated simply by e_x instead of the rather complicated symbol $e_{N(x)}$. If x is in P we have $e_x e_P = e_P$ and hence the unique idempotent in $N(xe_P)$ is e_P. Therefore, $xe_P = (xe_P)e_P \in N(xe_P) \subset H(e_P)$ for all x in P. To show $H(e_P)$ is an ideal of P, we first show that $H(e_P) \subset P$. This is true since there is a χ_P in $(S, T)^\wedge$ such that an element x of S is in P if and only if $\chi_P(x) \neq z$;

so $\chi_P(e_P) \neq z$ (consequently $\chi_P(e_P) = u$) and thus $\chi_P(x) \neq z$ for all x in $H(e_P)$. Now $PH(e_P) = PH(e_P)e_P \subset Pe_P \subset H(e_P)$ shows that $H(e_P)$ is an ideal of P; since it is a group it must be the minimal ideal of P. The fact that $r_P\colon P \to H(e_P)$ is a continuous retraction onto is then evident.

LEMMA 4. *Under the hypothesis of Theorem* 2, *for any nonvoid* P *in* $\mathscr{P}(S)$, $\sigma(P) = \{\chi : \chi \in (S, T)^\wedge, sp(\chi) = P\}$ *is isomorphic to the generalized character group* $(\hat{H}(e_P), T)$ *of the maximal subgroup* $H(e_P)$ *of* S. *In particular, if* $T \subset C$, $\sigma(P)$ *is also homeomorphic with the character group* $\hat{H}(e_P)$ *of* $H(e_P)$.

Proof. Let $h\colon \sigma(P) \to (\hat{H}(e_P), T)$ be the mapping which takes each χ in $\sigma(P)$ to $\chi \mid H(e_P)$. Clearly $\chi \in \sigma(P)$ implies $\chi \mid H(e_P) \in (\hat{H}(e_P), T)$. We have $h(\chi_1 \cdot \chi_2) = (\chi_1 \cdot \chi_2) \mid H(e_P) = (\chi_1 \mid H(e_P)) \cdot (\chi_2 \mid H(e_P)) = h(\chi_1) \cdot h(\chi_2)$ so that h is a homomorphism. To show h is an isomorphism, we show each φ in $(\hat{H}(e_P), T)$ may be extended, uniquely, to a χ in $\sigma(P)$. To this end, we define, for φ in $(\hat{H}(e_P), T)$.

$$\chi = \begin{cases} z, & \text{on } S\backslash P\,, \\ \varphi \circ r_p & \text{on } P\,. \end{cases}$$

This is continuous because r_P is continuous and P is open and closed. A routine verification shows that χ is an extension of φ to an element in $\sigma(P)$. Such an extension is unique as we shall now see. If χ' is any element in $\sigma(P)$ with $\chi' \mid H(e_P) = \varphi$, then $\chi'(x) = z = \chi(x)$ for all x in $S\backslash P$ and $\chi'(x) = \chi'(x) \cdot u = \chi'(x) \cdot \chi'(e_P) = \chi'(xe_P) = \varphi(xe_P) = \varphi \circ r_{p\ P}(x) = \chi(x)$, for all x in P. Therefore, h is an isomorphism of $\sigma(P)$ onto $(\hat{H}(e_P), T)$. It remains to show, in the case $T \subset C$, that $h\colon \sigma(P) \to \hat{H}(e_P)$ is also a homeomorphism. This follows from the fact that h is one-to-one, continuous and that $\hat{H}(e_P)$ is discrete [7].

COROLLARY. *If* S *satisfies the hypotheses of Theorem* 2, *and if* S *is connected, then* $\mathscr{P} = \{S, \Box\}$ *and hence* $(S, T)^\wedge = (\hat{H}(e_s), T) \cup \{0\} = (H(e_s), T)^\wedge$.

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REPRESENTATIONS OF LATTICE-ORDERED GROUPS HAVING A BASIS

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A convex l-subgroup C of a lattice-ordered group G is said to be a prime subgroup provided the collection $L(C)$ of left cosets of G by C is totally-ordered by the relation: $xC \leqq yC$ if and only if there exists $c \in C$ such that $xc \leqq y$. A collection $\bar{C}$ of prime subgroups of G is called a representation for G if $\bigcap \bar{C}$ contains no proper l-ideal of G. A representation $\bar{C}$ is said to be irreducible if the intersection of any proper subcollection of $\bar{C}$ does contain a proper l-ideal of G. $\bar{C}$ is a minimal representation if each element of $\bar{C}$ is a minimal prime subgroup. A representation $\bar{C}$ is *-irreducible if $\bigcap \bar{C} = \{1\}$ while $\bigcap (\bar{C} - \{C\}) \neq \{1\}$ for every $C \in \bar{C}$. In this paper it is shown that an l-group with a basis admits a minimal irreducible representation and that such a representation can be chosen in essentially only one way. In particular, an l-group with a normal basis has a unique minimal irreducible representation. In addition, two properties equivalent to the existence of a basis are derived; namely the existence of a representation $\bar{C}$ such that each element of $\bar{C}$ has a nontrivial polar and the existence of a *-irreducible representation.

For a linearly-ordered set L, let $P(L)$ denote the collection of all order-preserving permutations of L. $P(L)$ is a group under the operation of composition of functions, and is an l-group if $f \in P(L)$ is defined to be positive provided $f(x) \geqq x$ for all $x \in L$. C. Holland [2] has related an arbitrary l-group G to l-groups of the form $P(L)$ in the following way: Letting C be a prime subgroup of G, the collection $L(C)$ of left cosets of G by C is totally-ordered (by the relation mentioned above) and the map $g \rightarrow \bar{g}$ where $\bar{g}(xC) = gxC$ for all $xC \in L(C)$ is an l-homomorphism from G into $P(L(C))$. This map is called the *natural* l-homomorphism. If $C = \{C_i \mid i \in I\}$ is a represention for G and if δ_i denotes the natural l-homomorphism of G into $P(L(C_i))$, then the large cardinal product $\prod$ of the $\delta_i(G)$ contains an l-isomorphic copy of G as an l-subgroup and subdirect product. (This l-isomorphism is defined by $g \rightarrow (\cdots, \delta_i(g), \cdots)$.) It is for this reason that $\bar{C}$ is called a representation. The main result of [2] is that every l-group has a representation. If $\bar{C} = \{C_i \mid i \in I\}$ is a representation for G and if each

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C_i is an l-ideal of G, then $\bar{C}$ is called a *realization* of G. In this case, each $\delta_i(G)$ is a totally-ordered group and G is l-isomorphic to an l-subgroup and subdirect product of a cardinal product of o-groups. If $\bar{C}$ is an irreducible representation consisting of l-ideals, then $\bar{C}$ is called an *irreducible realization*.

2. Minimal irreducible representations of l-groups with basis.

An element s of an l-group G is *basic* provided $s > 1$ and $\{x \in G \mid 1 \leqq x \leqq s\}$ is totally-ordered by the order relation in G. A basic element s of G is *normal* if s and $g^{-1}sg$ are comparable ($g^{-1}sg \geqq s$ or $s > g^{-1}sg$) for all $g \in G$. For $x \in G$, the *absolute value* of x is defined by $|x| = x \vee x^{-1}$. Two elements x and y of G are said to be *disjoint* if $|x| \wedge |y| = 1$. A subset S of G is a *basis* for G if S is a maximal set of pairwise disjoint elements and each element of S is basic. A basis S is *normal* if each element of S is normal.

P. Lorenzen [4] has shown that an l-group G has a realization if and only if no positive element of G is disjoint from one of its conjugates. P. Jaffard [3] has proven that an abelian l-group has an irreducible realization if and only if it has a basis. F. Sik [5] generalized this result by showing that for an l-group G, the possession of a normal basis is equivalent to the existence of an irreducible realization of G. Using this result along with Lorenzen's, it is easily seen that an l-group G with a basis has a realization if and only if it has a normal basis.

It will now be shown that an l-group G with a basis has a minimal irreducible representation which can be chosen in essentially only one way. The construction depends upon those prime subgroups of G having nontrivial polars and not upon the choice of a basis. It will be shown further that the concept of a minimal irreducible representation is a direct generalization of the concept of an irreducible realization.

LEMMA 2.1. (*P. Conrad, unpublished*) *A convex l-subgroup C of an l-group G is prime if and only if the conditions $a, b \in G$ and $a \wedge b = 1$ imply $a \in C$ or $b \in C$.*

For an element x of an l-group G, let $D(x) = \{y \in G \mid |x| \wedge |y| = 1\}$. For a subset B of G, let $D(B) = \bigcap D(x)$ $(x \in B)$. Since each $D(x)$ is a convex l-subgroup of G, $D(B)$ is also a convex l-subgroup of G.

LEMMA 2.2. *Let C be a prime subgroup of the l-group G where $D(C) \neq \{1\}$. Let $s \in D(C)$, $s > 1$. Then s is basic, $C = D(s)$ and C is minimal prime. Conversely, if s is basic, then $D(s)$ is a prime subgroup of G and $s \in DD(s)$.*

Proof. Let $s \geqq x \geqq 1$ and $s \geqq y \geqq 1$. Then $x(x \wedge y)^{-1}, y(x \wedge y)^{-1} \in D(C)$ and $x(x \wedge y)^{-1} \wedge y(x \wedge y)^{-1} = 1$. It follows from Lemma 2.1 that $x(x \wedge y)^{-1} \in C \cap D(C)$ or $y(x \wedge y)^{-1} \in C \cap D(C)$ and so $x(x \wedge y)^{-1} = 1$ or $y(x \wedge y)^{-1} = 1$. Thus $y \geqq x$ or $x \geqq y$ and so s is basic. Since $s \notin C$, Lemma 2.1 implies that $D(s) \subseteqq C$ and since $s \in D(C)$ it is immediate that $C \subseteqq D(s)$. Thus $C = D(s)$. Since $D(s)$ is contained in any prime subgroup which does not contain s, it is clear that C is minimal.

Suppose s is basic and let $a, b \in G$ be such that $a \wedge b = 1$. If $a, b \notin D(s)$, then $s \geqq a \wedge s > 1$ and $s \geqq b \wedge s > 1$. Since s is basic, $a \wedge s \geqq b \wedge s$ or $b \wedge s > a \wedge s$. In either case it follows that $a \wedge b \wedge s > 1$, contradicting the assumption that $a \wedge b = 1$. Thus $a \wedge b = 1$ implies that $a \in D(s)$ or $b \in D(s)$ and so $D(s)$ is prime by Lemma 2.1. It is clear that $s \in DD(s)$.

LEMMA 2.3. *Let C_1 and C_2 be distinct prime subgroups of the l-group G. Then $D(C_1) \cap D(C_2) = \{1\}$.*

Proof. If $D(C_1) \cap D(C_2) \neq \{1\}$, let $s \in D(C_1) \cap D(C_2)$ where $s > 1$. By Lemma 2.2., $C_1 = D(s) = C_2$; and this contradicts the supposition that $C_1 \neq C_2$.

THEOREM 2.1. *Let C' be the collection of all prime subgroups C of the l-group G such that $D(C) \neq \{1\}$. For each $C \in C'$, let $s(C) \in D(C)$ where $s(C) > 1$. Then the following are equivalent:*

(a) $\{s(C) \mid C \in C'\}$ *is a basis for G.*

(b) $\bigcap C' = \{1\}$.

(c) C' *is a representation for G.*

In case any of these conditions hold, a subset $\bar{C}$ of C' is an irreducible representation if and only if $\bar{C}$ contains exactly one group from each conjugate class in C'.

Proof. Suppose that (a) holds and let $x \in G$ where $x > 1$. Then there exists $C \in C'$ such that $x \wedge s(C) > 1$. By Lemma 2.2., $D(s(C)) = C$ and so $x \notin C$. Thus $\bigcap C' = \{1\}$. (b) implies (c) by definition. If (c) holds and if $1 < x \in G$, then there exists $g \in G$ and $C \in C'$ such that $g^{-1}xg \notin C$. Thus $x \notin gCg^{-1}$ while $gCg^{-1} \in C'$. It follows from Lemma 2.1. that $x \wedge s(gCg^{-1}) > 1$. Therefore $\{s(C) \mid C \in C'\}$ is a basis for G.

It is clear that an irreducible representation cannot contain distinct conjugate subgroups. Suppose then that $\bar{C}$ contains exactly one group from each conjugate class in C'. Let $1 < x \in G$ and let $C \in C'$ be such that $x \wedge s(C) > 1$. There exists $g \in G$ such that $g^{-1}Cg \in \bar{C}$ and since

$x \notin C$ it follows that $g^{-1}xg \notin \bigcap \bar{C}$. Thus $\bigcap \bar{C}$ contains no proper l-ideal of G and so $\bar{C}$ is a representation of G. If E is a proper subcollection of $\bar{C}$, let $C \in C'$ be such that no conjugate of C is in E. If there exists $C_1 \in E$ and $g \in G$ such that $g^{-1}s(C)g \notin C_1$, then $s(C) \notin gC_1g^{-1}$ while $gC_1g^{-1} \in C'$. The only element of C' not containing $s(C)$ is C and so $gC_1g^{-1} = C$ contradicting the supposition that no conjugate of C is in E. Thus $\bigcap E$ does contain a proper l-ideal and so $\bar{C}$ is an irreducible representation of G.

COROLLARY 2.1. *If S is a basis of the l-group G, then $\{D(s) \mid s \in S\}$ is the set C' of all prime subgroups C of G which satisfy $D(C) \neq \{1\}$.*

Proof. By Lemma 2.2. $\{D(s) \mid s \in S\} \subseteqq C'$. Thus $\bigcap C' = \{1\}$ and so it follows from the Theorem that $\{D(s) \mid s \in S\} = C'$.

COROLLARY 2.2. *Every l-group with a basis admits a minimal irreducible representation.*

COROLLARY 2.3. *An l-group G has a representation $\bar{C}$ such that $D(C) \neq \{1\}$ for each $C \in \bar{C}$ if and only if G has a basis.*

The above results show one way in which a minimal irreducible representation can be chosen for an l-group with a basis. The following shows that this is the only way in which such a representation can be chosen.

THEOREM 2.2. *If an l-group G has a basis S and if $\bar{C}$ is a minimal irreducible represntation for G, then $\bar{C} \subseteqq C' = \{D(s) \mid s \in S\}$. Thus $\bar{C}$ contains exactly one group from each conjugate class in C'.*

Proof. Let $C \in \bar{C}$. Then $\bigcap (\bar{C} - \{C\})$ contains a proper l-ideal N of G. (For the purpose of the following argument, let $N = G$ in case $\bar{C}$ has only one element.) Let $1 < g \in N$ and choose $s \in S$ such that $1 < g \wedge s \leqq s$. Then $g \wedge s$ is basic and since $1 < g \wedge s \leqq g$, $h^{-1}(g \wedge s)h \in N$ for all $h \in G$. Since $\bigcap \bar{C}$ does not contain an l-ideal of G, there exists $k \in G$ such that $k^{-1}(g \wedge s)k \notin C$. Moreover, $k^{-1}(g \wedge s)k$ is basic. Since C is prime, it follows that $D(k^{-1}(g \wedge s)k \subseteqq C$. The minimality of C implies that $D(k^{-1}(g \wedge s)k) = C$. It follows from Corollary 2.1. that $C \in C'$.

It is easily seen that a basic element s is normal if and only if $D(s)$ is an l-ideal. The following is then immediate.

COROLLARY 2.4. *An l-group with a normal basis has a unique minimal irreducible representation $\bar{C}$ and each element of $\bar{C}$ is an l-ideal. Thus $\bar{C}$ is an irreducible realization.*

THEOREM 2.3. *A representation $\bar{C}$ of an l-group G is *-irreducible if and only if G has a basis S and $\bar{C} = \{D(s) \mid s \in S\}$.*

Proof. If G has a basis S and if $\bar{C} = \{D(s) \mid s \in S\}$, it is clear that $\bar{C}$ is a *-irreducible representation of G.

Suppose then that $\bar{C}$ is a *-irreducible representation and let C' denote the collection of prime subgroups C of G such that $D(C) \neq \{1\}$. Let $C_1 \in \bar{C}$ and let $1 < g \in \bigcap (\bar{C} - \{C_1\})$. If $1 < h \in C_1$ then $g \wedge h \in \bigcap \bar{C}$ and so $g \wedge h = 1$. Thus $D(C_1) \neq \{1\}$ and so $\bar{C} \subseteqq C'$. It follows that $\bigcap C' = \{1\}$ and therefore by Theorem 2.1. that $\{s(C) \mid C \in C'\}$ is a basis for G. By Corollary 2.1., $C' = \{D(s(C)) \mid C \in C'\}$. Since the intersection of any proper subcollection of C' is nontrivial, it follows that $\bar{C} = C'$.

COROLLARY 2.5. (*F. Sik* [5]) *An l-group G has a normal basis if and only if it has an irreducible realization.*

Proof. If G has a normal basis S then $C' = \{D(s) \mid s \in S\}$ is an irreducible realization.

If $\bar{C}$ is an irreducible realization of G, then $\bar{C}$ is a *-irreducible representation of G. It follows from the Theorem that G has a basis S and $\bar{C} = \{D(s) \mid s \in S\}$. Thus each $D(s)$ is an l-ideal of G and so S is a normal basis for G.

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ON RELATIVE COIMMUNITY

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The paper relates to questions raised by A. A. Muchnik in a 1956 Doklady abstract, namely, whether a noncreative r.e. set can be simple in a creative one, and whether a creative r.e. set can be simple in a noncreative one. We furnish a negative answer to the second question, and give a variety of partial results having to do with the first. Thus, we show that no universal set can have immune relative complement inside a noncreative r.e. set and that any r.e. set which is hyperhypersimple in a creative set must itself be creative; whereas, there exist three sets α, β, γ, $\alpha \subseteq \beta \subseteq \gamma$, such that β is creative, α and γ are nonuniversal, and both $\beta - \alpha$ and $\gamma - \beta$ are hyperhyperimmune.

In addition, we answer two questions of J. P. Cleave regarding the comparison of effectively inseparable (e.i.) and "almost effectively inseparable" (almost e.i.) sequences of r.e. sets. Thus: a sequence can be almost e.i. without being e.i.; and an almost e.i. sequence of disjoint r.e. sets may have a noncreative union.

1. In [7], Muchnik formulated (in slightly different language) the following two problems: given two r.e. sets Δ, Σ, with $\Delta \subseteq \Sigma$ and $\Sigma - \Delta$ immune, can we have

(1) Δ creative and Σ mesoic?

(2) Δ mesoic and Σ creative?

In the present paper, we consider these questions relative to not-necessarily-r.e. universal sets; and we make two or three applications of our results to matters considered in [7] and [1]. We are indebted to J. P. Cleave for providing us with a draft copy of [1], which has since been supplanted by a (forthcoming) joint paper of Cleave and C. E. M. Yates. (For an abstract of the Cleave-Yates paper, see [2].)

2. **Definitions and preliminary lemmas.** Basic terminology is essentially as in [3]. Notational departures from [3]: we use 'W_x' in place of 'ω_x', 'ϕ', in place of '0' for the null set, '$\cup$' for union, '$\cap$' for intersection, and '$-$' instead of a prime symbol for complementation. A set Δ of natural numbers is said to be *immune* just in case Δ is infinite and, for all i, if $W_i \subseteq \Delta$ then W_i is finite. If Δ,

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Σ are sets of numbers such that $\Delta \subseteqq \Sigma$ and $\Sigma - \Delta$ is immune, we say that Δ is *coimmune in* Σ. (In case $\Delta = W_j$, $\Sigma = W_k$, for some j and k, we say instead that Δ is *simple in* Σ.) Similarly, if $\Delta \subseteqq \Sigma$ and $\Sigma - \Delta$ is hyperhyperimmune, we say that Δ is *cohyperhyperimmune* in Σ, or that Δ is *hyperhypersimple in* Σ, in case both Δ and Σ are r.e. (For definition and discussion of the notion of hyperhyperimmunity, the reader may consult [9] or [10]; the existence of hyperhypersimple sets is known from [5].)

LEMMA 1. *There exists a set of numbers, α, such that both α and its complement, $\bar{\alpha}$, are hyperhyperimmune.*

Proof. This follows from the definition of hyperhyperimmunity ([10]) by a straightforward diagonal argument, since there are only countably many recursive sequences of pairwisedisjoint nonempty finite sets.

The terms 'creative', 'productive' 'contraproductive' 'mesoic' and 'simple', as applied to number sets, have their customary significance (see [3]). A mesoic set Δ is said to be *pseudosimple* just in case, for some number j, $W_j \subseteqq \bar{\Delta}$ and $\Delta \cup W_j$ is simple. We will make use of the (more or less) standard notations '$\leqq_{m-1}$' and '$\leqq_{1-1}$' for the relations of (recursive) many-one and one-to-one reducibility, respectively. By a *universal* set is meant a set Δ of numbers such that $W_j \leqq_{1-1} \Delta$ for all j (or, equivalently as it happens, $W_j \leqq_{m-1} \Delta$ for all j).

Lemma 2. ([11, *Chapter* 5, *Proposition* 2 *and Theorem* 6], *noting that g and t can be one-to-one in the cited Theorem* 6; *see also* [4, *Proposition* 1.12]). *Δ is universal if and only if $\bar{\Delta}$ is productive.*

Let an infinite set Δ be given. Suppose there is a partial recursive function p such that, for all j, if $W_j \subseteqq \Delta$ then j is in the domain of p and $(\forall i)(i \in W_j \Rightarrow p(j) > i)$. Then (and only then), we say that Δ is *strongly effectively immune*. An r.e. set with a strongly effectively immune complement is called *strongly effectively simple*. An example of a strongly effectively simple set: the simple-but-not-hypersimple set of Post [9]. The following fact is easy to establish, using a trick due to Myhill ([4]):

LEMMA 3. *If Δ is strongly effectively immune, then there is a recursive function, r, such that* $(\forall_j)(W_j \subseteqq \Delta \Rightarrow (\forall i)(i \in W_j \Rightarrow r(j) > i))$.

In [1] and [6], it has been noted that Friedberg's procedure ([5]) for decomposing a nonrecursive r.e. set into two nonrecursive, r.e., disjoint subsets can be extended to provide K(r.e.) components, for any K such that $2 \leqq K \leqq \aleph_0$, in such a way that, in the case $K = \aleph_0$, the components are presented *in a recursive sequence* (i.e., in a sequence indexed by a recursive function). In [1], extending an important observation of Yates, Cleave shows that if decomposition of a nonrecursive r.e. set Σ into K r.e. components ($2 \leqq K \leqq \aleph_0$) is carried out according to this extension of Friedberg's construction, then any two of the resulting components are recursively inseparable in a remarkably strong sense: namely, if W_j is any one of the components, then, for arbitrary k, $W_k \subseteqq \bar{W}_j \Rightarrow W_k - \Sigma$ is r.e.

In general, suppose $\{W_r\}_{r \in R}$ is an indexing of the set of components in a K-component decomposition of the nonrecursive r.e. set Σ into r.e. subsets ($2 \leqq K \leqq \aleph_0$), where R is understood to be r.e. in case $K = \aleph_0$. Then, we shall say that the decomposition in question is a *$CFY(K)$-decomposition* just in case, for any such index set R, $r \in R \Rightarrow (\forall_j)(W_j \subseteqq \bar{W}_r \Rightarrow W_j - \Sigma$ is r.e.).

Suppose that, in fact, there is a partial recursive function p such that $r \in R \Rightarrow (\forall_j)(W_j \subseteqq \bar{W}_r \Rightarrow p(r, j)$ is defined and $W_{p(r,j)} = (W_j - \Sigma) \cup$ (a finite subset of Σ). We shall, under these circumstances, say that the $CFY(K)$-decomposition of Σ whose components are given by the set $\{W_r\}_{r \in R}$ is a *strong* $CFY(K)$-decomposition of Σ.

The fundamental observation of Cleave and Yates is then just this:

LEMMA 4. *Let Σ be a nonrecursive r.e. set, and suppose $2 \leqq K \leqq \aleph_0$. Then Σ admits a strong $CFY(K)$-decomposition.*

The next two lemmas express simple but useful properties of $CFY(K)$-decompositions.

LEMMA 5. *Let Σ be an r.e., nonrecursive set, and W_j a component in a $CFY(K)$-decomposition of Σ. Then W_j is not simple in any r.e. set.*

Proof. Suppose, to the contrary, that $W_j \subseteqq W_k$, where $W_k - W_j$ is immune. The union of the components other than W_j is an r.e. set, say W_e; hence, since $W_k - W_j$ is immune and $W_j \cap W_e = \phi$, we have $W_e \cap W_k =$ a finite set. Therefore, $W_k - (W_e \cap W_k)$ is r.e., includes W_j, and misses W_e (and hence misses each of the components going to make up W_e). Thus, $(W_k - (W_e \cap W_k)) - W_j$ must be an r.e. set. But here is an absurdity, since $(W_k - (W_e \cap W_k)) - W_j$ must be immune. The lemma follows.

LEMMA 6. *Let Δ be either a creative set or a nonpseudosimple mesoic set. Let Δ_1, Δ_2 be the components in a $CFY(2)$-decomposition* *of Δ. Then, at least one of Δ_1, Δ_2 has the property of being neither pseudosimple nor many-one reducible to a simple set.*

Proof. Let Σ_1, Σ_2 be pseudosimple mesoic sets. Suppose $\Sigma_1 \cup \Sigma_2$ is neither simple nor recursive. Let Σ_1', Σ_2' be r.e. sets such that $\Sigma_1' \subseteq \bar{\Sigma}_1$, $\Sigma_2' \subseteq \bar{\Sigma}_2$, and $\Sigma_1 \cup \Sigma_1'$, $\Sigma_2 \cup \Sigma_2'$ are simple. Now, $(\Sigma_1 \cup \Sigma_1') \cap (\Sigma_2 \cup \Sigma_2')$ is simple, and is a subset of $(\Sigma_1 \cup \Sigma_2) \cup (\Sigma_1' \cap \Sigma_2')$. Hence, since $\Sigma_1 \cup \Sigma_2$ is not recursive, $\overline{(\Sigma_1 \cup \Sigma_2) \cup (\Sigma_1' \cap \Sigma_2')}$ is infinite and so $(\Sigma_1 \cup \Sigma_2) \cup (\Sigma_1' \cap \Sigma_2')$ is simple. Therefore $\Sigma_1 \cup \Sigma_2$ is pseudosimple. Thus, we see that either Δ_1 or Δ_2 must be nonpseudosimple. It is an evident feature of $CFY(K)$-decompositions that the components are pairwise recursively inseparable; and from this it follows that neither Δ_1 nor Δ_2 can be many-one reducible to a simple set. The lemma follows. (We will see later, in Theorem 6, that Δ_1, Δ_2 must be mesoic when Δ is creative, as well as when Δ is noncreative.)

Recall that W_i, W_j are termed *effectively inseparable* just in case $W_i \cap W_j = \phi$ and there is a partial recursive function p such that, for all k and m, if $W_i \subseteq W_k$, $W_j \subseteq W_m$, and $W_k \cap W_m = \phi$, then $p(k, m)$ is defined and lies outside $W_k \cup W_m$. In [1], Cleave considers the following two sequential variations on this concept:

Let $\{W_{r(i)}\}$ be a recursive sequence (i.e., the indexing function r is recursive) of pairwise-disjoint, nonrecursive r.e. sets. $\{W_{r(i)}\}$ is *e.i.* (Cleave) just in case, for $i \neq j$, there is a partial recursive function $P_{i,j}$ such that, if $W_{r(i)} \subseteq W_k$, $W_{r(j)} \subseteq W_m$, and $W_k \cap W_m = \phi$, then $P_{i,j}(k, m)$ exists and lies outside $W_k \cup W_m \cup (\cup_n W_{r(n)})$. Again, Cleave calls $\{W_{r(i)}\}$ *almost e.i.* just in case, whenever $i \neq j$, there is a partial recursive function $p_{i,j}$ such that, if $W_{r(i)} \subseteq W_k$, $W_{r(j)} \subseteq W_m$, and $W_k \cap W_m = \phi$, then $p_{i,j}(k, m)$ is defined and $W_{p_{i,j}(k,m)}$ is an infinite recursive set whose intersection with $W_k \cup W_m \cup (\cup_n W_{r(n)})$ is finite. Cleave shows, in [1], that the $CFY(\aleph_0)$-decomposition of the creative set $\{x \mid x \in W_x\}$ given by the extended Friedberg construction presents an almost e.i. sequence; his argument, in fact, is valid for *any strong* $CFY(\aleph_0)$-decomposition of a creative set[1]. He then asks:

(1) Do there exist almost e.i., non-e.i. sequences?

(2) Must the union of the terms of an almost e.i. sequence be creative?

In § 3 we shall provide pleasantly straight-forward proofs that the answers to these two questions are, respectively, "yes" and "no".

One other concept, of Muchnik's ([7]), will receive a little of our attention in § 3: the notion of "sets-of-a-pair in an r.e. set". We

[1]This proof of Cleave's, showing that any strong $CFY(\aleph_0)$-decomposition of a creative set presents an almost e.i. sequence, will, presumably, appear in the paper corresponding to [2].

rephrase Muchnik's original definition as follows: Disjoint r.e. sets Δ_0, Δ_1 are said to be *sets-of-a-pair in the r.e. set* Σ just in case $\Delta_0 \cup \Delta_1 \subseteq \Sigma$, $\Sigma - (\Delta_0 \cup \Delta_1)$ is infinite, Σ is indeed r.e., and, for all i and for $j = 0, 1$, $\Delta_j \subseteq W_i \subseteq \Sigma \Rightarrow [(W_i - \Delta_j$ is finite) or $(W_i \cap \Delta_{1-j} \neq \phi)]$. We shall say that an r.e. set Δ is *SOPRE* just in case there exist two other r.e. sets, Σ_1 and Σ_2, such that Δ, Σ_1 are sets-of-a-pair in the r.e. set Σ_2.

REMARK. It is not hard to show that any creative set is SOPRE. In Theorem 9 we will put forward an additional bit of information on SOPREness.

One further lemma will prove handy in § 3.

LEMMA 7. *The question whether a mesoic set can be coimmune in a universal set reduces to the question whether a mesoic set can be simple in a creative set.*

Proof. Suppose Δ is universal, Σ mesoic, and Σ is coimmune in Δ. Let Σ_1, Σ_2 be an effectively inseparable pair of disjoint creative sets. Since Δ is universal, there is a one-to-one recursive function f such that $f(\Sigma_1) \subseteq \Delta$, $f(\Sigma_2) \subseteq \bar{\Delta}$. Now, $f(\Sigma_1)$, $f(\Sigma_2)$ are themselves effectively inseparable ([**11**, Chapter 5, *Proposition* 4]). Hence, since $\Sigma \subseteq \overline{f(\Sigma_2)}$, the sets $f(\Sigma_1) \cup \Sigma$, $f(\Sigma_2)$ are effectively inseparable. Therefore, $f(\Sigma_1) \cup \Sigma$ is creative. Hence $(f(\Sigma_1) \cup \Sigma) - \Sigma$ must be infinite; and so Σ is simple in the creative set $f(\Sigma_1) \cup \Sigma$, proving the lemma.

3. Theorems.

THEOREM 1. *A universal set cannot be coimmune in a mesoic set.*

Proof. Suppose that Δ is universal, Σ mesoic, $\Delta \subseteq \Sigma$, and $\Sigma - \Delta$ is immune. Let Σ_1, Σ_2 be disjoint, effectively inseparable r.e. sets. Let f be a one-to-one recursive function reducing Σ_1 to Δ. Then $f(\Sigma_2) \subseteq \bar{\Delta}$; hence, since $\Sigma - \Delta$ is immune, $f(\Sigma_2) \cap \Sigma$ must be finite. Therefore, $\Sigma - (f(\Sigma_2) \cap \Sigma) = \Sigma_3$ is a mesoic superset of $f(\Sigma_1)$ which is disjoint from $f(\Sigma_2)$. But $f(\Sigma_1)$, $f(\Sigma_2)$ are effectively inseparable; and so also Σ_3, $f(\Sigma_2)$ are effectively inseparable. But this is impossible, since Σ_3 is mesoic. The theorem follows.

THEOREM 2. *If Δ is a universal set, then there are two non-universal sets Σ_1 and Σ_2 such that Σ_1 is cohyperhyperimmune in Δ and Δ is cohyperhyperimmune in Σ_2.*

Proof. Applying Lemma 1, let Δ_1 be a hyperhyperimmune set whose complement is likewise hyperhyperimmune. Since both a universal set and its complement have infinite r.e. subsets, the sets $\Delta \cap \Delta_1$, $\bar{\Delta} \cap \Delta_1$ must be infinite and therefore immune (indeed, hyperhyperimmune); and we have, clearly, $\Sigma_1 = \Delta \cap \Delta_1$ cohyperhyperimmune in Δ cohyperhyperimmune in $\Sigma_2 = \Delta \cup (\bar{\Delta} \cap \Delta_1)$. It remains to see that $\bar{\Sigma}_1$, $\bar{\Sigma}_2$ are not productive. Now, $\bar{\Sigma}_2 = \bar{\Delta}_1 \cap \bar{\Delta}$ cannot be productive, since it is immune. If $\bar{\Sigma}_1$ were productive, it would be contraproductive (Myhill); hence, since a contraproductive set has a nonimmune complement, $\bar{\Sigma}_1$ is not productive, and the proof is complete.

THEOREM 3. (i) *A pseudosimple set cannot be coimmune in a universal set.*

(ii) *If Δ is a mesoic set such that $\Delta \leqq_{m-1} \Sigma$ for some simple set Σ, then Δ cannot be coimmune in a universal set.*

(iii) *There are mesoic sets Δ, neither pseudosimple nor many-one reducible to a simple set, such that Δ is not coimmune in any universal set.*

Proof. It follows from Lemma 7 that we need only prove (i), (ii), and (iii) with 'universal' replaced by 'creative'. Then (i) becomes evident, since a simple set cannot have a creative superset; (ii) is an easy consequence of the (easily proved) *Theorem* 5 of [7] together with the fact that any creative set is recursively inseparable from some r.e. subset of its complement; and (iii) results at once from Lemmas 5 and 6.

THEOREM 4. *If an r.e. set Δ is hyperhypersimple in Σ, where Σ is creative, then Δ must also be creative.*

Proof. It was pointed out by Yates, in [12], that an r.e. set Δ, with infinite complement, is hyperhypersimple if and only if there is no recursive sequence $\{W_{r(i)}\}$, of pairwise-disjoint r.e. sets (*finite or infinite*), such that $W_{r(i)} \cap \bar{\Delta} \neq \phi$ holds for all i. It readily follows from consideration of inverse images of r.e. sets under one-to-one recursive functions that, for r.e. sets Δ and Σ, Δ is hyperhypersimple in Σ if and only if $\Delta \subseteqq \Sigma$, $\Sigma - \Delta$ is infinite, and there is no recursive sequence $\{W_{r(i)}\}$ of pairwise-disjoint r.e. subsets (finite or infinite) of Σ such that $(\forall i)(W_{r(i)} \cap (\Sigma - \Delta) \neq \phi)$. Now, it follows straightforwardly from Myhill's isomorphism theorem ([8]) that if Σ is creative, then there is a recursive sequence $\{W_{r(i)}\}$ of pairwise-disjoint *creative* sets such that $\Sigma = \bigcup_i W_{r(i)}$. Let $\{W_{r(i)}\}$ be such a sequence, relative to the given creative set Σ; and suppose Δ is an r.e. set hyperhypersimple in Σ. It follows that there is at least one i such that $W_{r(i)} \cap (\Sigma - \Delta) = \phi$; i.e., $W_{r(i)} \subseteqq \Delta$. But then Δ is the disjoint union of the r.e. sets $W_{r(i)}$ and $\Delta \cap (\bigcup_{j \neq i} W_{r(j)})$; and hence, since $W_{r(i)}$ is creative, Δ is creative.

THEOREM 5. *If an r.e. set Δ is strongly effectively simple in Σ, where Σ is creative, then Δ must also be creative.*

Proof. Applying Lemma 3, Let s be a recursive function such that, for all numbers i, $W_i \subseteqq \Sigma - \Delta \Rightarrow (\forall x)(x \in W_i \Rightarrow s(i) > x)$. (It is not really essential to our purposes to have a *total* function s, but the proof is just a bit less cumbersome if we do.) Now, there exist a recursive function r, and a *strictly increasing* recursive function q, such that, for all i, $W_{r(i)} = W_i \cap \Sigma$ and $W_{q(i)} = W_i - \{x \mid x < sr(i)\}$. Let p be productive for $\bar{\Sigma}$: by results of Myhill, we may assume p to be strictly increasing and recursive. Let h be a 2-place recursive function such that $W_{h(i,j)} = W_i$ and $h(i, j) > j$, for all i and j. Then, the function p^* defined by $p^*(x) = p(h(x, sr(x)))$ is productive for $\bar{\Sigma}$, and has the property that $p^*(i) > sr(i)$, for all i. Since p^* and q are strictly increasing, then, we see that $p^*q(i) > sr(i)$, for all i. We now claim, and the reader will easily check, that the function p^*q is productive for $\bar{\Delta}$. This completes the proof.

THEOREM 6. *Each component of a $CFY(K)$-decomposition of a creative set, $2 \leqq K \leqq \aleph_0$, is mesoic.*

Proof. Suppose, to the contrary, that Σ is a component in a $CFY(K)$-decomposition of a creative set Δ, and that Σ is creative. Now, it is easily verified that if f is a one-to-one recursive function generating Σ, and Σ_1 is a hyperhypersimple (strongly effectively simple) set, then $f(\Sigma_1)$ is hyperhypersimple (strongly effectively simple) in Σ. Hence, by Theorem 4 or Theorem 5, $f(\Sigma_1)$ is creative. It follows from the Myhill isomorphism theorem that Σ itself is simple in a creative set (consider a recursive permutation mapping $f(\Sigma_1)$ onto Σ). But, by Lemma 5, Σ cannot be simple in *any* r.e. set; and from this contradiction, the theorem follows.

THEOREM 7. *Let $\{W_{r(i)}\}$ be a recursive sequencing of the components of a strong $CFY(\aleph_0)$-decomposition of Σ, Σ a creative set. Then, the sequence $\{W_{r(i)}\}$ is almost e.i. but not e.i.*

Proof. $W_{r(0)}, W_{r(1)}, \cdots$ is an almost e.i. sequence by the result of Cleave ([1]) cited in § 2. It is clear that if $W_{r(0)}, W_{r(1)}, \cdots$ were an e.i. sequence, then, for $i \neq j$, the terms $W_{r(i)}, W_{r(j)}$ would be effectively inseparable. But hence, $W_{r(i)}, W_{r(j)}$ would be creative; whereas, by Theorem 6, they must be mesoic. From this contradiction in the subjunctive mood, we conclude to Theorem 7.

THEOREM 8. *The union of the terms of an almost e.i. recursive sequence of pairwise-disjoint r.e. sets need not be creative. Indeed: if Σ is a creative set, then Σ is the disjoint union of two mesoic* *sets Δ_1, Δ_2, each of which is the union of the terms of such a sequence.*

Proof. Again, let $\{W_{r(i)}\}$ be a recursive indexing of the components of a strong $CFY(K)$-decomposition of Σ, so that the sequence $W_{r(0)}$, $W_{r(1)}, \cdots$ is almost e.i. Now, it is easy to see that each of the subsequences $W_{r(0)}$, $W_{r(2)}$, $W_{r(4)}, \cdots$, $W_{r(1)}$, $W_{r(3)}$, $W_{r(5)}, \cdots$ is likewise almost e.i. Since $\{W_{r(i)}\}$ is a $CFY(\aleph_0)$-decomposition of Σ, the pair of sets $\bigcup_{2|i} W_{r(i)}$, $\bigcup_{2\nmid i} W_{r(i)}$ are the components of a $CFY(2)$-decomposition of Σ. Hence, by Theorem 6, each of $\bigcup_{2|i} W_{r(i)}$, $\bigcup_{2\nmid i} W_{r(i)}$ is mesoic, and the theorem is proved.

THEOREM 9. *Suppose Δ_1, Δ_2 are sets-of-a-pair in the r.e. set Σ. If Δ_1 is creative, then Σ is creative. On the other hand, if Σ is creative, there exist two mesoic sets Δ_1, Δ_2 such that Δ_1, Δ_2 are sets-of-a-pair in Σ.*

Proof. For the first assertion: if Δ_1 is creative, so is $\Delta_1 \cup \Delta_2$. But $\Delta_1 \cup \Delta_2$ is simple in Σ. Hence, by Theorem 1, Σ must be creative. For the second assertion: Let Σ' be any r.e. set which is simple in Σ, and let Δ_1, Δ_2 be the components of any $CFY(2)$-decomposition of Σ'. It is then easily checked that Δ_1, Δ_2 are sets-of-a-pair in Σ; and, by Theorem 6, Δ_1 and Δ_2 must be mesoic.

REMARK. The first assertion of Theorem 9 extends and completes *Theorem* 8 of [7].

Notice that, in the proof of the second part of Theorem 9, we proceeded in such a way that at least one of Δ_1, Δ_2 must be nonpseudosimple; this follows from Theorem 3(i) and Lemma 6. It is not hard to insure that *both* Δ_1, Δ_2, be nonpseudosimple mesoic sets. For choose Σ' to be a creative set, and apply the following general result.

THEOREM 10. *Let Σ be a creative set, and Δ_1, Δ_2 two mesoic sets (not necessarily disjoint) such that $\Delta_1 \cup \Delta_2 = \Sigma$. Then neither Δ_1 nor Δ_2 can be pseudosimple.*

Proof. Suppose, to the contrary, that (say) Δ_1 is pseudosimple: let j be a number such that $W_j \subseteqq \bar{\Delta}_1$, $\Delta_1 \cup W_j = \Delta_3$ is simple. Let f be a one-to-one recursive function generating Δ_3. Now, creative sets intersect simple sets creatively ([3, *Theorem* T2.6(2)]); so, $\Sigma \cap \Delta_3 = \Delta_1 \cup (\Delta_2 \cap \Delta_3) =$ a creative set. Again, as is easy to verify, mesoic sets intersect simple sets mesoically; thus, $\Delta_2 \cap \Delta_3$ is mesoic. Hence, by [3, *Theorem* T2.6(2)] and the fact that removal of any recursive subset from a creative set leaves a creative residue, we see that $\Delta_2 \cap W_j$ is mesoic. Now, $f^{-1}(\Sigma \cap \Delta_3) =$ a creative set. This follows from

[3, *Theorem* T2.6(1)] and the (easily verified) fact that if $r(\Delta)$ is productive, r a 1-to-one recursive function, then also Δ is productive. But $f^{-1}(\Sigma \cap \Delta_3) = f^{-1}(\Delta_1) \cup f^{-1}(\Delta_2 \cap W_j)$. Furthermore, $f^{-1}(\Delta_1)$ is recursive, since its complement is $f^{-1}(W_j)$. Hence, $f^{-1}(\Delta_2 \cap W_j)$ must be creative. But therefore, since f is one-to-one recursive, $ff^{-1}(\Delta_2 \cap W_j) = \Delta_2 \cap W_j =$ a creative set: contradiction. The theorem follows.

REMARK. Theorem 10 can also be proved with the word 'pseudosimple' replaced by the words 'many-one reducible to a simple set'; however, the latter result does not interest us here.

The following two assertions, related to Theorem 3, may also be proven: (i) a mesoic set Δ which is many-one reducible to a *pseudosimple* set cannot be coimmune in a universal set; and (ii) if Δ is r.e. and is coimmune in a creative set, then Δ is almost effectively inseparable from some creative set.

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Φ-BOUNDED HARMONIC FUNCTIONS AND CLASSIFICATION OF RIEMANN SURFACES

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Let $\Phi(t)$ be a nonnegative real valued function defined for t in $[0, \infty)$ such that $\Phi(t)$ is unbounded in $[0, \infty)$ and bounded in a neighborhood of a point in $[0, \infty)$. A harmonic function u on a Riemann surface R is said to be Φ-bounded if the composite function $\Phi(|u|)$ has a harmonic majorant on R. Denote by $O_{H\Phi}$ the class of all Riemann surfaces on which every Φ-bounded harmonic function reduces to a constant. The main result in this paper is the following: $O_{H\Phi} = O_{HP}$ (resp. O_{HB}) if and only if $d(\Phi) < \infty$ (resp. $d(\Phi) = \infty$), where $d(\Phi) = \limsup_{t\to\infty} \Phi(t)/t$. This is the best possible improvement of a result of M. Parreau.

We also prove a similar theorem for the classification of subsurfaces of Riemann surfaces using Φ-bounded harmonic functions vanishing on the relative boundaries of subsurfaces.

The chief tool of our proof is the theory of Wiener compactifications of Riemann surfaces.

Consider a nonnegative real valued function $\Phi(t)$ defined for all real numbers t in $[0, \infty)$. A harmonic function u on a Riemann surface R is said to be Φ-bounded if the composite function $\Phi(|u|)$ has a harmonic majorant on R. The totality of Φ-bounded harmonic functions on R is denoted by $H\Phi(R)$, or simply $H\Phi$. We denote by $O_{H\Phi}$ the class of all Riemann surfaces R on which every Φ-bounded harmonic function reduces to a constant. Our problem is to determine $O_{H\Phi}$ for every Φ.

First assume that $\Phi(t)$ is bounded on $[0, \infty)$. Then every harmonic function is Φ-bounded. Hence R belongs to $O_{H\Phi}$ if and only if there exists no nonconstant harmonic function on R. Thus *the class $O_{H\Phi}$ consists of all closed Riemann surfaces if Φ is bounded*. Soon we see that the converse is also valid. Hence, hereafter, we always assume that

(1) *$\Phi(t)$ is unbounded on $[0, \infty)$.*

We say that $\Phi(t)$ is bounded at a point t_0 in $[0, \infty)$ if there exists a neighborhood of t_0 relative to $[0, \infty)$ in which $\Phi(t)$ is bounded. Now assume that $\Phi(t)$ is not bounded at any point of $[0, \infty)$. Let u be a nonconstant harmonic function on R. Then $\Phi(|u|)$ is not bound at any neighborhood of any point of R and so u is not Φ-bounded. Thus *the class $O_{H\Phi}$ consists of all Reimann surfaces if $\Phi(t)$ is not bounded at*

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any point of $[0, \infty)$. Soon we see that the converse is also true. Hence, hereafter, we always assume that

(2) *$\Phi(t)$ is bounded at least at one point in* $[0, \infty)$.

Now our problem which is left is to determine $O_{H\Phi}$ for functions Φ satisfying the two conditions (1) and (2). For the aim, we put

$$d(\Phi) = \limsup_{t \to \infty} \Phi(t)/t .$$

Clearly $O \leqq d(\Phi) \leqq \infty$. Our result is stated as follows:

THEOREM 1. *Assume that Φ satisfies* (1) *and* (2). *If $d(\Phi)$ is finite (resp. infinite), then $O_{H\Phi} = O_{HP}$ (resp. O_{HB}).*

Since the restrictions on Φ are exclusive each other, we also see that $O_{H\Phi} = O_{HP}$ (resp. O_{HB}) implies that Φ satisfies (1) and (2) and $d(\Phi)$ is finite (resp. infinite). This theorem is proved by Parreau [3] for the special Φ which is increasing and convex (and so continuous) (see also Ahlfors-Sario's book [1], pp. 216–219). Parreau's proof keenly uses the increasingness and convexity of Φ and one might suspect that these assumptions are inevitable. We are interested in the fact that for the validity of Parreau's result, no assumption is needed for Φ except the inevitable conditions (1) and (2). Thus our Theorem 1 is the best possible generalization of Parreau's result at least in the above formulation.

2. Before entering the proof of Theorem 1, for convenience, we explain an outline of the *Wiener compactification* of a Riemann surface and its some properties which we use in the proof of Theorem 1. For details, consult Constantinescu-Cornea's book [2], § 6, 8 and 9.

Let F be a Riemann surface not belonging to O_G and f be a real valued function on F. Let $\overline{W}_f^F$ (resp. $\underline{W}_f^F$) be the totality of superharmonic (resp. subharmonic) functions s on F such that there exists a compact subset K_s of F with the property that $f \leqq s$ (resp. $f \geqq s$) on $F - K_s$. If $\overline{W}_f^F$ and $\underline{W}_f^F$ are nonvoid, then $\overline{W}_f^F$ and $\underline{W}_f^F$ are Perron's families and so

$$\bar{h}_f^F(p) = \inf\,(s(p);\, s \in \overline{W}_f^F) \text{ and } \underline{h}_f^F(p) = \sup\,(s(p);\, s \in \underline{W}_f^F)$$

are harmonic and $\bar{h}_f^F \geqq \underline{h}_f^F$. If $\bar{h}_f^F = \underline{h}_f^F$ on F, then we write $h_f^F = \bar{h}_f^F = \underline{h}_f^F$ and we call f to be harmonizable on F.

Let R be an arbitrary Riemann surface. A real-valued function f on R is said to be a continuous Wiener function if (a) for any subsurface F of R with $F \notin O_G$ as a Riemann surface, the restriction of f on F is harmonizable on F and the restriction of $|f|$ on F has a superharmonic majorant on F; and if (b) f is finitely continuous on R. We denote by $WC = WC(R)$ the totality of continuous Wiener functions on R. We also denote by $WB = WB(R)$ the totality of bounded members in WC. Observe that WC (resp. WB) is a vector space and closed under max and min operations. Any continuous superharmonic function on R which has a harmonic majorant clearly belongs to WC. Hence $HP \subset WC$ and $HB \subset WB$.

There exists a unique compact Hausdorff space R^* containing R as its open and dense subset such that $C(R^*)|R = WB(R)$, where $C(R^*)$ is the totality of finitely continuous functions on R^* and $C(R^*)|R$ is the totality of restrictions of functions in $C(R^*)$ to R. We call R^* the Wiener compactification of R. By the obvious identification, we may simply write as $C(R^*) = WB(R)$. It is clear that any function in $WC(R)$ is (not necessarily finitely) continuous on R^*, or more accurately, is continuously extended to R^*. Hereafter, we use topological notions relative to R^* only. For example, $\bar{A}$ for $A \subset R$ means the closure of A in R^*. But the notation ∂A for $A \subset R^*$ is the only exceptional. ∂A means the boundary of $A \cap R$ relative to R.

Let $W_0C(R) = (f \in WC;\ h_f^R = 0)$ if $R \notin O_G$ and $W_0C(R) = WC$ if $R \in O_G$. We set $\Delta = (p \in R^*;\ f(p) = 0$ for any f in $W_0C)$. This is a compact subset of $\Gamma = R^* - R$ and called the (Wiener) harmonic boundary of R. It is seen that $W_0C = (f \in WC;\ f = 0$ on $\Delta)$. From the definition, it is obvious that $R \in O_G$ if and only if $\Delta = \varphi$. Moreover,

LEMMA 1. *$R \in O_{HB} - O_G$ if and only if Δ consists of only one point.*

Let F be an open subset of R each boundary point of which is regular for Dirichlet problem and $\partial F \neq \phi$. Such an F is called a regular open subset of R. We say that $F \in SO_{HB}$ if any connected component of F does not carry any nonconstant bounded harmonic functions vanishing continuously at ∂F. The most important is the following

LEMMA 2. *$F \notin SO_{HB}$ if and only if $\bar{F} - \overline{\partial F}$ contains a point of Δ.*

As an corollary of this, we can easily see the following useful

LEMMA 3. *Let F be a regular open subset of R and s be a superharmonic function on F bounded from below. If*

$$\liminf_{F \ni p \to q} s(p) \geqq 0$$

for any q in $\partial F \cup (\bar{F} \cap \Delta)$, then $s \geqq 0$ on F.

3. Proof of Theorem 1 for $d(\Phi) < \infty$. Since $d(\Phi) < \infty$, we can find a positive number c and a point t_0 in $[0, \infty)$ such that $\Phi(t) \leqq ct$ for any $t \geqq t_0$. Assume that there exists a nonconstant HP-function

u_1 on R. Then $u = u_1 + t_0$ is also a nonconstant harmonic function on R with $u \geqq t_0 \geqq 0$ on R. Thus $\Phi(|u|) \leqq c|u| = cu$ and cu is an HP-function on R. Hence $O_{H\Phi} \subset O_{HP}$.

Conversely, assume that there exists a nonconstant $H\Phi$-function u on R. We have to prove the existence of a nonconstant HP-function on R. By the definition, there exists an HP-function v on R with $\Phi(|u|) \leqq v$ on R. If v is not a constant or u is bounded, then nothing is left to prove and so we assume that v is a constant and u is not bounded. Then the connected open set $D = (|u(p)|;\ p \in R)$ in $[0, \infty)$ does not contain 0. Contrary to the assertion, assume that $D \ni 0$. Then $D = [0, \infty)$ and so $(\Phi(|u(p)|);\ p \in R) = (\Phi(t);\ t \in [0, \infty))$ is unbounded in $[0, \infty)$ by the assumption (1) for Φ. But this is impossible, since $\Phi(|u|) \leqq v$(constant) on R. Thus $0 \notin D$. This shows that u does not change sign on R. Hence u or $-u$ is a nonconstant HP-function on R. Therefore, $O_{H\Phi} \supset O_{HP}$. Thus $O_{H\Phi} = O_{HP}$ for Φ with $d(\Phi) < \infty$.

4. **Proof of Theorem 1 for $d(\Phi) = \infty$.** First assume that there exists a nonconstant HB-function u on R. By the assumption (2) for Φ, there exists an interval $(a, b) \subset [0, \infty)$ in which $\Phi(t) \leqq c$ (constant). By choosing a suitable constants A and B, the range of $v = Au + B$ is contained in (a, b). Then $\Phi(|v|) = \Phi(v) \leqq c$ on R. Thus v is a nonconstant $H\Phi$-function on R. Hence $O_{HB} \supset O_{H\Phi}$.

Next we prove the converse inclusion $O_{HB} \subset O_{H\Phi}$, or equivalently, $R \notin O_{H\Phi}$ implies $R \notin O_{HB}$. Assume that there exists a nonconstant $H\Phi$-function u on R. We have to prove that $R \notin O_{HB}$. Contrary to the assertion, assume that $R \in O_{HB}$. By the definition, there exists an HP-function v such that $\Phi(|u|) \leqq v$ on R. From this, we see that $R \notin O_{HP}$. For, if $R \in O_{HP}$, then $\Phi(|u|) \leqq v$ (constant) and since $d(\Phi) = \infty$, $|u|$ is bounded. This contradicts $R \in O_{HB}$. Hence $R \notin O_{HP}$ and a fortiori $R \notin O_G$. Thus $R \in O_{HB} - O_G$ and so by Lemma 1, the harmonic boundary Δ of R consists of only one point δ, i.e. $\Delta = (\delta)$. By $d(\Phi) = \infty$, we can find a stricly increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers such that

$$\lim_{n\to\infty} \Phi(r_n)/r_n = \infty \text{ and } \lim_{n=\infty} r_n = \infty .$$

Let $G_n = (p \in R;\ |u(p)| < r_n)$. Since u is not a constant and u is unbounded by $R \in O_{HB}$, G_n is a regular open subset of R with $\partial G_n \neq \phi$ and $G_n \nearrow R$. We see that $G_n \notin SO_{HB}$ for some n. For, if this is not the case, then $G_n \in SO_{HB}$ for all $n = 1, 2, \cdots$. Let $a_n = r_n/\Phi(r_n)$. Then $a_n \searrow 0 (n \to \infty)$. Consider the function $a_n v - |u|$, which is superharmonic and bounded from below on G_n and continuous in $G_n \cup \partial G_n$. If $q \in \partial G_n$, then

$$| u(q) | = r_n = (r_n/\Phi(r_n))\, \Phi(r_n) = a_n \Phi(| u(q) |) \leqq a_n v(q) .$$

Thus $a_n v - | u | \geqq 0$ on ∂G_n. Hence $a_n v - | u | \geqq 0$ in G_n. For, if $a_n v(p_0) - | u(p_0) | < d < 0$ for some p_0 in G_n, then $G'_n = (p \in G_n;\ a_n v(p) - | u(p) | < d)$ is a nonempty regular open subset with $G'_n \cup \partial G'_n \subset G_n$. The function $d - (a_n v - | u |)$ is a positive and bounded (with bound $d + r_n$) subharmonic function in G'_n vanishing continuously at $\partial G'_n$. So $G'_n \notin SO_{HB}$. But this is a contradiction, since $G_n \supset G'_n \cup \partial G'_n$ and $G_n \in SO_{HB}$. Hence $a_n v - | u | \geqq 0$ in G_n. Now let p be an arbitrary point in R. There exists an n_0 such that $p \in G_n$ for all $n \geqq n_0$. Then $| u(p) | \leqq a_n v(p)$ for all $n \geqq n_0$. Thus by making $n \nearrow \infty$, $| u(p) | = 0$, i.e. $u \equiv 0$ on R, which is a contradiction. Hence $G_{n_1} \notin SO_{HB}$ for some n_1 and so $G_n \notin SO_{HB}$ for all $n \geqq n_1$ and so without loss of generality, we may assume that $G_n \notin SO_{HB}$ for all $n = 1, 2, \cdots$. In particular, $G_1 \notin SO_{HB}$ implies that $\bar{G}_1 - \overline{\partial G_1}$ contains δ by Lemma 2 (recall that $\Delta = (\delta)$), i.e. $\bar{G}_1$ is a neighborhood of δ in the Wiener compactification R^* of R. Hence in the topology of R^*,

$$(*) \qquad \limsup_{R \ni p \to \delta} | u(p) | = \limsup_{G_1 \ni p \to \delta} | u(p) | \leqq r_1 .$$

Now consider the function $f_n = a_n v + r_1 - | u |$, which is superharmonic and bounded from below on G_n and continuous in $G_n \cup \partial G_n$. If $q \in \partial G_n$, then as before,

$$| u(q) | = r_n = (r_n/\Phi(r_n))\, \Phi(r_n) = a_n \Phi(| u(q) |) \leqq a_n v(q) \leqq a_n v(q) + r_1$$

and so $f_n(q) \geqq 0$ on ∂G_n. This with (*) gives that

$$\liminf_{G_n \ni p \to q} f_n(p) \geqq 0$$

for any q in $\partial G_n \cup (\delta) = \partial G_n \cup (\bar{G}_n \cap \Delta)$. Hence by Lemma 3, $f_n \geqq 0$ in G_n, or

$$| u | \leqq a_n v + r_1$$

in G_n. Let p be an arbitrary point in R. There exists an n_0 such that $p \in G_n$ for all $n \geqq n_0$. Thus $| u(p) | \leqq a_n v(p) + r_1$ for all $n \geqq n_0$. Hence by making $n \nearrow \infty$, $| u(p) | \leqq r_1$, i.e. $| u | \leqq r_1$ on R. Hence $R \notin O_{HB}$. This is a contradiction, since we assumed that $R \in O_{HB}$. Thus $R \in O_{HB}$.

5. Finally we make a few remark to the classification of Riemann surfaces with regular boundaries. Let $\Phi(t)$ be a non-negative real-valued function defined in $[0, \infty)$. Let R be a Riemann surface and F be a regular open subset of R. We denote by $H_0\Phi = H_0\Phi(R, F)$ the totality of harmonic functions u in F vanishing continuously at ∂F such that $\Phi(| u |)$ admits a harmonic majorant in F. We say that

$F \in SO_{H\Phi}$ if $H_0\Phi$ contains only zero. We want to determine $SO_{H\Phi}$ for every Φ. As before, unless Φ satisfies (1), then $F \in SO_{H\Phi}$ if and only if F does not carry any nonzero harmonic function in F vanishing continuously at ∂F. Thus *$SO_{H\Phi}$ consists of all relatively compact regular open subsets of Riemann surfaces if $\Phi(t)$ is bounded in* $[0, \infty)$. Similarly as before, *$SO_{H\Phi}$ consists of all regular open subsets of Riemann surfaces if $\Phi(t)$ is not bounded at $t = 0$.* Hence we have only to consider the problem of determining $SO_{H\Phi}$ under the condition

(3) *$\Phi(t)$ is bounded at $t = 0$ and unbounded in* $[0, \infty)$.

As before $d(\Phi) = \limsup_{t\to\infty} \Phi(t)/t$. By (3), $SO_{H\Phi} \subset SO_{HB}$ is always valid. Without assuming (3), we can show $SO_{H\Phi} \supset SO_{HB}$ if $d(\Phi) = \infty$ (see the proof of Theorem 2 below). If $d(\Phi) < \infty$, then we cannot get any definite conclusion in general. So we prove only the following

THEOREM 2. *Assume that Φ satisfies* (3) *and $d(\Phi) = \infty$. Then $SO_{H\Phi} = SO_{HB}$.*

Proof. Assume that there exists a nonconstant $H_0\Phi$-function u in F. Then $\Phi(|u|) \leqq v$ in F for some harmonic function v in F. We want to show that $F \notin SO_{HB}$. Contrary to the assertion, assume that $F \in SO_{HB}$. By $d(\Phi) = \infty$, there exists an increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers such that $a_n = r_n/\Phi(r_n) \searrow 0$ and $r_n \nearrow \infty$ as $n \nearrow \infty$. Let $F_n = (p \in F;\ |u(p)| < r_n)$. Clearly $F_n \nearrow F$ and $F_n \in SO_{HB}$. As in the proof of Theorem 1 for $d(\Phi) = \infty$, $a_n v - |u| \geqq 0$ on ∂F_n and $a_n v - |u|$ is lower bounded superharmonic function in F_n and so $F_n \in SO_{HB}$ implies that $a_n v \geqq |u|$ in F_n and finally $u = 0$ in F. This is a contradiction and so $F \notin SO_{HB}$, or $SO_{H\Phi} \supset SO_{HB}$.

Now we change the definition of $H_0\Phi = H_0\Phi(R, F)$ as follows: $H_0\Phi$ is the totality of harmonic functions u in F vanishing continuously at ∂F such that $\Phi(|u|)$ admits a harmonic majorant in R, where we define $u = 0$ in $R - F$. Under this new definition, Theorem 2 is again valid. In fact, $SO_{H\Phi} \subset SO_{HB}$ is clear by (3) and the above proof for $SO_{H\Phi} \supset SO_{HB}$ for $d(\Phi) = \infty$ can be applied with an obvious modification to the present case. Moreover, we can show the following

THEOREM 3. *Assume that Φ satisfies* (3). *If F is a regular open subset of R with the compact complement in R, then $F \in SO_{H\Phi}$ if and only if $F \in SO_{HB}$, or equivalently, $R \in O_G$.*

Proof. Clearly $F \in SO_{H\Phi}$ implies $F \in SO_{HB}$ by the condition (3). Hence we have to show that $F \notin SO_{H\Phi}$ implies $F \notin SO_{HB}$. Evidently, $F \notin SO_{HB}$ is equivalent to $R \notin O_G$. Let u be a nonconstant $H_0\Phi$-function in F. Then there exists an HP-function v in R such that $\Phi(|u|) \leqq v$ on R, where we define $u = 0$ in $R - F$. Contrary to the assertion, assume that $F \in SO_{HB}$, or equivalently $R \in O_G$. Then the inclusion $O_G \subset O_{HP}$ implies that v is a constant, i.e. $\Phi(|u|)$ is a bounded function on R. Let $D = (|u(p)|;\ p \in R)$. Since D is connected and $|u|$ is not bounded, $D = [0, \infty)$. Thus $(\Phi(|u(p)|);\ p \in R) = (\Phi(t);\ t \in [0, \infty))$. From this, the boundedness of $\Phi(|u|)$ implies the boundedness of $\Phi(t)$, which contradicts the assumption (3).

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ON n-ORDERED SETS AND ORDER COMPLETENESS

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In this paper, the notion of an n-ordered set is introduced as a natural generalization of that of a totally ordered set (chain). Two axioms suffice to describe an n-order on a set, which induces three associated structures called respectively: the incidence, the convexity, and the topological structures generated by the order. Some properties of these structures are proved as they are needed for the final theorems. In particular, the existence of natural k-orders in the "flats" of an n-ordered set and the fact that (as it happens for chains) the topological structure is Hausdorff.

The idea of Dedekind cut is extended to n-ordered sets and the notions of strong-completeness, completeness, and conditional completeness are introduced. It is shown that the S^n sphere is s-complete when considered as an n-ordered set. It is also proved that E^n, the n-dimensional euclidean space, fails to be s-complete or complete, but that it is conditionally complete. It is also proved that every s-complete set is compact in its order topology but that the converse is not true. These results generalize classical ones about the structure of chains and lattices.

II. n-Ordered sets. An element of the cartesian product X^{n+1} of a set X will be called an n-simplex and denoted by $\sigma^n = (s_0, s_1, \cdots, s_n)$ where $s_i \in X$ for every i. The class of even permutations of this sequence is called an oriented n-simplex and denoted by $|\sigma^n| = |s_0, s_1, \cdots, s_n|$. The class of odd permutations is another oriented n-simplex denoted by $|-\sigma^n| = |-(s_0, s_1, \cdots, s_n)|$. The set of all oriented n-simplexes of X will be denoted by $|X^n|$. In what follows n-simplex will mean oriented n-simplex.

The *join* of two simplexes $|\sigma^h| = |s_0,s_1, \cdots, s_h|$ and $|\tau^k| = |t_0, t_1, \cdots, t_k|$ is the $h+k-1$ — simplex $|s_0, s_1, \cdots, s_h, t_0, t_1, \cdots, t_k|$ and will be denoted by $|\sigma^h, \tau^k|$.

An *n-ordered set* is a pair (X, φ_n), where X is a set and φ_n is a function from $|X^n|$ to the set $\{-1, 0, 1\}$ and which satisfies A_1 and A_2.

A_1.—*For every* $|\sigma^n| \in |X^n|$; $\varphi_n|-\sigma^n| = -\varphi_n|\sigma^n|$.

Before stating A_2 we introduce the following notation:

$$\Phi_i(\sigma^n, \tau^n) = \varphi_n|t_i, s_1, s_2, \cdots, s_n|\varphi_n|t_0, t_1, \cdots, t_{i-1}, s_0, t_{1+1}, \cdots, t_n|$$

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A_2.—*If* $\Phi_i(\sigma^n, \tau^n) \geqq 0$ *for* $i = 0, 1, \cdots n$; *then* $\varphi_n \mid \sigma^n \mid \varphi_n \mid \tau^n \mid \geqq 0$.

D_1.—The simplex $\mid \pi^{n-1} \mid$ is said to be an *upper bound* for the set $\{x_\alpha; \alpha \in I\} \subset X$ if $\varphi_n \mid x_\alpha, \pi^{n-1} \mid \geqq 0$ for every $\alpha \in I$. If all the relations are strictly $>$ then $\mid \pi^{n-1} \mid$ is a *proper upper bound.* Similar definitions for *lower bounds* using $\leqq$ and $<$.

D_2.—The n-order φ_n is *open from above* (*from below*) if every finite subset of X has a proper upper bound (lower bound).

T_1.—*If* φ_n *is an open from above* (*or from below*) *n-order of X then the following transitive property holds*:

If $\varphi_n \mid s_0, s_1, \cdots, s_{i-1}, x, s_{i+1}, \cdots, s_n \mid \geqq 0$ *for all i and some $x \in X$ then*: $\varphi_n \mid \sigma^n \mid \geqq 0$.

Proof. Apply A_2 to the pair $\mid x, \pi^{n-1} \mid, \mid \sigma^n \mid$ where $\mid \pi^{n-1} \mid$ is a proper bound for $\{s_i\} \cup \{x\}$

EXAMPLES.

(a) In the vector space V^n over the reals define:

$$\varphi_{n-1} \mid v_0, v_1, \cdots, v_{n-1} \mid = \text{sign of det.} \mid v_0, v_1, \cdots, v_{n-1} \mid$$

The function φ_{n-1} is an $n-1$-order of V^n.

(b) In the same space define:
$\varphi_n \mid v_0, v_1, \cdots, v_n \mid = \text{sign of det.} \mid v_i - v_0 \mid$, $i = 1, 2, \cdots, n$. φ_n is an n-order of V^n.

(c) The function of example (a) restricted to the sphere $\mid V \mid = 1$ gives an n-1 order of the n-1-sphere.

(d) Any 1-order satisfying the transitive property of T_1 is equivalent to a chain if we define: $\varphi_1 \mid a, b \mid$ to be $-1, 0$ or 1 according to $a > b$, $a = b$ and $a < b$ respectively.

(e) A field G is said to be n-ordered if it is also an n-ordered set and the mappings: $f_a: x \to ax$ and $g_a: x \to a + x$ are order-automorphisms for any $a \neq 0$.
If we call: $\mid \sigma^n \mid = \mid S_0, S_1, \cdots, S_n \mid$; $\mid a\sigma^n \mid = \mid aS_0, aS_1 \cdots aS_n \mid$, and $\mid a + \sigma^n \mid = \mid a + S_0, a + S_1, \cdots, a + S_n \mid$, then the definition means exactly that $\varphi_n \mid \sigma^n \mid \varphi_n \mid a\sigma^n \mid$ and $\varphi_n \mid \sigma^n \mid \varphi_n \mid a + \sigma^n \mid$ depend only on a. The following examples can be given:

(e_1) The real numbers field is a 1-ordered (open) field. (This is a well known result).

(e_2) The complex numbers field is a 2-ordered (open) field if we define for any $\mid \sigma^2 \mid = \mid \alpha_0, \alpha_1, \alpha_2 \mid$:

$$\varphi_2(\sigma^2) = \frac{i\Delta(\sigma^2)}{|\Delta(\sigma^2)|} \qquad \text{where } \Delta(\sigma^2) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \bar{\alpha}_0 & \bar{\alpha}_1 & \bar{\alpha}_2 \end{vmatrix};$$

$\bar{\alpha}$ being the complex conjugate of α, $|\alpha|$ the modulus of α.

(e_3) The field of quaternions, considered as a 4-dimensional vector space over R and with the 4-order of example (b) above becomes a 4-ordered (noncommutative) field.

(f) The n-order of V^n given in example (b) makes an n-ordered vector space out of V^n in the sense that the mappings f_a: $x \longrightarrow ax$ and g_y: $x \longrightarrow x + y$ are order-isomorphisms for any $a \in R$, $a \neq 0$ and any $y \in V^n$. This example can be generalized as follows:

(g) Let V be any linear space over the ordered commutative field K, and $B \subset V$ any Hamel base for V. If $N = \{b_1, b_2, \cdots, b_n\}$ is any finite subset of B: we can make V into an n-ordered vector space by defining $\varphi_n(V_0, V_1, \cdots, V_n) = +1, -1$ or 0 whenever det $(V_i^j - V_0^j)$ is $>$, $<$ or $= 0$ in $K(V_i^j$ is the coefficient of b_j in the expression of V_i in terms of the base B)

The independence of the axioms follows from the following examples:

In the set $\{a, b, c\}$ define: $\varphi_2 | a, b, c | = \varphi_2 | b, a, c | = 1$ and $\varphi_2 = 0$ elsewhere. This system satisfies A_2 but not A_1.

In the set $\{a, b, c, d, e\}$ define:

$$\begin{aligned} &\varphi_2 | e, c, d | = \varphi_2 | e, c, a | = \varphi_2 | e, c, b | = \varphi_2 | d, a, b | \\ &= \varphi_2 | d, b, c | = \varphi_2 | d, c, a | = \varphi_2 | a, c, b | = 1 \end{aligned}$$

and define φ_2 on the remaining simplexes according to A_1. This system satisfies A_1 but not A_2.

III. Consequences of the Axioms.

D_3.—Two elements x, y of W are said to be *equivalent* if for every $|\pi^{n-1}| \in |X^{n-1}|$ we have: $\varphi_n | x, \pi^{n-1} | = \varphi_n | y, \pi^{n-1} |$. They are *conjugate* if $\varphi_n | x, \pi^{n-1} | = -\varphi_n | y, \pi^{n-1} |$. The relation between equivalent elements is an equivalence relation and the set of equivalence classes can be n-ordered in the usual way. For this set the following axiom holds.

A_3.—*There are no distinct equivalent points.*

From now on we assume (X, φ_n) satisfies A_1, A_2 and A_3 and call (X, φ_n) a reduced n-ordered system. An easy consequence of A_3 is:

C_3.—An element $x \in X$ has at most one conjugate x^*.

D_4.—A simplex $|\sigma^k|$, $k \leqq n$, is said to be *singular* if for every $|\pi^{n-k-1}|$ we have: $\varphi_n | \sigma^k, \pi^{n-k-1} | = 0$. In particular $|\sigma^n|$ is singular if: $\varphi_n | \sigma^n | = 0$.

The following theorems follow easily and are stated without proof:

T_2.—*x^* nonsingular, is the conjugate of x, if and only if $|x, x^*|$ is singular.*

T_3.—*Any simplex with repeated elements is singular.*

T_4.—*There is at most one singular 0-simplex.*

T_5.—*If $x \neq y$, for some $|\pi^{n-1}|$: $\varphi_n|x, \pi^{n-1}| \neq \varphi_n|y, \pi^{n-1}|$.*

We have also:

T_6.—*If $\Phi_i(\sigma^n, \tau^n) \leqq 0$ for $i = 0, 1, 2, \cdots, n$ then $\varphi_n|\sigma^n|\varphi_n|\tau^n| \leqq 0$.* (Compare A_2)

C_6.—If $\Phi_i(\sigma^n, \tau^n) = 0$ for $i = 0, 1, 2, \cdots, n$ then $\varphi_n|\sigma^n|\varphi_n|\tau^n| = 0$.

T_7.—*If $\Phi_i(\sigma^n, \tau^n) \geqq 0$ for $i = 0, 1, 2, \cdots n$ and $\varphi_n|\sigma^n|\varphi_n|\tau^n| = 0$ then: $\Phi_i(\sigma^n, \tau^n) = 0$ for every i.*

IV. Flats and relative orders.

D_5.—Given a nonsingular k-simplex $|\pi^k|$, $k < n$, the set $F|\pi^k| = \{x; |x, \pi^k| \text{ is singular}\}$ will be called the *flat* determined by π^k

T_8.—*If $s_i \in F|\pi^{n-1}|$, $i = 0, 1 \cdots n$ then $|\sigma^n|$ is singular.*

Proof. Apply C_6 to the pair $|\sigma^n|$, $|x, \pi^{n-1}|$ where the last simplex is nonsingular (Such an x exists by D_4)

C_8.—If $|\sigma^n|$ and $|\tau^n|$ are both nonsingular, then for some i: $|t_i, s_1, s_2, \cdots, s_n|$ is not singular.

T_9.—*If $|\mu^k, \pi^k|$, $h + k = n - 1$, is nonsingular, the function $\varphi_h|\sigma^h| = \varphi_n|\sigma^h, \pi^k|$ is a reduced h-order defined on the h-simplexes $|\sigma^h|$ of the set $F|\mu^h|$, φ_h is called the order of $F|\mu^h|$ relative to $|\pi^k|$.* The proof is straightforward.

T_{10}.—*(Invariance of the relative order).—If φ_h and ψ_h are the relative orders of $F|\mu^h|$ by $|\pi^k|$ and $|\tau^k|$ respectively, then:*

$$\varphi_h|\sigma^h| = \psi_h|\mu^h|\varphi_h|\mu^h|\psi_h|\sigma^h| \quad \textit{for any } |\sigma^h| \subset F|\mu^h|.$$

Proof. We consider first the case where $|\pi^k|$ and $|\tau^k|$ differ by only one element. Let $|\pi^k| = |a, \xi^{k-1}|$ and $|\tau^k| = |\pm(b, \xi^{k-1})|$ and apply A_2 to the pair: $|b, \xi^{k-1}, \mu^h|$ and $|a, \xi^{k-1}, \sigma^h|$. It is easily seen that the only Φ_i different from 0 is:

$$\varphi_n|a, \xi^{k-1}, \mu^h|\varphi_n|b, \xi^{k-1}, \sigma^h|.$$

Hence: $\varphi_n|\tau^k, \mu^h|\varphi_n|\pi^k, \sigma^h| = \varphi_n|\pi^k, \mu^h|\varphi_n|\tau^k, \sigma^h|$ and the theorem follows since: $\varphi_n|\tau^k, \mu^h| \neq 0$.

For the general case we construct inductively, using C_8, the sequence: $|\pi^k_{-1}| = |\pi^k|$; $|\pi^k_j| = |t_{i_0}, t_{i_1}, \cdots, t_{i_j}, p_{j+1}, p_{j+2}, \cdots, p_n|$ where the t_i are elements of $|\tau^k|$ and apply the previous result several times to the h-orders relative to $|\pi^k_j|$ and $|\pi^k_{j+1}|$ for $j = -1, 0, 1, \cdots, n$.

Since the previous result is independent of $|\tau^k|$ and $|\pi^k|$ we have:

C_{10}.—*The orders induced in* $F|\mu^h|$ *by* $|\pi^k|$ *and* $|\tau^k|$ *are either identical or opposite and we may speak of the two "natural" orders in any flat* $F|\mu^h|$.

V. Convexity theorems.

D_6.—The element x is said to be *contained* in the nonsingular simplex $|\pi^h|$ if for some natural order of $F|\pi^h|$ we have:

$$a_i = \varphi_h | p_0, p_1, \cdots, p_{i-1}, x, p_{i+1}, \cdots, p_n | \varphi_h | \pi^h | \geqq 0$$

for every $0 \leqq i \leqq h$.

If every $a_i > 0$ we say that x is *interior* to $|\pi^h|$.

D_7.—The *segment* $(\bar{a}, \bar{b})$ is the set of interior points of the non-singular 1-simplex $|a, b|$.

D_8.—A set $C \subset X$ is said to be *convex* if for every $a, b \in C$, such that $|a, b|$ is not singular we have: $(\bar{a}, \bar{b}) \subset C$.

From the definitions follows:

T_{11}.—*If* x *is contained in* (*interior to*) $|\sigma^h|$ *it is also contained in* (*interior to*) $|-(\sigma^h)|$

T_{12}.—*If* x *is contained in* (*interior to*) $|\sigma^n|$ *and every* s_i *satisfies*: $\varphi_n | s_i, \pi^{n-1}| \geqq 0$ *for some* $|\pi^{n-1}|$, *then* $\varphi_n | x, \pi^{n-1} | \geqq 0$ (> 0).

Proof. We assume $\varphi_n|\sigma^n| > 0$ and apply A_2 to the pair:

$$|x, \pi^{n-1}|, |\sigma^n| \text{ to get } \varphi_n | x, \varphi^{n-1} | \geqq 0 .$$

Now if x is interior to $|\sigma^n|$, $\varphi_n | x, \pi^{n-1}|$ cannot be 0, otherwise by C_6 and T_8 we would have $\varphi_n | \sigma^n | = 0$ which contradicts our assumption.

D_9.—We say that $|\sigma^k|$ is contained in (interior to) $|\pi^h|$ if every s_i is contained in (interior to) $|\pi^h|$.

Using the previous theorem we now can prove:

T_{13}.—*If* x *is contained in* $|\sigma^n|$ *and* $|\sigma^n|$ *is contained in* (*interior to*) $|\pi^n|$ *then* x *is contained in* (*interior to*) $|\pi^n|$.

This theorem can be extended in a natural way to the case of two simplexes $|\sigma^h|$ and $|\pi^k|$ where h and k can be different from n. We omit the details. As a corollary of these theorems we have:

T_{14}—*The sets* $Ct|\sigma^h|$ *and* $Int|\sigma^h|$ *formed by the elements which are contained in and interior to* $|\sigma^h|$ *respectively, are convex.*

VI. The induced structures. Given an n-ordered set (X, φ_n) the following structures are said to be induced by the order:

(a) The *incidence structure*, $(X, \mathscr{R})$ where $\mathscr{R}$ is the family of flats of (X, φ_n).

(b) The *convexity structure* $(X, \mathscr{G})$ where $\mathscr{G}$ is the family of convex subsets of (X, φ_n).

(c) The *topological structure* $(X, \mathscr{F})$ where $\mathscr{F}$ is the family of closed sets generated by the sub-base $\mathscr{B}$. The elements of $\mathscr{B}$ are the sets $\bar{B}^+_{\pi^{n-1}} = \{x; \varphi_n \mid x, \pi^{n-1} \mid \geqq 0\}$ for any nonsingular $\mid \pi^{n-1} \mid$, together with the $\bar{B}^-_{\pi^{n-1}} = \{x, \varphi_n \mid x, \pi^{n-1} \mid \leqq 0\}$. We prove the following theorem concerning the topological structure $(X, \mathscr{F})$

T_{15}.—*The topological space* $(X, \mathscr{F})$ *is Hausdorff, provided* (X, φ_n) *contains no singular point.*

Proof. If $\mid x, y \mid$ is singular then by T_2, $x = y^*$. Since x is not singular, for some $\mid \pi^{n-1} \mid$ we have $\varphi_n \mid x, \pi^{n-1} \mid > 0$, and therefore $\varphi_n \mid y, \pi^{n-1} \mid < 0$. The sets $B^+_{\pi^{n-1}} = \{z; \varphi_n \mid z, \pi^{n-1} \mid > 0\}$ and $B^-_{\pi^{n-1}} = \{z; \varphi_n \mid z, \pi^{n-1} \mid < 0\}$ are disjoint (open) neighborhoods of x and y respectively.
If $\mid x, y \mid$ is not singular, for some π^{n-2}, $\varphi_n \mid x, y, \pi^{n-2} \mid > 0$. Assume first that for some z, we have: $0 \neq \varphi_n \mid z, x, \pi^{n-2} \mid \neq \varphi_n \mid z, y, \pi^{n-2} \mid \neq 0$. To be precise let $\varphi_n \mid z, x, \pi^{n-2} \mid < 0$ and $\varphi_n \mid z, y, \pi^{n-2} \mid > 0$ and call $\mid \pi^{n-1} \mid = \mid z, \pi^{n-2} \mid$. Then $B^+_{\pi^{n-1}}$ and $B^-_{\pi^{n-1}}$ are the required neighborhoods. If such a z does not exist, call $\mid \tau^{n-1} \mid = \mid x, \pi^{n-2} \mid$ and $\mid \sigma^{n-1} \mid = \mid y, \pi^{n-2} \mid$. It is easily verified that $B^-_{\tau^{n-1}}$ and $B^+_{\sigma^{n-1}}$ satisfy the requirement. The above theorem is an extension of a well known result in the topology of chains. (See [1] p. 39)
The following result is important and will be needed in the sequel:

T_{16}.—*If* x, y *are contained in* $\mid \sigma^n \mid$ *then* x *is contained in some* $\mid \sigma^n_i \mid = \mid s_0, s_1, \cdots, s_{i-1}, y, s_{i+1}, \cdots, s_n \mid$.

Proof. Call $P_i = \varphi^n \mid \sigma^n_i \mid$ and $P_{ij} = \varphi^n \mid \sigma^n_{ij} \mid = \varphi^n \mid s_0, s_1, \cdots, s_{j-1}, x, s_{j+1}, \cdots, s_{i-1}, y, s_{i+1}, \cdots, s_n \mid$ for $i \neq j$. Clearly $P_{ij} = -P_{ji}$. We put $P_{ii} = \varphi^n \mid s_0, s_1, \cdots, s_{i-1}, x, s_{i+1}, \cdots, s_n \mid$.
Applying A_2 to the pair:

$$\mid \sigma^n_i \mid \text{ and } \mid \sigma^n_{jk} \mid \text{ we get:}$$

If $P_j P_{ki}$ and $P_k P_{ij}$ are both $\geqq 0$ then $P_i P_{kj} \geqq 0$. We may assume $\varphi^n \mid \sigma^n \mid > 0$. Then by D_6 all P_r are $\geqq 0$. Hence we have transitively: $P_{ki} \geqq 0$ and $P_{ij} \geqq 0$ imply $P_{kj} \geqq 0$. Using this, we can prove easily, by induction on n, that for a certain value of K, say $k = k_0$ all $P_{k_0 j} \geqq 0$, $j = 0, 1, 2, \cdots, n$, and this means that x is contained in $\mid \sigma^n_{k_0} \mid$.

VI. ***n*-Order completeness.** In the theory of ordered sets a lattice is said to be complete if every subset of it has a L.U.B. and a G.L.B. This notion is equivalent to that of compactness of the associated topological space (interval topology) when applied to chains. (See [3]) In this sense the lattice of real numbers fails to be complete.

(See [3], p. 51) On the other hand it is *conditionally complete* because every *bounded* subset has a L.U.B. and a G.L.B. This property is equivalent to the fact that every Dedekind cut has a separation element. We proceed to extend these ideas to n-ordered sets. Let (x, φ_n) be a reduced n-ordered set. Every element $x \in X$ determines an n-1-order in X by defining: $\varphi_{n-1}|\pi^{n-1}| = \varphi_n|x, \pi^{n-1}|$. Consider now the subsets of $|X^{n-1}|$ defined by: $C_x^+ = \{|\pi^{n-1}|; \varphi_{n-1}|\pi^{n-1}| \geqq 0\}$ and $C_x^- = \{|\pi^{n-1}|; \varphi_{n-1}|\pi^{n-1}| \leqq 0\}$ It is clear that $C_x^+ \cup C_x^- = |X^{n-1}|$ and x is called the separation element of the pair (C_x^+, C_x^-). We also have for every nonsingular $|\pi^{n-1}|: |\pi^{n-1}| \in C_x^+ \cap C_x^-$ if and only if: $x \in F|\pi^{n-1}|$. We now extend the notion of "cut" to n-ordered sets. Let C^+ and C^- be two subsets of $|X^{n-1}|$ such that $C^+ \cup C^- = |X^{n-1}|$ and γ any object not in X. Let X^* be the set $X \cup \{\gamma\}$. We extend the function φ_n to the set $|X^{*n}|$ by defining: $\varphi_n^*|\gamma, \pi^{n-1}| = +1, -1$ or 0 whenever $|\pi^{n-1}|$ is in $C^+ - C^-, C^- - C^+$ or in $C^+ \cap C^-$, resp. Then $\varphi_n^*|\pi^n| = \varphi_n|\pi^n|$ for $|\pi^n| \in |X^n|$. We call γ the ideal element defined by (C^+, C^-).

D_{10}.—A pair (C^+, C^-) of subsets of $|X^{n-1}|$ is said to be a *cut* if the following properties are satisfied:

(a) $C^+ \cup C^- = |X^{n-1}|$

(b) (X^*, φ_n^*) is an n-ordered set. (Satisfies A_1 and A_2)

D_{11}.—A cut (C^+, C^-) is said to be *interior* or a *Dedekind cut* if the ideal element γ defined by the cut is interior to some $|\sigma^n|$ of X. This means that for some $|\sigma^n|$ and every i we have:

$$\varphi_n^*|s_0, s_1, \cdots, s_{i-1}, \gamma, s_i, \cdots, s_n|\varphi_n^*|\sigma^n| > 0 .$$

D_{12}.—An n-ordered set (X, φ_n) is said to be *strongly complete* (s-complete) if every cut has a separation element in X. It is *conditionally s-complete* if every interior cut has a separation element. It is *order complete* if the topological space $(X, \mathscr{T})$ is compact.

T_{17}.—*If (X, φ_n) is s-complete, then every element has a conjugate.*

Proof. For every $x \in X$ the sets C_x^+ and C_x^- obviously form a cut (C_x^+, C_x^-). It is also clear that the pair (C_x^-, C_x^+) is also a cut defining x^*.

T_{18}.—*The S^n sphere with the n-order defined in II, example c, is strongly complete.*

We give only an idea of the proof: For any nonsingular n-simplex $|\pi^n|$ in S^n and taking antipodal points, we have a decomposition of S^n into 2^{n+1} simplexes. Given a cut (C^+, C^-), the ideal element, γ is order-contained in one of them say $|\pi_0^n|$. The repeated barycentric subdivisions of $|\pi_0^n|$ furnish, (because of T_{16}) a sequence of simplexes

$|\pi_i^n|$, $i = 0, 1, 2 \cdots$ such that γ is interior to all of them and their diameters tend to 0. There is also a unique point p of S^n common to all the $|\pi_i^n|$. It is easily shown that p is the separation element of the cut.

It follows from T_{17} that E^n, the euclidean n-space with the n-order of example II(b) is not *s-complete* and from D_{11} that it is not *order complete.* This is not surprising if we recall the initial remark of this section. But we can prove:

T_{19}.—*E^n with the n-order of example* II(b) *is conditionally s-complete.*

We omit the proof since it is entirely similar to that of T_{18}. The relationship between order-completeness, s-completeness, and compactness is established in the following theorem which is similar to the classical result for partially ordered sets and chains. (See [3] and [2])

T_{20}.—*If the ordered set (X, φ_n) is s-complete, then it is order complete i.e. the space $(X, \mathscr{T})$ is compact.*

Proof. Let $\mathfrak{S}$ be a collection of closed sets of $(X, \mathscr{T})$ with the finite intersection property. It follows from a well known theorem of Alexander that we may restrict ourselves to the case where $\mathscr{S}$ consists of elements from the sub-base $\mathscr{B}$. (See VI a) Let $\mathscr{M}$ be a maximal extension of $\mathscr{S}$ in $\mathscr{T}$ with respect to the property. Then an element of $\mathscr{T}$ belongs to $\mathscr{M}$ if and only if it meets every element of $\mathscr{M}$. (See [4])
Using the notation of T_{15} we now define (C^+, C^-):

$$|\pi^{n-1}| \in C^+ \text{ if } \bar{B}^+_{\pi^{n-1}} \in \mathscr{M} \quad \text{and} \quad |\pi^{n-1}| \in C^- \text{ if } \bar{B}^-_{\pi^{n-1}} \in \mathscr{M} .$$

We shall prove that (C^+, C^-) satisfies D_{10} and is therefore a cut. If $|\pi^{n-1}|$ is not in C^+ for some $M_0 \in \mathscr{M}$ we have:

$$M_0 \subset X - \bar{B}^+_{\pi^{n-1}} = B^-_{\pi^{n-1}} \subset \bar{B}^-_{\pi^{n-1}} .$$

It follows that every $M \in \mathscr{M}$ meets $\bar{B}^-_{\pi^{n-1}}$ since it meets M_0. Or $|\pi^{n-1}| \in C^-$. Therefore $C^+ \cup C^- = |X^{n-1}|$. In order to show that D_{10}(b) holds, we first prove the following result:

If γ is the ideal element defined by (C^+, C^-) and γ satisfies a finite system of equalities: $\varphi_n | \gamma, \sigma_i^{n-1} | = e_i$; $i = 1, 2, \cdots, l$, then there is some $z \in X$ which, when substituted for γ, also satisfies the equalities.

Proof. If $e_i = 1$, then $|\sigma_i^{n-1}|$ is in C^+ but not in C^- and therefore $\bar{B}^-_{\sigma_i{}^{n-1}}$ fails to meet at least one element of $\mathscr{M}$. Denote it by M_i. Similarly if $e_j = -1$, $\bar{B}^+_{\sigma_j{}^{n-1}}$ does not meet $M_j \in \mathscr{M}$. And if $e_k = 0$, both $\bar{B}^+_{\sigma_k{}^{n-1}}$ and $\bar{B}^-_{\sigma_k{}^{n-1}}$ belong to $\mathscr{M}$. We call $M_k = \bar{B}^+_{\sigma_k{}^{n-1}} \cap \bar{B}^-_{\sigma_k{}^{n-1}}$;

clearly $M_k \in \mathscr{M}$.

We consider now $I = \bigcap_r M_r$, $r = 1, 2, \cdots, l$. Since I is not empty, we take any $z \in I$. It can be readily seen that z satisfies all the equalities. To show now that (C^+, C^-) is a cut, it suffices to check A_2 since A_1 is obviously satisfied. But if some pair $|\sigma^n|, |\tau^n|$ of n-simplexes of X^* fails to satisfy A_2, by the previous result the same is true when we put $z \in X$ instead of γ, and this leads to a contradiction. Let s be the separation element of the cut (C^+, C^-) and G any element of $\mathscr{G}$. Since G belongs to the sub-base $\mathscr{B}$ and T_{17} holds, it can be written $G = \bar{B}^+_{\tau n-1}$ for some $|\tau^{n-1}|$: This means $|\tau^{n-1}| \in C^+$ and $\varphi_n |s, \tau^{n-1}| \geqq 0$, or equivalently, $s \in \bar{B}^+_{\tau n-1} = G$. This completes the proof.

That the converse of the above theorem is not true, can be seen by means of the following example:

Let (S^2, φ_2) be the 2-sphere with the 2-order of Example II(c) and K the finite subset of six elements $(\pm i, \pm j, \pm k)$, 2-ordered by the restriction of φ_2 to K. Then K is compact in the induced topology but the cut generated by the elements $\pm(1/3)(i + j + k)$ of S^2 in (S^2, φ_2), restricted to (K, φ_2), have no separation elements in (K, φ_2) and therefore it is not s-complete.

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SOME CONSIDERATIONS ON CONVERGENCE IN ABELIAN LATTICE-GROUPS

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We define α-convergence in an abelian l-group as follows: The net $(x_i)_{i \in I}$ α-converges to x if x is the only element such that $x = \bigvee_{i \geqq i_0} (x_i \wedge x) = \bigwedge_{i \geqq i_0} (x_i \vee x)$ for every $i_0 \in I$. In an Archimedean l-group (x_i) α-converges to x if and only if for every a and b the net $(a \vee x_i) \wedge b$ order-converges (in the ordinary sense) to $(a \vee x) \wedge b$. In general α-convergence is weaker than this latter condition and is considerably more natural in the non-Archimedean case. The algebraic operations of an arbitrary abelian l-group G are continuous relative to α-convergence. If G is completely distributive its α-convergence derives from a Hausdorff group-topology. Three sufficient conditions are given for the preservation of the α-convergence of an l-group G when it is embedded in another l-group E. In an appendix, we formulate a necessary and sufficient condition in order that an abstract sequential convergence derive from a topology.

The present paper is supplementary to [9] and concludes the investigation begun there. We note here that α-convergence is weaker than the concepts of convergence studied in that paper. In the next few paragraphs we review briefly some of the basic definitions and recall some of the results of [9] which will be needed below. The elementary theory of lattice groups is assumed known; we refer the reader to [2, Chap. XIV] or [4]. We shall employ the additive notation and use the standard abbreviation "l-group" for "lattice-group."

If A is a subset of an l-group G and if A has a least upper bound in G, we shall denote this l.u.b. by $\bigvee^{(G)}_{a \in A} a$ or $\sup^{(G)} A$; dually the g.l.b. is denoted by $\bigwedge^{(G)}_{a \in A} a$ or $\inf^{(G)} A$. In the case of a family $(x_\alpha)_{\alpha \in I}$ the notation is $\bigvee^{(G)}_{\alpha \in I} x_\alpha$ or $\sup^{(G)} \{x_\alpha : \alpha \in I\}$ and dually for greatest lower bounds. We shall omit subscripts and superscripts whenever confusion is unlikely. The term "positive" will be used for "$\geqq 0$." Throughout the present paper R will denote the real line, R^X (where X is an arbitrary set) the l-group of all real functions on X. M will be reserved for the l-group of all *bounded* real functions on [0, 1].

In [9] we investigated several types of order-convergence, the main ones being o-convergence, natural convergence and L-convergence. We repeat here the definitions of the latter two. Let G be an abelian

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l-group and let $(x_i)_{i\in I}$ be a directed net in G (in the sense of [6]). An element $u \in G$ is said to be a *superelement* of (x_i) if $x_i \leqq u$ eventually; a *subelement* is defined dually. The net (x_i) is said to be *eventually bounded in* G if it has a superelement and a subelement. For ordinary sequences "bounded" and "eventually bounded" are equivalent.

We say that an eventually bounded net (x_i) *converges naturally to* $x \in G$ *relative to* G (denoted: $\nu\text{-}\lim_{i\in I}^{(G)} x_i = x$) if $\inf^{(G)} U = \sup^{(G)} V = x$, where U is the set of superelements and V the set of subelements of (x_i) in G. The operations $+$, $\vee$, $\wedge$, etc., of the l-group G are continuous with respect to this convergence.

If M is the l-group of all bounded real functions on $[0, 1]$ and if we define $\sigma_n(x) = n^2x(1 - x^2)^n$, $x \in [0, 1]$, $n = 1, 2, \cdots$ then the sequence (σ_n) is pointwise convergent to 0 but is not eventually bounded in M. It is therefore natural to extend our definition of convergence so as to obtain non-eventually-bounded convergent nets, and one way to do this (discussed in § 7 and § 8 of [9]) is the following:

1.1. DEFINITION. The net $(x_i)_{i\in I}$ *L-converges to x relative to G* (denoted: $L\text{-}\lim_{i\in I}^{(G)} x_i = x$) if and only if for each pair $a, b \in G$ $\nu\text{-}\lim_{i\in I}^{(G)} (a \vee x_i) \wedge b = (a \vee x) \wedge b$.

1.2. PROPOSITION. ([9, Prop. 7.2]). *$L\text{-}\lim^{(G)} x_i = x$ if and only if for every $b \geqq 0$ in G $\nu\text{-}\lim^{(G)} | x_i - x | \wedge b = 0$.*

Our L-convergence is not related to Rennie's L-topology ([10]).

The following lemma, which will be needed later, is contained in Lemma 6.3 of [9].

1.3. LEMMA. *Let G be an Archimedean l-group, a an element of G and $(x_i)_{i\in I}$ a net in G which is eventually bounded. Then the following statements are equivalent:*

(i) $\bigvee_{i\geqq i_0} (x_i \wedge a) = a$ *for every* $i_0 \in I$.

(ii) $a \leqq u$ *for every superelement u of* (x_i).

2. **The α-convergence in an abelian lattice-group.** The sequence (σ_n) defined above L-converges to 0 in M and in this respect Definition 1.1 is effective. Suppose however that we embed M in the non-Archimedean l-group $J \circ M$ (where J denotes the ordered group of integers and $\circ$ denotes lexicographic product) by means of the "canonical" mapping $f \to (0, f)$. The sequence $(0, \sigma_n)$ is now bounded in $J \circ M$ and our trick fails: $(0, \sigma_n)$ is not L-convergent in $J \circ M$.

There is however another, more intrinsic, way of describing the pointwise convergence of (σ_n) in M which remedies the defect in this particular case. This is achieved by means of Def. 2.1 below; this definition may seem a little sophisticated at first but is in fact very natural, as a closer examination will show.

2.1. DEFINITION. *The net* $(x_i)_{i\in I}$ *α-converges to* $x \in G$ *relative to* G (denoted: $\alpha\text{-}\lim^{(G)}_{i\in I} x_i = x$) if x is the only element of G satisfying:

$$(1) \qquad x = \bigvee_{i\geqq i_0}^{(G)} (x_i \wedge x) = \bigwedge_{i\geqq i_0}^{(G)} (x_i \vee x) \quad \text{for every } i_0 \in I\,.$$

Compare Def. 2.1 with Löwig's Thm. 42 in [7]; see also Lemma 1.3 above. It will be convenient to call an element x satisfying (1) a *central element of* (x_i) *relative to* G. The definition then reads: $\alpha\text{-}\lim^{(G)} x_i = x$ if and only if x is the only central element *of* (x_i) *relative to* G.

Before studying α-convergence and its connection to L-convergence, we introduce another concept of convergence for purposes of comparison only and as an auxiliary tool. In fact, it proves to be very defective in the case of abelian l-groups, despite the fact that it arises from a close imitation of the method so successfully employed by H. Löwig in the case of Boolean rings. If a net (x_i) is eventually bounded in G, then an element $x \in G$ is said to be an *interelement* of (x_i) relative to G if $\upsilon \leqq x \leqq u$ for every subelement υ and every superelement u of (x_i) in G. If (x_i) is not eventually bounded, then x is said to be an interelement of (x_i) relative to G if and only if for every a, b $(a \vee x) \wedge b$ is an interelement of $(a \vee x_i) \wedge b$ in the preceding sense.

DEFINITION. The net (x_i) *L^*-converges to x relative to G* (denoted: $L^*\text{-}\lim^{(G)} x_i = x$) if x is the only interelement of (x_i) relative to G.

If (x_i) is eventually bounded then $\nu\text{-}\lim x_i = x$, $L\text{-}\lim x_i = x$ and $L^*\text{-}\lim x_i = x$ are of course equivalent.

2.2. LEMMA. *If x is an interelement of (x_i) and if $y \in G$ and $i_0 \in I$ are such that $x_i \wedge x \leqq y \leqq x$ for all $i \geqq i_0$, then y too is an interelement of (x_i) and dually.*

Proof. If (x_i) is eventually bounded, if u is a superelement and υ a subelement of (x_i), then there is some $i \geqq i_0$ such that $\upsilon \leqq x_i \wedge x \leqq y \leqq x \leqq u$, hence $\upsilon \leqq y \leqq u$. Suppose now that (x_i) is not eventually bounded and fix a, b. If $\upsilon \leqq (a \vee x_i) \wedge b \leqq u$ for all $i \geqq k$ say, then $\upsilon \leqq (a \vee x) \wedge b \leqq u$ by the definition of interelement. We therefore have on one hand $(a \vee y) \wedge b \leqq (a \vee x) \wedge b \leqq u$ and on the other hand $v \leqq [(a \vee x_i) \wedge b] \wedge [(a \vee x) \wedge b] = [a \vee (x_i \wedge x)] \wedge b$ for all $i \geqq k$; if i is chosen to be $\geqq i_0, k$, then $v \leqq [a \vee (x_i \wedge x)] \wedge b \leqq (a \vee y) \wedge b$. Thus $v \leqq (a \vee y) \wedge b \leqq u$; we infer that $(a \vee y) \wedge b$ is an interelement of $(a \vee x_i) \wedge b$, $i \in I$.

2.3. LEMMA. *If x is a central element of (x_i) then x is an interelement of (x_i). If moreover (x_i) is not eventually bounded, then the converse is also true.*

Proof. Let x be a central element of (x_i). If (x_i) is eventually bounded, it is immediately seen that x is also an interelement of (x_i). If (x_i) is not eventually bounded we show, using the infinite distributive laws, that for every $a, b \in G$ $(a \vee x) \wedge b$ is a central element (and hence also an interelement) of $(a \vee x_i) \wedge b, i \in I$.

Assume now that (x_i) is not eventually bounded and let x be an interelement of (x_i). Fix $i_0 \in I$. Obviously $x \leqq x_i \vee x$ for all $i \geqq i_0$. If $y \leqq x_i \vee x$ for all $i \geqq i_0$, then $(x \vee x_i) \wedge y = y$ eventually. Since $(x \vee x) \wedge y = x \wedge y$ is, by the definition, an interelement of $(x \vee x_i) \wedge y$, $i \in I$, and since the latter net is eventually equal to y, we have $y = x \wedge y$, hence $y \leqq x$. Thus $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$. Similarly we prove the dual equality.

2.4. COROLLARY. *If G is Archimedean, then x is an interelement of (x_i) if and only if it is a central element of (x_i).*

Proof. If (x_i) is eventually bounded this is a direct consequence of Lemma 1.3. If (x_i) is not eventually bounded the result is included in the above Lemma.

2.5. THEOREM. *L-$\lim x_i = x$ implies L^*-$\lim x_i = x$, and L^*-$\lim x_i = x$ implies α-$\lim x_i = x$. If (x_i) is eventually bounded then L^*-$\lim x_i = x$ is equivalent with L-$\lim x_i = x$ (and with ν-$\lim x_i = x$). If (x_i) is not eventually bounded L^*-$\lim x_i = x$ is equivalent with α-$\lim x_i = x$.*

Proof. If L-$\lim x_i = x$, where (x_i) is not eventually bounded, then ν-$\lim (a \vee x_i) \wedge b = (a \vee x) \wedge b$ for every a, b. Hence x is an interelement of (x_i). If y were another interelement, then (with $a = x \wedge y$, $b = x \vee y$ in the definition) $[(x \wedge y) \vee y] \wedge (x \vee y) = y$ would be an interelement of $[(x \wedge y) \vee x_i] \wedge (x \vee y)$, $i \in I$, which however converges naturally to $[(x \wedge y) \vee x] \wedge (x \vee y) = x$. Thus $y = x$. We have shown that L-$\lim x_i = x$ implies L^*-$\lim x_i = x$.

If $L^*\text{-lim}\, x_i = x$ then $x = \bigvee_{i \geqq i_0} (x_i \wedge x)$ for every i_0; in fact if $y \geqq x_i \wedge x$ for all $i \geqq i_0$, then $x \wedge y$ is an interelement of (x_i) by Lemma 2.2, therefore $x \wedge y = x$, i.e. $y \geqq x$. Dually we show that $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$. That x is the only central element of (x_i) follows from Lemma 2.3. The final part of the theorem is also a consequence of Lemma 2.3.

None of the converse implications is valid. To show that $L^*\text{-lim}\, x_i = x$ does not imply $L\text{-lim}\, x_i = x$ consider the direct product $M \times (J \circ M)$ of the l-groups M and $J \circ M$ and set $x_n = (\sigma_n; 0, \sigma_n)$ (the sequence (σ_n) was defined above). It can easily be shown that $\alpha\text{-lim}\, x_n = 0$ and since (x_n) is not eventually bounded in $M \times (J \circ M)$ we infer from the preceding theorem that $L^*\text{-lim}\, x_n = 0$. However $L\text{-lim}\, x_n = 0$ is false; in fact if $x = (f; 1, g)$ then $(0 \vee x_n) \wedge x$ fails to converge naturally to $(0 \vee 0) \wedge x = 0$, since every superelement $u = (h; m, h')$ of $(0 \vee x_n) \wedge x = (\sigma_n \wedge f; 0, \sigma_n)$ must necessarily have $m \geqq 1$. Finally to show that $\alpha\text{-lim}\, x_i = x$ does not imply $L^*\text{-lim}\, x_i = x$ consider the sequence $(0, \sigma_n)$ in $J \circ M$. We thus see that α-convergence is in general weaker than L-convergence, both for bounded as well as for unbounded sequences. However, in an Archimedean l-group they are equivalent:

2.6. THEOREM. *In an Archimedean l-group L-convergence and α-convergence are equivalent.*

Proof. Assume $\alpha - \lim x_i = x$. Then, by Thm. 3.8 of the following section, for every $a, b \in G$ $\alpha - \lim (a \vee x_i) \wedge b = (a \vee x) \wedge b$. By Corollary 2.4 this means $(a \vee x) \wedge b$ is the only interelement of $(a \vee x_i) \wedge b$, $i \in I$ and since the latter net is bounded, $\nu - \lim (a \vee x_i) \wedge b = (a \vee x) \wedge b$. Hence $L - \lim x_i = x$.

The entire machinery of [9, § 8] is now at our disposal for the "completion" of an Archimedean l-group relative to its α-convergence.

3. Continuity of the algebraic operations. The operations $+, -, \vee, \wedge$ etc. are continuous relative to L-convergence; this follows from Prop. 1.2 (see [9, Prop. 7.4]). It is much less trivial to show that they are continuous relative to α-convergence too. This will be our next goal: The proof of Thm. 3.8 below goes via a number of auxiliary propositions, most of them covering special cases. Let us however remark at this point that L^*-convergence violates this natural requirement of continuity. Setting $x_n = (\sigma_n; 0, \sigma_n)$ in the l-group $M \times (J \circ M)$ as before, and $c = (0; 1, 0)$ we see that $L^*\text{-lim}\, x_n = 0$ does not imply $L^*\text{-lim}\, x_i \wedge c = 0 \wedge c$. Hence the mapping $x \to x \wedge c$, with c fixed, may fail to be continuous. The mapping $G \times G \ni (x, y) \to x + y \in G$ may also fail to be jointly continuous as is seen from the consideration of the sequences $x_n = (\sigma_n; 0, \sigma_n)$ and $y_n = (-\sigma_n; 0, 0)$ in $M \times (J \circ M)$. These are overwhelming disadvantages and we have to reject L^*-convergence. It is of course true that $L^*\text{-}\lim x_i = x$ implies $L^*\text{-}\lim(-x_i) = -x$ and $L^*\text{-}\lim(x_i + c) = x + c$ but this offers little consolation.

Notice the following useful facts:

(2) If $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$ and if $x^* \geqq x$, then $x^* = \bigwedge_{i \geqq i_0} (x_i \vee x^*)$; and dually.

(3) If $\alpha\text{-}\lim x_i = x$ and if $x_i \leqq y$ eventually, then $x \leqq y$; and dually.

(4) If x, y are central elements of (x_i), then so are $x \vee y$ and $x \wedge y$.

(5) If (y_j) is a subnet of (x_i) (in the sense of [6]) and if x is a central element of (y_j), then x is a central element of (x_i) also.

(6) The following three statements are equivalent:
(i) x is a central element of (x_i);
(ii) $-x$ is a central element of $(-x_i)$;
(iii) $x + c$ is a central element of $(x_i + c)$.

The easy proofs are left to the reader. From (6) in particular it follows that $\alpha\text{-}\lim x_i = x$, $\alpha\text{-}\lim(-x_i) = -x$ and $\alpha\text{-}\lim(x_i + c) = x + c$ are equivalent.

3.1. PROPOSITION. *If* $\alpha\text{-}\lim_{i\in I} x_i = x$ *and if* $(y_j)_{j\in J}$ *is a subnet of* $(x_i)_{i\in I}$, *then* $\alpha\text{-}\lim_{j\in J} y_j = x$.

Proof. We first show that x is a central element of (y_j), i.e.

$$x = \bigvee_{j \geqq j_0} (y_j \wedge x) = \bigwedge_{j \geqq j_0} (y_j \vee x) \quad \text{for each } j_0 \in J. \tag{7}$$

Assume $z \geqq y_j \wedge x$ for every $j \geqq j_0$ and define $y = x \wedge z$. We shall show that y is a central element of (x_i). In fact fix i_0. We have $x \geqq y$, hence by (2) $y = \bigvee_{i \geqq i_0} (x_i \wedge y)$. To show the dual equality $y = \bigwedge_{i \geqq i_0} (x_i \vee y)$ assume

$$u \leqq x_i \vee y \quad \text{for all } i \geqq i_0. \tag{8}$$

Then $u \leqq x_i \vee y \vee x = x_i \vee x$ for all $i \geqq i_0$, hence

$$u \leqq \bigwedge_{i \geqq i_0} (x_i \vee x) = x. \tag{9}$$

Suppose now $y_j = x_{n(j)}$, $j \in J$, and choose $j' \geqq j_0$ such that $n(j') \geqq i_0$. Then by (8) $u \leqq x_{n(j')} \vee y$ and combining with (9)

$$u \leqq x \wedge (x_{n(j')} \vee y) = y \vee (x_{n(j')} \wedge x) = y ,$$

since $y = z \wedge x \geqq y_j \wedge x$ for all $j \geqq j_0$. Thus $y = \bigwedge_{i \geqq i_0} (x_i \vee y)$.

We infer that y is a central element of (x_i), hence $y = x$, i.e. $x \wedge z = x$, $z \geqq x$ and this proves the first half of (7). The dual is proved analogously. It follows now from (5) that x is the only central element of (y_j).

3.2. PROPOSITION. If $x_i \geqq 0$ for every i, then the following are equivalent:

(i) $\alpha\text{-lim}\, x_i = 0$

(ii) If $x \geqq 0$ and $x = \bigvee_{i \geqq i_0} (x_i \wedge x)$ for every i_0, then $x = 0$

(iii) For each $x > 0$ there exist $i_0 \in I$ and $u_0 \in G$ such that $x > u_0 \geqq x_i \wedge x$ for all $i \geqq i_0$.

Proof. That (i) implies (ii) is obvious. (iii) is only a restatement of (ii). We now show that (ii) implies (i). Assume (ii) is true. Then 0 is a central element of (x_i). In fact fix i_1; obviously $0 = \bigvee_{i \geqq i_1} (x_i \wedge 0)$. To verify the dual equality $0 = \bigwedge_{i \geqq i_1} (x_i \vee 0) = \bigwedge_{i \geqq i_1} x_i$ suppose $y \leqq x_i$ for all $i \geqq i_1$. Defining $x = y \vee 0$ we have $0 \leqq x \leqq x_i$ for all $i \geqq i_1$, i.e., $x = x_i \wedge x$ for all $i \geqq i_1$. But then $x = \bigvee_{i \geqq i_0} (x_i \wedge x)$ for *any* i_0, since there is always an $i \geqq i_0, i_1$. By hypothesis (ii) $x = 0$, i.e., $y \leqq 0$ and this means $0 = \bigwedge_{i \geqq i_1} x_i$.

If x were another central element of (x_i), then $x = \bigwedge_{i \geqq i_0} (x_i \vee x) \geqq 0$ and on the other hand $x = \bigvee_{i \geqq i_0} (x_i \wedge x)$ for every i_0. By (ii) $x = 0$.

3.3. PROPOSITION. If $x_i \geqq 0\, (i \in I)$, $y_j \geqq 0\, (j \in J)$, $\alpha\text{-lim}\, x_i = 0$ and $\alpha\text{-lim}\, y_j = 0$, then $\alpha\text{-lim}_{(i,j) \in I \times J} (x_i + y_j) = 0$. (Here $I \times J$ is directed by the cartesian ordering).

Proof. We shall apply the preceding proposition. Let $z \geqq 0$ be such that

$$z = \bigvee_{i \geqq i_0,\, j \geqq j_0} [(x_i + y_j) \wedge z] \quad \text{for every } i_0, j_0 . \tag{10}$$

We shall first show that

$$z = \bigvee_{i \geqq i_0} (x_i \wedge z) \quad \text{for every } i_0 . \tag{11}$$

Let $y \geqq x_i \wedge z$ for all $i \geqq i_0$. Define $y_0 = z \wedge y$. Then $z \geqq y_0 \geqq x_i \wedge z$ for all $i \geqq i_0$. From this we shall deduce that

$$z - y_0 = \bigvee_{j \geqq j_0} [y_j \wedge (z - y_0)] \quad \text{for every } j_0 . \tag{12}$$

In fact if $u \geqq y_j \wedge (z - y_0)$ for all $j \geqq j_0$, then

$$u + y_0 \geqq (y_j + y_0) \wedge z \geqq (y_j + x_i \wedge z) \wedge z$$

for all $i \geqq i_0$ and all $j \geqq j_0$, hence $u + y_0 \geqq (y_j + x_i) \wedge (y_j + z) \wedge z = (y_j + x_i) \wedge z$ for all $i \geqq i_0$ and all $j \geqq j_0$. By (10) $u + y_0 \geqq z$, i.e., $u \geqq z - y_0$ and thus (12) is established. This implies, by the preceding proposition, $z - y_0 = 0$, $z = y_0 = z \wedge y$, $y \geqq z$ which proves (11). Finally, by the preceding proposition again, (11) implies $z = 0$.

3.4. PROPOSITION. If $\alpha\text{-}\lim x_i = 0$ then $\alpha\text{-}\lim x_i^+ = 0$ and $\alpha\text{-}\lim x_i^- = 0$ (where $x_i^+ = x_i \vee 0$, $x_i^- = (-x) \vee 0$).

Proof. $x_i^+ \geqq 0$, therefore we can apply Prop. 3.2. Let $x \geqq 0$ be such that

$$x = \bigvee_{i \geqq i_0} (x_i^+ \wedge x) \quad \text{for every } i_0 . \tag{13}$$

We shall show that

$$x = \bigvee_{i \geqq i_0} (x_i \wedge x) \quad \text{for every } i_0 . \tag{14}$$

If $y \geqq x_i \wedge x$ for all $i \geqq i_0$, then $y \geqq x_i \wedge 0 \wedge x$ for all $i \geqq i_0$, $y \geqq [\bigvee_{i \geqq i_0} (x_i \wedge 0)] \wedge x = 0 \wedge x (=0)$. Thus $y \geqq (x_i \wedge x) \vee (0 \wedge x) = x_i^+ \wedge x$ for all $i \geqq i_0$, hence $y \geqq x$ by (13); (14) is established. That $x = \bigwedge_{i \geqq i_0} (x_i \vee x)$ follows from (2) and the relation $x \geqq 0$. Now x being a central element of (x_i) must be equal to 0.

Finally $\alpha\text{-}\lim x_i = 0$ implies $\alpha\text{-}\lim (-x_i) = 0$ and by what was proved $\alpha\text{-}\lim x_i^- = 0$.

3.5. COROLLARY. $\alpha\text{-}\lim x_i = 0$ *implies* $\alpha\text{-}\lim |x_i| = 0$.

Proof. $\alpha\text{-}\lim x_i = 0$ implies $\alpha\text{-}\lim x_i^+ = 0$ and $\alpha\text{-}\lim x_j^- = 0$. By Prop. 3.3 $\alpha\text{-}\lim_{i,j} (x_i^+ + x_j^-) = 0$. Now $|x_i| = x_i^+ + x_i^-$, $i \in I$, being a subnet of $(x_i^+ + x_j^-)$, $(i, j) \in I \times J$, α-converges to 0.

3.6. PROPOSITION. $\alpha\text{-}\lim |x_i| = 0$ implies $\alpha\text{-}\lim x_i = 0$.

Proof. $\alpha\text{-}\lim |x_i| = 0$ implies $\alpha\text{-}\lim (-|x_i|) = 0$. Since $0 \leqq x_i \vee 0 \leqq |x_i| \vee 0$ we have $0 = \bigwedge_{i \geqq i_0} (x_i \vee 0)$ for every i_0. Dually $0 \geqq x_i \wedge 0 \geqq (-|x_i|) \wedge 0$ implies $0 = \bigvee_{i \geqq i_0} (x_i \wedge 0)$ and thus 0 is a central element of (x_i). Let x be another central element. Then $-x$ is a central element of $(-x_i)$, therefore

(15) $$x = \bigvee_{i \geqq i_0} (x_i \wedge x) \quad \text{for every } i_0$$

(16) $$-x = \bigvee_{i \geqq i_0} (-x_i \wedge -x) \quad \text{for every } i_0 .$$

We infer from (15) and (16) that

(17) $$|x| = \bigvee_{i \geqq i_0} (|x_i| \wedge |x|) \quad \text{for every } i_0 .$$

(In fact if $y \geqq |x_i| \wedge |x|$ then $y \geqq x_i \wedge x$ and $y \geqq (-x_i) \wedge (-x)$, hence $y \geqq x$ and $y \geqq -x$, i.e. $y \geqq x \vee -x = |x|$).
By Prop. 3.2 $|x| = 0$, $x = 0$.

3.7. THEOREM. *α-lim $x_i = x$ if and only if α-lim $|x_i - x| = 0$.*
In fact both are equivalent to α-lim $(x_i - x) = 0$. Notice that, by Prop. 3.2 $\alpha\text{-lim}_{i \in I}\, x_i = 0$ and $0 \leqq y_i \leqq x_i$ for all $i \in I$ imply $\alpha\text{-lim}_{i \in I}\, y_i = 0$.

3.8. THEOREM. *If $\alpha\text{-lim}_{i \in I}\, x_i = x$ and $\alpha\text{-lim}_{j \in J}\, y_j = 0$, then $\alpha\text{-lim}_{i \in I}\, (-x_i) = -x$, $\alpha\text{-lim}_{i \in I}\, |x_i| = |x|$, $\alpha\text{-lim}_{(i,j) \in I \times J} (x_i + y_j) = x + y$, $\alpha\text{-lim}_{(i,j) \in I \times J} (x_i \vee y_j) = x \vee y$ and dually.*

Proof. $0 \leqq |(x_i + y_j) - (x + y)| \leqq |x_i - x| + |y_j - y|$. By Prop. 3.3, Thm. 3.7 and the remark preceding the present theorem:

$$\alpha\text{-}\lim_{i,j} (x_i + y_j) = x + y .$$

Similarly

$$\begin{aligned} |x_i \vee y_j - x \vee y| &= |x_i \vee y_j - x \vee y_j + x \vee y_j - x \vee y| \\ &\leqq |x_i \vee y_j - x \vee y_j| + |x \vee y_j - x \vee y| \leqq |x_i - x| + |y_j - y| \end{aligned}$$

etc.

4. **Subspaces and product spaces.** If an abelian l-group G is embedded in another abelian l-group E with preservation of all existing joins and meets, then the α-convergence of E can be relativized to G. It is natural to ask under what conditions this relative convergence coincides with the α-convergence of G itself. Theorem 4.1 below gives three sufficient conditions.

Let E be an l-group and G an l-subgroup. G is said to be *regular in* E (equivalently E is said to be *regular over* G or a *regular extension of* G) if $A \subset G$ and $\inf^{(G)} A = 0$ imply $\inf^{(E)} A = 0$. It is then true that $x = \sup^{(G)} X (X \subset G)$ implies $x = \sup^{(E)} X$, and dually.

4.1. THEOREM. *If the abelian l-group E is a regular extension of G, (x_i) and x are in G, then $\alpha\text{-}\lim^{(E)} x_i = x$ implies $\alpha\text{-}\lim^{(G)} x_i = x$. If moreover either*

(i) G is Archimedean

or (ii) E is completely distributive (see Definition 5.1 below)

or (iii) for every $e \in E$, $e > 0$ there is $g \in G$ such that $0 < g \leqq e$

then $\alpha\text{-}\lim^{(G)} x_i = x$ and $\alpha\text{-}\lim^{(E)} x_i = x$ are equivalent.

Proof. That $\alpha\text{-}\lim^{(E)} x_i = x$ implies $\alpha\text{-}\lim^{(G)} x_i = x$ follows from the definition of α-convergence and the regularity of E over G. We here prove the sufficiency of conditions (i) and (iii) for the converse implication; the sufficiency of condition (ii) will be proved below (§ 5).

Let $\alpha\text{-}\lim^{(G)} x_i = x$. Without loss of generality we can assume $x = 0, x_i \geqq 0$. Then 0 is a central element of (x_i) relative to E too. Let $e_0 \in E$ be another central element of (x_i) in E; then $e_0 = \bigwedge^{(E)}_{i \geqq i_0} (x_i \vee e_0) \geqq 0$ and

$$e_0 = \bigvee^{(E)}_{i \geqq i_0} (x_i \wedge e_0) \quad \text{for every } i_0 . \tag{18}$$

Case (i) (G Archimedean). We first show that $e_0 \wedge a = 0$ for every positive $a \in G$. In fact $\alpha\text{-}\lim^{(G)} x_i = 0$ implies $L\text{-}\lim^{(G)} x_i = 0$ by Thm. 2.6, hence $\nu\text{-}\lim^{(G)} x_i \wedge a = 0$. Then $\nu\text{-}\lim^{(E)} x_i \wedge a = 0$ (see [9, Prop. 6.2]), therefore $\nu\text{-}\lim^{(E)} x_i \wedge e_0 \wedge a = 0$. If $\tilde{u}$ is any superelement of $x_i \wedge e_0 \wedge a, i \in I$, in E then by (18)

$$e_0 \wedge a = \bigvee^{(E)}_{i \geqq i_0} (x_i \wedge e_0 \wedge a) \leqq \tilde{u}$$

for suitable i_0. Thus $e_0 \wedge a$, being a lower bound to the set of superelements of $x_i \wedge e_0 \wedge a$, $i \in I$, in E, must be 0.

In particular $e_0 \wedge x_i = 0$ for every $i \in I$, hence by (18) $e_0 = 0$.

Case (iii). (For every $e > 0$, $e \in E$, there is $g \in G$ such that $0 < g \leqq e$). Assume $e_0 > 0$ in (18) and let $g \in G$ be such that $0 < g \leqq e_0$. Then

$$g = e_0 \wedge g = \left[\bigvee^{(E)}_{i \geqq i_0} (x_i \wedge e_0)\right] \wedge g = \bigvee^{(E)}_{i \geqq i_0} (x_i \wedge e_0 \wedge g) = \bigvee^{(E)}_{i \geqq i_0} (x_i \wedge g)$$

hence $g = \bigvee^{(G)}_{i \geqq i_0} (x_i \wedge g)$. By Prop. 3.2 we then have $g = 0$, a contradiction. Thus e_0 must be 0.

Condition (iii) in the above theorem covers the case of the Everett extension G^* of G by means of "Cauchy" cuts, as well as the extension $\tilde{G}$ (see [9]). It seems improbable that the implication $\alpha\text{-}\lim^{(G)} x_i = x \Rightarrow \alpha\text{-}\lim^{(E)} x_i = x$ remains valid if we merely assume that E is regular over G.

We close this section with a theorem on cartesian products, whose proof is easy.

4.2. THEOREM. *If G is a direct union $G = \times_{\tau\in T} G^\tau$ of abelian l-groups $G^\tau, \tau \in T$ and if $x \in G, x_i \in G$, then $\alpha\text{-}\lim_i^{(G)} x_i = x$ if and only if $\alpha\text{-}\lim_i^{(G^\tau)} x_i^\tau = x^\tau$ for every $\tau \in T$.* (Here x^τ denotes the τ-th "coordinate" of x).

5. The case of completely distributive abelian l-groups. There is a very neat characterization of α-convergence in a completely distributive abelian l-group.

5.1. DEFINITION. An abelian l-group G is said to be *completely distributive* if it satisfies the following condition:

(P) *If, for each index α in a set A, $(x_{\alpha j})_{j\in I_\alpha}$ is a family in G and if all joins and meets exhibited in equality* (19) *below exist, then this equality is valid:*

$$\bigwedge_{\alpha\in A} \bigvee_{j\in J_\alpha} x_{\alpha,j} = \bigvee_{\varphi\in\phi} \bigwedge_{\alpha\in A} x_{\alpha,\varphi(\alpha)} \; ; \tag{19}$$

here $\phi \equiv \times_{\alpha\in A} J_\alpha$, i.e., ϕ is the set of all choice-functions $\varphi(.)$ on A with $\varphi(\alpha) \in J_\alpha$ for each $\alpha \in A$.

For equivalent formulations of complete distributivity see [12, Thm. 2.6].

5.2. THEOREM. *If G is completely distributive and $(x_i)_{i\in I}$ is a directed net in G, then the following are equivalent:*

(i) $\alpha\text{-}\lim\limits_{i\in I} x_i = 0$

(ii) *For each cofinal subset J of I* $\bigwedge\limits_{j\in J} |x_j| = 0$.

Proof. If $\alpha\text{-}\lim_{i\in I} x_i = 0$ then $\alpha\text{-}\lim_{i\in I} |x_i| = 0$ and since $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$ we must have $\alpha\text{-}\lim_{j\in J} |x_j| = 0$, hence $\bigwedge_{j\in J} |x_j| = 0$. Conversely assume that (ii) holds. To show that $\alpha\text{-}\lim x_i = 0$ (equivalently $\alpha\text{-}\lim |x_i| = 0$) it is sufficient, by Prop. 3.2, to show that if $x \geqq 0$ and $x = \bigvee_{j\geqq i} (|x_j| \wedge x)$ for each $i \in I$ then $x = 0$. But if $x = \bigvee_{j\geqq i} (|x_j| \wedge x)$ for each $i \in I$ then

$$x = \bigwedge_{i\in I} \bigvee_{j\geqq i} (|x_j| \wedge x) \, .$$

On the other hand by hypothesis (ii) $\bigwedge_{i\in I} (|x_{\varphi(i)}| \wedge x) = 0$ for each choice function $\varphi(.) \in \times_{i\in I} J_i$, where $J_i = \{j \in I : j \geqq i\}$. Thus

$$\bigvee_{\varphi} \bigwedge_{i\in I} (|x_{\varphi(i)}| \wedge x) = 0$$

and by the complete distributivity of G, $x = 0$.

Notice that (i) implies (ii) in any abelian l-group. We can now proceed to the

Completion of the proof of Theorem 4.1. Let E be a completely distributive regular extension of G. If $\alpha\text{-}\lim_{i\in I}^{(G)} x_i = 0$ then $\bigwedge_{j\in J}^{(G)} |x_j| = 0$ for every cofinal subset $J \subset I$. By the regularity of E over G $\bigwedge_{j\in J}^{(E)} |x_j| = 0$ for every cofinal $J \subset I$, hence by Thm. 5.2 $\alpha\text{-}\lim^{(E)} x_i = 0$.

Our next result is that in a completely distributive abelian l-group the α-convergence derives from a group topology.

5.3. THEOREM. *If G is completely distributive then its α-convergence derives from a topology $\mathfrak{T}$ on G which makes G into a Hausdorff topological group.*

This means that $\alpha\text{-}\lim x_i = x$ if and only if (x_i) is eventually in each $\mathfrak{T}$-neighborhood of x.

Proof. To show that the α-convergence is a topological convergence, it is sufficient, by [6, p. 74] to show that it has the following properties:

(i) If $x_i = x$ for every $i \in I$, then $\alpha\text{-}\lim x_i = x$.

(ii) If $\alpha\text{-}\lim x_i = x$ and if (y_j) is a subnet of (x_i), then $\alpha\text{-}\lim y_j = x$.

(iii) If $\alpha\text{-}\lim x_i = x$ is false, then there is a subnet (y_j) of (x_i) no subnet of which α-converges to x.

(iv) If, for each i in a directed set I, $(x_{i,j})_{j\in R_i}$ is a net in G such that $\alpha\text{-}\lim_{j\in R_i} x_{i,j} = x_i$ and if $\alpha\text{-}\lim x_i = x$ then

$$\alpha\text{-}\lim y_{(i,f)} = x$$

where $y_{(i,f)} \equiv y_{(i,f(\cdot))}$, $(i, f(.)) \in I \times \bigtimes_{i\in I} R_i \equiv \sum$ is the net defined by $y_{(i,f(\cdot))} = x_{i,f(i)}$ $\left(\sum \equiv I \times \bigtimes_{i\in I} R_i \text{ is directed coordinatewise}\right)$.

For a variation on these conditions see [1].

(i) is obvious and (ii) was proved earlier (Prop. 3.1). To show (iii) we assume (without loss of generality) that $x = 0$. If $\alpha - \lim x_i = 0$ is false then by Thm. 5.2 there is a cofinal subset J of I and some $z \in G$ such that $0 < z \leq |x_j|$ for every $j \in J$. Then $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$ no subnet of which can α-converge to 0.

Finally to establish (iv) we need a lemma.

LEMMA. *Let D be a cofinal subset of $\sum \equiv I \times \bigtimes_{i\in I} R_i$. For each*

$i_0 \in I$ *let* $A_{i_0} = \{j \in R_{i_0}$: *there exists a choice-function* $f(.) \in \times_{i\in I} R_i$ *with* $f(i_0) = j$ *and* $(i_0, f(.)) \in D\}$, *i.e.* $A_{i_0} = \{f(i_0): f$ *is such that* $(i_0, f) \in D\}$. *Then the set*

$$\Omega = \{i_0 \in I : A_{i_0} \text{ is cofinal in } R_{i_0}\}$$

is cofinal in I.

In fact suppose there is $k_0 \in I$ such that, for every $i \geqq k_0$ A_i is not cofinal in R_i. Then for each $i \geqq k_0$ there is $j(i) \in R_i$ such that no element j of A_i is $\geqq j(i)$. Define $f_0(.)$ in $\times_{i\in I} R_i$ by:

$$f_0(i) = \begin{cases} \text{an arbitrary element of } R_i \text{ if } i \ngeqq k_0 \\ j(i) \text{ if } i \geqq k_0 . \end{cases}$$

Then there is no element $(i, f(.)) \in D$ with $(i, f(.)) \geqq (k_0, f_0(.))$, which contradicts the fact that D is cofinal in $I \times \times_{i\in I} R_i$.

Having established the lemma we now turn to the proof of (iv). Suppose that $\alpha\text{-lim}_{j\in R_i} x_{i,j} = x_i$ for each $i \in I$ and that $\alpha\text{-lim}_{i\in I} x_i = 0$. (There is no loss of generality in assuming $x = 0$). We shall show that $\alpha\text{-lim}_{(i,f(\cdot))\in\Sigma} y_{(i,f(\cdot))} = 0$ by applying Thm. 5.2. If D is cofinal in $\sum$ and $z \leqq |y_{i,f(\cdot)}|$ for every $(i, f(.)) \in D$, i.e. if $z \leqq |x_{i,f(i)}|$ for every i and $f(.)$ such that $(i, f(.)) \in D$, then in particular, for a fixed i in Ω (Ω is defined as in the lemma) $z \leqq |x_{i,f(i)}|$ for every $f(.) \in \times_{i\in I} R_i$ such that $(i, f(.)) \in D$, hence

$$z - |x_i| \leqq |x_{i,f(i)}| - |x_i| \leqq |x_{i,f(i)} - x_i|$$

$$z - |x_i| \leqq \inf\left\{|x_{i,f(i)} - x_i| : f(.) \in \underset{i\in I}{\times} R_i \quad \text{and} \quad (i, f(.)) \in D\right\} = 0 .$$

(That the *inf* is zero is a consequence of the equality $\alpha\text{-lim}_{j\in R_i}|x_{i,j} - x_i| = 0$ and the fact that $A_i = \{f(i) : f(.) \in \times_{i\in I} R_i$ and $(i, f(.)) \in D\}$ is cofinal in R_i).

Thus $z \leqq |x_i|$ for each $i \in \Omega$ and since Ω is cofinal in I, $z \leqq \bigwedge_{i\in\Omega} |x_i| = 0$ by the same reasoning. In other words

$$\bigwedge_{(i,f(\cdot))\in D} |y_{(i,f(\cdot))}| = 0$$

for each cofinal subset D of $\sum$; hence $\alpha\text{-lim}_{(i,f(\cdot))\in\Sigma} y_{(i,f(\cdot))} = 0$.

We conclude that there is a topology $\mathfrak{T}$ on G such that $\alpha\text{-lim}\, x_i = x$ if and only if, for each $\mathfrak{T}$-neighborhood U of x, $x_i \in U$ eventually. This topology is Hausdorff since limits of arbitrary α-convergent nets are unique. Finally by Thm. 3.8 the operations of the l-group are continuous and the proof is complete.

5.4. COROLLARY. *If G is a regular l-subgroup of a direct union*

$\times_{\tau\in T} G^\tau$ of simply ordered abelian l-groups, then its α-convergence derives from a Hausdorff group-topology. In particular in R^X (X any set) α-convergence is pointwise convergence.

In fact such an l-group G is completely distributive. It is a little absurd to derive this corollary from Thm. 5.3, since we can prove it directly and in fact determine the topology. In each G^τ α-convergence is equivalent to the topological convergence which is defined by means of open intervals. By Theorem 4.2 the α-convergence of $\times_{\tau\in T} G^\tau$ derives from the product topology. Finally by Thm. 4.1, case (ii), ($\times_{\tau\in T} G^\tau$ is completely distributive) the α-convergence of G derives from the relative topology of the subspace $G \subset \times_{\tau\in T} G^\tau$. Notice that this argument serves to establish Thm. 5.3 in the particular case that G is Archimedean, for if G is Archimedean and completely distributive then it is representable as a regular l-subgroup of a direct union $\times_{\tau\in T} G^\tau$ of simply ordered (abelian) l-groups $G^\tau, \tau \in T$ (in fact as a "regular subdirect union"). See [13, Thm. 2.2].

Similar results are of course valid for Boolean algebras, where the analogue of α-convergence is simply the natural convergence as defined in § 1. For instance the "pathological" examples given in [7, p. 1192–93] and [3] are not completely distributive. K. Matthes [8] has given a condition on a lattice L which is necessary and sufficient in order that the natural convergence of L derive from a topology. If R^R is the l-group of all real functions on the real line, then the natural convergence of R^R does not derive from any topology (R^R is not "$\aleph_0$-regulär" [8]), whereas its α-convergence (L-convergence is a topological convergence. Notice however that for *sequences* natural convergence and α-convergence are equivalent in R^R.

An abelian l-group G is said to be $(\aleph_0, \aleph_0)$-distributive if it satisfies condition (P) of Def. 5.1 whenever the set A as well as each J_α are countable.

5.5. PROPOSITION. If G is $(\aleph_0, \aleph_0)$-distributive then its α-convergence of sequences derives from a T_1-topology $\mathfrak{T}(G)$.

This follows from the discussion of § 6 and the fact that in an $(\aleph_0, \aleph_0)$-distributive l-group the characterization of Thm. 5.2 is valid for ordinary sequences. As far as continuity of the group operations is concerned we can affirm that the mapping $G \ni x \to -x \in G$ is continuous relative to $\mathfrak{T}(G)$ and that for each $y \in G$ the mapping $G \ni x \to x + y \in G$ is also continuous. The l-group $G \times G$ is $(\aleph_0, \aleph_0)$-distributive, hence its sequential α-convergence derives from a T_1-topology $\mathfrak{T}(G \times G)$. If $G \times G$ is topologized with $\mathfrak{T}(G \times G)$ and G is topologized with $\mathfrak{T}(G)$ then the mapping $\varphi: G \times G \ni (x, y) \to x + y \in G$ is continuous. In fact using Thm. 4.2

we can easily show that the inverse image $\varphi^{-1}(K) \subset G \times G$ of a $\mathfrak{T}(G)$-closed set $K \subset G$ is $\mathfrak{T}(G \times G)$-closed.

6. Appendix on abstract sequential convergence. In this section we give an elementary theorem on a necessary and sufficient condition in order that a given abstract sequential convergence be equivalent to a sequential convergence defined by means of a topology. The argument establishing this result is essentially due to Löwig [7]. Though the present appendix is only loosely connected with the rest of the paper, it is attached here because the main result is used in establishing Proposition 5.5.

Let X be an arbitrary set and $\mathfrak{C}$ an assignment of "limits" to certain sequences of elements of X. If the element $x \in X$ is assigned to the sequence (x_n), we write $\mathfrak{C}\text{-}\lim_n x_n = x$ and say that (x_n) $\mathfrak{C}$-converges to x. We say that $\mathfrak{C}$ is an *abstract sequential convergence in X with unique limits* if it satisfies the following conditions:

(20) To each sequence at most one "limit" is assigned.

(21) If $x_n = x$ for every n, then $\mathfrak{C}\text{-}\lim x_n = x$.

(22) If $\mathfrak{C}\text{-}\lim\limits_n x_n = x$ and if $(x_{k(n)})$ is a subsequence of (x_n), then

$$\mathfrak{C}\text{-}\lim_n x_{k(n)} = x.$$

The *star-convergence* corresponding to $\mathfrak{C}$ is defined as follows: $\mathscr{C}^*\text{-}\lim x_n = x$ if and only if every subsequence $(x_{k(n)})$ of (x_n) contains a sub-subsequence $(x_{k(\lambda(n))}) = x$ such that $\mathscr{C}\text{-}\lim x_{k(\lambda(n))} = x$. The notion of star-convergence was introduced by Urysohn in [11]. A T_1-topology $\mathfrak{T}(\mathscr{C})$ can also de defined in X by means of $\mathfrak{C}$; it is called the *derivative topology* of $\mathfrak{C}$:

(τ) A set $K \subset X$ is closed relative to $\mathfrak{T}(\mathfrak{C})$ if
$\mathfrak{C}\text{-}\lim x_n = x$ and $x_n \in K$ for all n imply $x \in K$.

$\mathfrak{T}(\mathfrak{C})$ determines a new sequential convergence which we shall call the *derivative topological convergence*: $\mathfrak{T}(\mathfrak{C})\text{-}\lim x_n = x$ if and only if, for each $\mathfrak{T}(\mathfrak{C})$-neighborhood U of x, (x_n) is eventually in U. In ordinary cases star-convergence is known to be equivalent with the derivative topological convergence. According to theorem 6.1 below this is actually true in the most general case, provided we stick to our reasonable assumption of uniqueness of limits. The first to observe the connection between star-convergence and derivative topological convergence was P. Urysohn [11] who proved a restricted form of Thm. 6.1, under the severe assumption that $\mathfrak{C}$ satisfies the following condition:

(I) The operator $A \to \tilde{A}$ defined by $\tilde{A} = \{x \in X$: there exists a sequence (x_n) in A with $\mathfrak{C}$-lim $x_n = x\}$ for every $A \subset X$, is idempotent, i.e. $\tilde{\tilde{A}} = \tilde{A}$.

The same condition was involved in the proof of Satz 29 of [5], which dealt with a particular kind of order-convergence in lattice groups introduced by Kantorovitch in the same paper. Löwig [7, pp. 1191–1192] removed condition (I) proving the same equivalence for another particular concept of order-convergence in Boolean algebras. However, it is easily seen that Löwig's argument, with slight modifications, can serve to establish the following general theorem.

6.1. THEOREM. (Urysohn [11], Löwig [7]). *Let* $\mathfrak{C}$ *be an abstract sequential convergence with unique limits in a set* X. *Then the corresponding star-convergence is equivalent to the derivative topological convergence, i.e.,* $\mathfrak{C}^*$-lim $x_n = x$ *if and only if* $\mathfrak{T}(\mathfrak{C})$-lim $x_n = x$.

Löwig's argument can be found in [7, pp. 1191–92]. What follows here is an outline of this argument, with some modifications necessitated by the fact that the property expressed by condition (22) above is assumed here for subsequences only and not for rearrangements (see Löwig's Thm. 29). First, one shows that if $\mathfrak{C}$-lim $x_n = x$, then $\mathfrak{T}(\mathfrak{C})$-lim $x_n = x$ and x is the only limit of (x_n) under the topological convergence $\mathfrak{T}(\mathfrak{C})$. This latter assertion (uniqueness) is established as follows: If $y \neq x$ then the set $K = \{x_n: x_n \neq y\} \cup \{x\}$ is $\mathfrak{T}(\mathfrak{C})$-closed; in fact if (a_n) is a sequence in K with $\mathfrak{C}$-lim $a_n = a$ and if the range of (a_n) is infinite then by the definition of K there is a subsequence of (a_n) which is of the form $x_{\lambda(1)}, x_{\lambda(2)}, x_{\lambda(3)}, \cdots$ with $\lambda(1) < \lambda(2) < \lambda(3) < \cdots$; in other words there exists a sequence which is both a subsequence of (a_n) and a subsequence of (x_n). This implies $a = x$, hence $a \in K$. If the range of (a_n) is finite the same conclusion (i.e. $a \in K$) is trivial. Thus K is indeed $\mathfrak{T}(\mathfrak{C})$-closed. The complement of K is a $\mathfrak{T}(\mathfrak{C})$-neighborhood of y which fails to contain eventually the terms of the sequence (x_n).

Next we show that $\mathfrak{T}(\mathfrak{C})$-lim $x_n = x$ implies $\mathfrak{C}^*$-lim $x_n = x$. Assume, by way of contradiction, that there exists a subsequence $(x_{k(n)})$ of (x_n) no sub-subsequence of which $\mathfrak{C}$-converges to x. The above result then can be seen to imply that no subsequence (or rearrangement of a subsequence) of $(x_{k(n)})$ $\mathfrak{C}$-converges at all. Then the set $K = \{x_{k(n)}: x_{k(n)} \neq x\}$ is $\mathfrak{T}(\mathfrak{C})$-closed and its complement is a $\mathfrak{T}(\mathfrak{C})$-neighborhood of x, but $(x_{k(n)})$ is eventually outside this neighborhood; a contradiction.

Finally the implication $\mathfrak{C}^*$-lim $x_n = x \Rightarrow \mathfrak{T}(\mathfrak{C})$-lim $x_n = x$ is easy to establish. This completes the argument.

It follows from Thm. 6.1 that limits of sequences are unique under the derivative topological convergence. Notice however that the topology $\mathfrak{T}(\mathfrak{C})$ is not necessarily Hausdorff, i.e., limits of nets may fail to be unique. Consider for instance the extended real line $\bar{R}$ and set $\mathfrak{C}$-$\lim x_n = x$ whenever $x_n = x$ for every n (x may be $+\infty$ or $-\infty$), $\mathfrak{C}$-$\lim x_n = +\infty$ whenever (x_n) is strictly increasing and $\mathscr{C}$-$\lim x_n = -\infty$ whenever (x_n) is strictly decreasing. Let A, B be $\mathfrak{T}(\mathfrak{C})$-open sets containing $+\infty$ and $-\infty$ respectively; then $A \cap B \neq \varnothing$ since the complements A^c and B^c are countable. In fact an uncountable set contains both a strictly increasing and a strictly decreasing sequence, hence an uncountable $\mathfrak{T}(\mathfrak{C})$-closed set contains both $-\infty$ and $-\infty$. An immediate consequence of 6.1 is the following theorem:

6.2. THEOREM. *An abstract sequential convergence $\mathscr{C}$ with unique limits derives from a topology if and only if it satisfies the following condition*

(23) *If (x_n) does not converge to x under $\mathfrak{C}$, then there is a subsequence $(x_{k(n)})$ no sub-subsequence of which converges under $\mathfrak{C}$ to x.*

Comparing this theorem with analogous results of Arnold [1] and Kelley [6] on convergence of arbitrary directed nets we see that, surprisingly, in the case of ordinary sequences the extra assumption of uniqueness of limits renders the condition on iterated limits (condition (iv) at the beginning of the proof of Thm. 5.3) superfluous.

Theorems 6.1 and 6.2 have been recorded here because the author has been unable to find an explicit statement of these general results in the literature. It seems that only the obvious implication $\mathfrak{C}^*$-$\lim x_n = x \Rightarrow \mathfrak{T}(\mathfrak{C})$-$\lim x_n = x$ is widely known. For instance it is stated in [2, p. 62] that if (x_n) star-converges to a then "it certainly converges to a in the star *topology*; moreover ... this special case is sufficient for the applications of star-convergence which we have in mind."

In connection with Thm. 6.2 we observe that in general there are more than one topologies determining the sequential convergence $\mathfrak{C}$. For instance if $\mathfrak{C}$ is pointwise convergence of sequences of real functions on [0, 1], then the class of Baire functions is $\mathfrak{T}(\mathfrak{C})$-closed but not closed relative to pointwise convergence of nets (i.e., relative to the product topology of $R^{[0,1]}$). The topology $\mathfrak{T}(\mathfrak{C})$ is the strongest topology determining $\mathfrak{C}$ and is T_1. If there is at least one Hausdorff topology determining $\mathfrak{C}$ then a fortiori $\mathfrak{T}(\mathfrak{C})$ is Hausdorff.

If $\mathfrak{C}$ is an abstract sequential convergence with unique limits on X which satisfies condition (23), then the sequential convergence $\mathfrak{C} \times \mathfrak{C}$

on $X \times X$ defined by:

$$\text{(a)} \quad \mathfrak{C} \times \mathfrak{C} - \lim (x_n, y_n) = (x, y) \text{ if and only if } \mathfrak{C} - \lim x_n = x \text{ and } \mathfrak{C} - \lim y_n = y,$$

also satisfies condition (23) and hence it derives from a T_1-topology $\mathfrak{T}(\mathfrak{C} \times \mathfrak{C})$. This topology is stronger than the product topology $\mathfrak{T}(\mathfrak{C}) \times \mathfrak{T}(\mathfrak{C})$ on $X \times X$. (Observe that if A, B are $\mathfrak{T}(\mathfrak{C})$-open in X then $A \times B$ is $\mathfrak{T}(\mathfrak{C} \times \mathfrak{C})$-open in $X \times X$). It may be strictly stronger. For instance if $\mathfrak{T}(\mathfrak{C})$ is not Hausdorff, then the diagonal of $X \times X$ is not closed under $\mathfrak{T}(\mathfrak{C}) \times \mathfrak{T}(\mathfrak{C})$ though it is obviously $\mathfrak{T}(\mathfrak{C} \times \mathfrak{C})$-closed.

The assumption of uniqueness of limits plays an important role in the considerations of the present section. (Let X be an infinite set and set $\mathfrak{C}$-$\lim x_n = x$ whenever the "range" of (x_n) is finite and x is any element of X. Under $\mathfrak{T}(\mathfrak{C})$ every sequence converges to every element).

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SOME REMARKS ON THE COEFFICIENTS USED IN THE THEORY OF HOMOLOGY MANIFOLDS

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In the theory of generalized n-manifolds (n-gms) or Čech cohomology manifolds (n-cms), as developed principally by Wilder, the ring of coefficients had been a field. Due to the influence of transformation groups interest was aroused for more general coefficicnt systems. However, it is usually simpler to deal algebraically with coefficients in a field. Thus it becomes desirable to have a theorem which implies the validity of a result for n-cms over principal ideal domains when it is known to be valid for fields. The main result of our paper is devoted to such a theorem.

In [1; Chap. 1], and [5], it was shown that under suitable restrictions X is an orientable n-cm over the integers Z if X is an orientable n-cm over Z_p, for all primes p and $p = 0$, (i.e. Z_0 is the field of quotients of Z). At the time it was not clear how to proceed to the general case of principal ideal domain L instead of the integers Z. Theorem 1 and the Corollary, here, is the extension of this result to the general case and is, in fact, a strengthening of the earlier result even when $L = Z$. The proof is similiar to arguments of related results in [7]. The previous argument in [5] was given only in outline form and can be adopted, although not easily, to yield a proof of Theorem 1. However, on the basis of what was sketched there the argument would not be any shorter than that which we give here.

As a consequence of the method we establish (Theorem 2) the equivalence of singular homology n-manifolds and n-cms for a wider class of space than was given in [7] and [8].

Before proceeding to the preliminaries we would like to illustrate several of the advantages of Theorem 1. As was mentioned above, arguments involving fields as coefficients are considerably easier than those with a principal ideal domain as a coefficient system. A particular case in mind would be spectral sequence arguments. Thus under certain conditions the establishment of a result valid for n-cms over principal ideal domains where it is known to be valid for fields would be automatic by appealing to Theorem 1 and the Corollary. For example,

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Wilder's monotone mapping theorem is thereby extended. Also the proofs of Separation and Union theorems for n-gms with boundaries (Michigan Mathematical Journal 7, (1960) 7–21) would be considerably simplified and the factorization theorem (Theorem 6) becomes valid for arbitrary principal ideal domains instead of only the integers Z and fields.

I wish to thank the referee for his helpful criticisms and suggestions on improving the exposition of a somewhat entagled earlier version of these results. I have incorporated his suggestions into the present version.

1. Preliminaries. All spaces will be Hausdorff. Two homology theories will eventually be used, the Borel-Moore homology theory with *compact supports* [2] and the singular homology theory. Coefficients will be taken in the L-module G, where L is a principal ideal domain. A superscript "s" will denote the singular homology if it becomes necessary to distinguish the two homology theories. The pth Čech or sheaf cohomology module with compact supports and coefficients in G of a locally compact space X will be denoted by $H_c^p(X; G)$. We shall assume familiarity with the concept of homology local connected up through dimension $n(lc_n$ or $lc_n^s)$ and cohomology locally connected (clc), (see [10] or [1]). For dimension of a locally compact space we shall use the Alexandroff-Cohen definition of cohomology dimension. Fundamental for the discussion will be the universal coefficient sequences.

$$(1.1)\quad 0 \longrightarrow H_p(X; L) \otimes G \longrightarrow H_p(X; G) \longrightarrow H_{p-1}(X; L) * G \longrightarrow 0\ ,$$

and

$$(1.2)\quad 0 \longrightarrow H_c^p(X; L) \otimes G \longrightarrow H_c^p(X; G) \longrightarrow H_c^{p+1}(X; L) * G \longrightarrow 0\ .$$

The sequences are exact and functorial.

(1.3) A *singular homology n-manifold* (n-shm) X *over an L-module* G is a space such that

1. $\dim_L X < \infty$,
2. $H_p^s(X, X - x; G) = \begin{cases} 0, & p \neq n \\ G, & p \neq n \end{cases}$, for each $x \in X$,
3. There exists a covering by open sets $\{U_\alpha\}$ of X such that

$$j_* : H_n^s(X, X - \bar{U}_\alpha; G) \longrightarrow H_n^s(X, X - x; G)$$

is bijective for each $x \in U_\alpha$.

This is what I called locally orientable singular homology n-manifold over G in [7]. It isn't hard to see that 2 and 3 imply that X is locally a Peano-space and consequently condition 1 makes sense, (that is, X is therefore locally compact and dimension is consequently well defined). We say that X is *orientable* if for each open connected set O, with compact closure, the homomorphism $j_*: H_n^s(X, X - \bar{O}; G) \to H_n^r(X, X - x; G)$ is bijective, for all $x \in O$. That is, 2 and 3 say that the orientation *sheaf* or the sheaf of local homology groups is trivial in all dimensions except n and there it is locally constant. It is orientable, therefore, if the orientation sheaf is constant.

By a *generalized n-manifold over L* (n-gm) we shall mean a *Čech cohomology n-maniford* over L (n-cm). This is a locally orientable generalized n-manifold in the sense of Wilder and with coefficients over L. See for example, [10], [1], [2], and [5].

(1.4) In [2], Borel and Moore defined a homology theory for locally compact spaces. This can be regarded as a single space theory in terms of Eilenberg and Steenrod [3; Chap. 10]. However, by looking at the homology theory as a relative homology theory (see below) we shall easily reformulate the definition of (Borel-Moore) homology manifolds to one that is analogous to that already given for singular homology manifolds. This enables us to use a single argument for the proof of Theorem 1 valid for both homology theories. Furthermore, the comparison between singular homology manifolds and homology manifolds there, in a sense, reduces to a comparison of the respective homology theories (Theorem 2). (An alternative would be to reformulate the definition of singular homology manifold in terms of locally finite singular homology leaving intact the Borel-Moore definition. But for various reasons we do not think this procedure is preferable.)

Let U be an open subset of the locally compact space X. Define the relative group

$$H_p(X, X - U; G) \quad \text{to be} \quad H_p(U; G) ,$$

where $H_p(U; G)$ denotes the Borel-Moore homology theory with closed supports, [cf. 2; 3.8 and § 5]. Thus, from the single space theory of Borel-Moore homology we obtain a homology theory, in the sense of Eilenberg and Steenrod, defined on the category of closed locally compact pairs and proper maps. We now follow Eilenberg and Steenrod and extend the relative theory on compact pairs to arbitrary Hausdorff pairs.

Let (X, Y) be an arbitrary Hausdorff pair. Define

$$H_p^c(X, Y; G) \quad \text{to be} \quad \text{Dir lim}_\lambda\, H_p(X_\lambda, A_\lambda; G)$$

where (X_λ, A_λ) runs through all compact pairs (direction is given by inclusion) in the arbitrary pair (X, Y). This procedure extends the homology theory from compact pairs to arbitrary pairs. See [3; p. 255] for outline of details. In particular, if X is locally compact and Y a subset whose closure is compact then

(i) $\text{Dir lim}_{U \supset Y} H_p(U) = \text{Dir lim}_{U \supset Y} H_p(X, X - U) = H_p^c(X, X - Y)$, where U runs through the open neighborhoods of the subspace Y. Clearly,

(ii) $H_p(X, X - U) = H_p^c(X, X - U)$,

when X is locally compact, U is open and $\bar{U}$ is compact, (use excision). We shall call this homology theory, H_*^c, defined on arbitrary Hausdorff pairs and continuous maps the *Borel-Moore homology theory with compact carriers*. For the pair (X, Y) we have, of course, the exact sequence

$$\text{(iii)} \quad \longrightarrow H_p^c(Y; G) \xrightarrow{i_*} H_p^c(X; G) \xrightarrow{j_*} H_p^c(X, Y; G) \xrightarrow{\partial^*} H_{p-1}^c(Y; G) \longrightarrow .$$

For locally compact spaces *homology n-manifolds* (n-hm) *over an L-module G are defined* exactly as singular homology n-manifolds over G were defined in 1.3. The homology to be used is the Borel-Moore homology theory with compact carriers. Putting $Y =$ point in (i) above we easily see that the definition is equivalent to that in [2; 7.5] when X is lc_0.

Poincaré duality holds for homology manifolds. The form of duality we shall need is that with compact supports. Thus it is necessary to identify the Borel-Moore homology with compact carriers with the homology of the Borel-Moore chains with *compact support*. If U is an open subset of the locally compact space X we have the exact sequence of Borel-Moore chains with compact support

$$0 \longrightarrow \Gamma_c(\mathscr{C}_H(X; L) \mid U \otimes G) \longrightarrow \Gamma_b(\mathscr{C}_H(X; L) \otimes G) \longrightarrow \Gamma_c(\mathscr{C}_H(X; L) \otimes G)/\Gamma_c(\mathscr{C}_H(X; L) \mid U \otimes G) \longrightarrow 0 ,$$

where $\mathscr{C}_H(X; L)$ denotes the *standard sheaf for homology* on X of [2; § 3]. The derived homology sequence is analogous to (iii) above, with Y replaced by U, compare [9; 2.2]. However, more is true. The chains with compact support in U, $\Gamma_c(\mathscr{C}_H(X; L) \mid U \otimes G)$, can be identified with the direct limit, directed by inclusion, of all the chains of X whose support lie in compact subsets of U. On the other hand, the chains of X whose support lie in a compact subset, F, have homology functorially equivalent to the homology of F. See [2; 3.4] and [9] for details. Similarly we may identify $H_*^c(X, U; G)$ as the homology of the chains of X with compact support mod those chains of X whose support, again compact, lies in U.

In what follows we shall only use the Borel-Moore homology theory for locally compact pairs (X, U), U open in X, and with compact carriers (or the equivalent, with compact supports). Therefore we shall simply abbreviate $H_*^c(X; U)$ to $H_*(X, U)$.

(1.5) Let q stand ambiguously for a prime ideal, $\neq 0$, or any of its generators of a principal ideal domain L. The ideal is maximal and the residue class ring is a field and will be denoted by L_q. Denote the field of quotients of L by L_0. Define the *family of all fields determined* by L to be the collection of fields just described. Denote the collection by $\mathscr{L}$.

(1.6) The following statements are well known consequences of elementary properties of tensor and Tor functors applied to L-modules.

Suppose that G is an L-module and that $G \otimes L_0 = 0$, then G contains only torsion elements.

Suppose that G is L-module, and q is a prime in L. Then G contains q-torsion, if and only if, $G * L_q \neq 0$. Hence, if G and H contain q-torsion, $G * H \neq 0$, because G contains a cyclic submodule isomorphic to L_q and $L_q * H \longrightarrow G * H$ is injective.

Let $g \in G$, and (g) be the cyclic submodule generated by g. From the exact sequence

$$G/(g) * G \longrightarrow (g) \otimes G \longrightarrow G \otimes G\ ,$$

and the fact that $G/(g) * G$ is a torsion module, (g) can not be isomorphic to L if $G = 0$. Hence, if $G \otimes G = 0$, *then G contains only torsion elements. In particular, if $G \otimes G = 0$ and $G \neq 0$, then $G * G \neq 0$.*

(1.7) Let X be locally compact, U and V open subsets of X such that $\bar{V} \subset U$, and $\bar{V}$ is compact. Suppose that X is homology locally connected up through dimension n in terms of the Borel-Moore homology theory with compact supports and coefficients in L, lc_n (respectively; X is lc_n^s over L). *Then the images of the homomorphisms*

$$i_* : H_p(V; L) \longrightarrow H_p(U; L)$$
$$j_* : H_p(X, X - \bar{V}; L) \longrightarrow H_p(X, X - \bar{U}; L)$$

are finitely generated, for all $p \leqq n$ in terms of the Borel-Moore homology theory with compact supports (respectively; in terms of the singular homology).

Proof. This argument has been communicated to the author by C. N. Lee. Both the Borel-Moore homology with compact supports and the singular homology theory satisfy the Mayer-Vietoris sequence in terms of open subsets. The absolute case for singular homology then becomes an exact copy of the absolute case for the Borel-Moore homology theory which is proved in [2; 6.8]. The relative case now follows from exactness and a little diagram chasing.

2. The family $\mathscr{L}^*$. Each field $L_p \in \mathscr{L} (p = 0, \text{or } q)$ is a vector space over the rationals or the integers modulo a prime. This is determined, of course, by the characteristic of each L_p. Denote the corresponding collection of fields isomorphic to the integers modulo a prime and possibly the rationals by $\mathscr{L}^*$. Clearly, L_p is a L_p^* module.

Let $\chi(L)$ denote the characteristic of L. In general we have three cases:

1. $\chi(L) = m, m \neq 0$.
2. $\chi(L) = 0$, $\chi(L_p) = 0$, for all $L_p \in \mathscr{L}$.
3. $\chi(L) = 0$, $\chi(L_p) \neq 0$, for some $L_p \in \mathscr{L}$.

The structure of $\mathscr{L}^*$ for these three cases are respectively:

1. $Z_m = \mathscr{L}^*$, and L is additively a vector space over Z_m.
2. $Z_0 = \mathscr{L}^*$, and L is additively a vector space over Z_0.
3. Z_0 and all Z_m for which there exists an $L_p \in \mathscr{L}$ for which $\chi(L_p) = m \neq 0$.

It is the presence of the third case which causes the difficulties in the main theorems.

The first observation to be made is that $\dim_G X \leqq \dim_L X$, for any L-module G. Also, if G is free over L, then $\dim_G X = \dim_L X$. This is a direct consequence of 1.2. Hence, $\dim_{L_p} X = \dim_{L_p^*} X$, and, in fact for Cases 1 and 2, $\dim_L X = \dim_{L_p^*} X$, for all $L_p^* \in \mathscr{L}^*$.

The next proposition is also an easy consequence of the universal coefficient Theorems 1.1 and 1.2.

(2.1) PROPOSITION. A space X is a n-cm, n-shm or an n-hm over L (respectively; orientable) where L is a field of characteristic m, if and only if, X is an n-cm, n-shm or an n-hm over Z_m, (respectively; orientable).

3. A dimension lemma. This section is concerned with the determination of $\dim_L X$ when one knows $\dim_{L_p} X$ or $\dim_{L_p^*} X$ for all $L_p \in \mathscr{L}$, or all $L_p^* \in \mathscr{L}^*$. Define $\dim_{\mathscr{L}^*} X = \max_{L_p^* \in \mathscr{L}^*} \{\dim_{L_p^*} X\}$. This will be called *the field dimension* of X. Clearly, $\dim_{\mathscr{L}^*} X = \dim_{\mathscr{L}} X$, where $\dim_{\mathscr{L}} X$ is defined analogously.

(3.1) LEMMA. *Let X be locally compact. Then* $\dim_L X \leqq (\dim_{\mathscr{L}^*} X) + 1$. *Furthermore, the strict inequality holds for Cases* 1 *and* 2 *of* § 2.

Proof. By the remarks in § 2, $\dim_L X = \dim_{L_p^*} X = \dim_{L_p} X = \dim_{\mathscr{L}^*} X$ for Cases 1 and 2.

Suppose now that $n = \dim_{\mathscr{L}^*} X < \dim_L X$. (We are necessarily in Case 3). Then $H_c^{n+k}(U; L_p) = 0$, for all $L_p \in \mathscr{L}$, all open $U \subset X$, and all $k > 0$. By 1.2, $H_c^{n+k}(U; L) \otimes L_p = 0$ and $H_c^{n+k+1}(U; L) * L_p = 0$, for all $L_p \in \mathscr{L}$. Hence by 1.6, $H_c^{n+k+1}(U; L) = 0$, for all $k > 0$, and all open subsets U, $U \subset X$. But if $H_c^{n+1}(U; L) = 0$, for all U, then $\dim_L X \leqq n$. Therefore, there exists some open subset U such that $H_c^{n+1}(U; L) \neq 0$. This implies that $H_c^{n+1}(U; L)$ is a torsion module. (In fact, $H_c^{n+1}(V; L)$ is a torsion module, possibly trivial, for all open subsets V of X.) We observe that if $\dim_L X > \dim_{\mathscr{L}^*} X = n$, $\dim_L X = n + 1$. Hence, there exists an $L_q \in \mathscr{L}$ such that $\dim_{\mathscr{L}^*} X = \dim_{L_q} X$ and there exists an open subset V such that $H_c^{n+1}(V; L)$ has nontrivial q-torsion. Note also that if $\dim_{L_p} X < \dim_{L_0} X$ for all $L_p \in \mathscr{L}$, $L_p \neq L_0$, then $\dim_L X = \dim_{L_0} X = \dim_{\mathscr{L}^*} X$.

Actually the lemma is stronger than we need. We shall only use in Theorem 1 the fact that $\dim_L X < \infty$ if $\dim_{\mathscr{L}^*} X < \infty$. However, it can be used to simplify and/or extend several arguments in dimension theory. The following is a sample of such a situation.

COROLLARY. (Bockstein, and Fary [4].) *Let X be a locally compact Hausdorff space such that* $\dim_L X = n < \infty$. *Then* $\dim_L X^k = kn$ *or* $k(n-1)+1$. *The latter case holds, if and only if,* $\dim_{\mathscr{L}^*} X < \dim_L X$.

Proof. If $\dim_L X = \dim_{\mathscr{L}^*} X$, then $\dim_L X^k = kn = \dim_{\mathscr{L}^*} X^k$. (It is known that $\dim_L X \times Y \leqq \dim_L X + \dim_L Y$, see [1, p. 15]. There are other published proofs of this fact that seem to be incomplete; however, they can be completed if one assumes $\dim_L X \times Y < \infty$. Now, $\dim_{\mathscr{L}^*} X^k \geqq k \dim_{\mathscr{L}^*} X$ follows from repeated use of the Künneth theorem and $\dim_L X^k = k \dim_L X$ from the fact that $\dim_L X^k \geqq \dim_{\mathscr{L}^*} X^k$). If $\dim_L X^k \neq kn$, then $\dim_L X > \dim_{\mathscr{L}^*} X$. Then there exists some "prime" $q \in L$ such that $\dim_{L_q^*} X = \dim_{\mathscr{L}^*} X = n - 1$.

In particular we can find an open subset $U \subset X$ such that $H_c^n(U; L)$ contains q-torsion. Thus, for $k = 1$, the theorem is proved. We proceed by induction. Assume $H_c^{r(n-1)+1}(U^r; L)$ has q-torsion, for all $r < k$. Then

$$H_c^{r(n-1)+1}(U^r; L) * H_c^n(U; L) \neq 0 , \qquad \text{by } 1.6.$$

Hence, by the Künneth theorem $H_c^{r(n-1)+1+n-1}(U^r \times U; L) \neq 0$. Setting $r = k - 1$, we obtain $H_c^{k(n-1)+1}(U^k; L) \neq 0$. Since $\dim_{\mathscr{L}^*} X^k = k(n-1)$, $\dim_L X^k = k(n-1) + 1$.

It is easily shown that the latter case cannot hold if X is clc over L, the point being $\dim_L X = \dim_{\mathscr{L}^*}$ when X is clc.

4. The main results.

THEOREM 1. *Let X be lc_n (respectively; lc^s_n) over L. Then X is an n-*hm *(respectively; an n-*shm*) over L, if and only if, X is an n-*hm *(respectively; an n-*shm*) over each $L^*_p \in \mathscr{L}^*$. Moreover, X is orientable, if and only if, X is orientable over each $L^*_p \in \mathscr{L}^*$.*

THEOREM 2. *Let X be lc_n (respectively; lc^s_∞) over L, then X is an n-*cm*, if and only if X is an n-*hm *(respectively; n-*shm*) over L. Moreover X is an orientatable n-*cm *if and only if X is an orientable n-*hm *(respectively; n-*shm*).*

COROLLARY. *Let X be lc_n over L then X is an n-*cm *over L if and only if X is an n-*cm *over every $L^*_p \in \mathscr{L}^*$.*

Summarizing these results we have:

Under lc_n over L, $n\text{-cm}_L \Leftrightarrow n\text{-cm}_{\mathscr{L}^*} \Leftrightarrow n\text{-cm}_{\mathscr{L}} \Leftrightarrow n\text{-hm}_L \Leftrightarrow n\text{-hm}_{\mathscr{L}} \Leftrightarrow n\text{-hm}_{\mathscr{L}^*}$.

Under lc^s_n over L, $n\text{-shm}_L \Leftrightarrow n\text{-shm}_{\mathscr{L}^*} \Leftrightarrow n\text{-shm}_{\mathscr{L}}$.

Under lc^s_∞ over L, $n\text{-cm}_L \Leftrightarrow n\text{-cm}_{\mathscr{L}^*} \Leftrightarrow n\text{-cm}_{\mathscr{L}} \Leftrightarrow n\text{-hm}_L \Leftrightarrow n\text{-hm}_{\mathscr{L}} \Leftrightarrow n\text{-hm}_{\mathscr{L}^*} \Leftrightarrow n\text{-shm}_L \Leftrightarrow n\text{-shm}_{\mathscr{L}^*} \Leftrightarrow n\text{-shm}_{\mathscr{L}}$.

Proof of Theorem 1. If X is an n-hm (respectively; an n-shm) over L, the universal coefficient theorems immediately imply that X is an n-hm (respectively; an n-shm) over each $L_p \in \mathscr{L}$, and over each $L^*_p \in \mathscr{L}^*$.

For the converse we shall not distinguish the two respective cases as the arguments are identical. The homology groups are to be interpreted as the singular homology or the Borel-Moore homology with compact supports as the case may be.

Assume that X is an n-hm (or an n-shm) over each $L^*_p \in \mathscr{L}^*$. Then as L_p is a vector space over L^*_p, we can (universal coefficient theorems) assume that X is an n-hm (or an n-shm) over each $L_p \in \mathscr{L}$.

Next, we show that $\dim_{L_p} X = n$, for all p. Hence by § 3, $\dim_L X \leqq n + 1$. In fact, after we show that X is an n-hm (or n-shm) over L, we may conclude that $\dim_L X = n$. These statements follow from the lemma:

*Let X be lc_0 over the principal ideal domain L or have a countable base for its topology (respectively: no assumptions if the singular homology is used). If X is an n-*hm *(respectively: n-*shm*) then $\dim_L X = n$.*

The proof of this general proposition depends upon the fact that for finite dimensional spaces over L, $\dim_L$ is a local property. That is, one need only check the vanishing of the cohomology modules for sufficiently small subsets of X. The hypothesis guarantees that the $(-1)^{st}$-homology modules for the Borel-Moore homology theory are 0 for open subsets of X. We now use Poincaré duality, (Poincaré duality is proved for singular homology manifolds in [7]), and obtain that $H_c^{n+k}(U; L) = H_{-k}(U; L) = 0$, for all open subsets U within an orientable part of X and all $k > 0$. This completes the proof of the lemma.

Let $x \in X$, then $H_k(X, X - x; L_p) = 0$, for all $k \neq n$. In particular, $H_k(X, X - x; L) \otimes L_0 = H_{k-1}(X, X - x; L) * L_q = 0$, for all $k \neq n$, and all q. Thus, for 1.6, $H_k(X, X - x; L) = 0$, for all $k \neq n$ or $n - 1$. Furthermore, for any open subset $P \subset X$, $H_n(X, P; L)$ is torsion free since $H_{n+1}(X, P; L_q) = 0$, by Poincaré duality. Let U be a connected neighborhood of x (with compact closure) such that U is orientable with respect to L_0. Let V be *any* connected open subset of U such that $\bar{V} \subset U$; let F denote the image of

$$j_* : H_n(X, X - \bar{U}; L) \longrightarrow H_n(X, X - \bar{V}; L) . \tag{4.1}$$

The module F is a finitely generated (by 1.7) torsion free L-module. Since

$$\begin{aligned} j_* : H_n(X, X - \bar{U}; L) \otimes L_0 &\longrightarrow H_n(X, X - \bar{V}; L) \otimes L_0 \\ &= H_n(X, X - \bar{V}; L_0) \end{aligned}$$

is bijective and has image isomorphic to L_0, it follows that F must be precisely isomorphic to L. Now tensor (4.1) by L_q and obtain the commutative diagram

$$\begin{array}{ccccc} H_n(X, X - \bar{U}; L) \otimes L_q & \longrightarrow & F \otimes L_q & \longrightarrow & 0 \\ & \searrow^{(j_* \otimes 1)} & \downarrow & & \\ & & H_n(X, X - \bar{V}; L) \otimes L_q & & \end{array}$$

which is exact on the top row. Since $F \otimes L_q \cong L_q$,

$$H_n(X, X - \bar{U}; L) \otimes L_q \neq 0 ,$$

and consequently $H_n(X, X - \bar{U}; L_q) \cong L_q$. Therefore, U is an *orientable* n-hm or n-shm over L_q, for all q. Note further that

$$H_{n-1}(X, X - \bar{U}; L) * L_q = 0 .$$

Therefore it follows that $H_{n-1}(X, X - x; L) = 0$, since it is the direct limit of torsion free modules (such as $H_{n-1}(X, X - \bar{U}; L)$) and $H_{n-1}(X, X - x; L) \otimes L_0 = 0$.

We show now that $j_* : H_n(X, X - \bar{U}; L) \to H_n(X, X - \bar{V}; L)$ is bijective with image, F, isomorphic to L, whence the orientation sheaf is locally isomorphic to L in dimension n, (i.e. $j_* : H_n) X, X - \bar{U}; L) \to H_n(X, X - x; L)$ is bijective for *all* $x \in U$).

First, $H_n(X - \bar{V}, X - \bar{U}; L)$ is torsion free, and $H_n(X - \bar{V}, X - \bar{U}; L_p) \cong H_c^0(\bar{U} - \bar{V}; L_p) = 0$. Therefore by 1.1 and 1.6 $H_n(X - \bar{V}, X - \bar{U}; L) = 0$. Exactness of the triple $(X, X - \bar{V}, X - \bar{U})$ implies that j_* is injective. A similar argument shows that $H_n(X, X - \bar{V}; L)$ is also isomorphic to L. Now, $H_n(X - \bar{V}, X - \bar{U}; L_q) = 0$, for all $L_q \in \mathscr{L}$, implies $H_{n-1}(X - \bar{V}, X - \bar{U}; L)$ is torsion free. Thus j_* must also be surjective.

We have shown that X is an n-hm (or n-shm) over L. That X is orientable over L if and only if it is orientable over each $L_p \in \mathscr{L}$ is clear from the above. In fact, orientability over L_0 is equivalent to orientability over L.

It is interesting to note that lc_0 was used to make the $(-1)^{st}$-Borel-Moore homology groups vanish and to guarantee sufficiently many open connected subsets of X. (These facts are implied by Condition 3 of 1.3 in the singular case). The lc_n (and lc_n^s) condition over L was used to imply the finite generation of the image of j_* in 4.1 (Added in proof: lc_n also guarantee the validity of change in rings.)

Proof of Theorem 2. Suppose X is an n-cm over L then Poincaré duality implies $H_k(X, X - x; L) \cong H^{n-k}(x; L) = 0$, for all integers $k \neq n$. In dimension, n, $H_n(X, X - \bar{V}; L) \cong H_c^0(\bar{V}; L) \cong L$, if V is connected contained within an orientable part of X and $\bar{V}$ compact. Furthermore, duality is functorial with respect to inclusion. Thus X is an n-hm and orientable if and only if X is orientable as an n-cm. [cf 2; 7.12 for a proof that essentially uses only the universal coefficient theorem]. Note also that *if X is an n-*cm *over L, then X is an n-*cm *over L_p and L_p^*, for all $L_p \in \mathscr{L}$, and $L_p^* \in \mathscr{L}^*$.* This follows trivially from the universal coefficient sequences. In particular, X is an n-hm over all $L_p \in \mathscr{L}^*$. In [8], I have shown that if X is lc_∞^s over L_p^*, and is an n-hm over L_p^*, then X is an n-shm over L_p^*. (The point being that the relative Borel-Moore homology groups $H_*(X, U; L_p^*)$ are naturally equivalent to the Čech homology groups (with compact carriers) which are naturally equivalent to the relative singular homology groups $H_*^s(X, U; L_p)$. Therefore, if X is a cm over L and is lc_∞^s over L (hence lc_∞^s over every L_p^*), then, by Theorem 1, X is an n-shm over L.

On the other hand, if X is an n-hm and is lc_n (or an n-shm and is lc_n^s) over L, then Poincaré duality (image $H_p(V; L) \to H_p(U; L)$ is isomorphic to image $H_c^{n-p}(V; L) \to H^{n-p}(U; L)$) implies that X is an n-cm over L. (The argument is that of [2; 7.12] with the observation that lc_n condition suffices instead of the full lc_∞ because H_c^{n-p} is 0 for negative p.)

Again orientability statements are clear.

Proof of the corollary. Trivially, if X is an n-cm over L then X is an n-cm over every coefficient L-module. Conversely, if X is an n-cm over L_p^* then X is an n-hm over L_p^*, by Theorem 2. By Theorem 1, if X is, in addition, lc_n over L, then X is an n-hm over L. Again by Theorem 2, X is an n-cm over L.

In [8] it was observed that Theorem 2 held for fields. In [6], the "if" part (singular homology) of Theorem 2 for $L = Z$ and X having a countable basis is proved. Their argument, however, does not appear to be amenable to the general case because of the reliance upon duality between compact topological groups and discrete abelian groups.

In Theorem 1, the integer n was kept fixed, i.e. n was independent of L_p, $L_p \in \mathscr{L}$. In order to free the theorem from this assumption a strengthening of the other hypotheses is needed, although no examples implying the contrary are known to me. An illustration of (possibly) stronger hypotheses which would imply the constancy of the integer n would be to assume that $H_{n_0}(X, X - x; L)$ is finitely generated for some $x \in X$, where $n_0 = \dim_{L_0} X$, or to assume that $\dim_{L_0} X = \dim_{\mathscr{L}^*} X$. (Cf. [1; I, 4.11]).

One could formulate these remarks as a question. *Let X be clc (respectively: lc^s_∞) over L, and* $\dim_L X < \infty$. *Suppose X is* n_p-cm *(respectively:* an n_p-shm) *over each $L_p \in \mathscr{L}$. Then, is X an* n-cm *(respectively:* an n-shm) *over L, for some integer n?* It seems likely that the answer is always affirmative. (We have already seen that if the characteristic of L satisfies 1 or 2 of § 2, the answer is affirmative.) However, if the local connectedness assumptions over L were removed it seems likely that the answer would be negative, (if not then it would follow that every compact group acting effectively on a manifold would necessarily be a Lie group).

Added in proof. If L' is a module over L and both are principal ideal domains then $H_p(X, X\text{-}\bar{U}; L')$ has the interpretations depending upon which ring, L or L', is used for the ground ring. The interpretations agree, that is the change of rings is valid, for the singular homology as is well known, and for the Borel-Moore homology theory, with compact carriers, if X is lc^p over L, (due to Bredon). We have implicitly used the validity of the change of rings in the proof of Theorem 1. When L' is a field, which is all that we need, the validity for the Borel-Moore homology can easily established as follows. Let the L'-modules

$$\mathrm{Hom}_{L'}(\Gamma_C(\mathscr{C}_H(X; L) \mid U \otimes_K L'), L') \quad \text{and} \quad \mathrm{Hom}_{L'}(\Gamma_C(\mathscr{C}_H(X; L') \mid U), L')$$

determine presheafs. Apply the arguments of [2; 6.4 and 6.6] to both presheafs. This implies that they both determine the sheaf-theoretic cohomology of U, with closed supports, over L as dual spaces of both homology theories. Hence the change of rings is valid. A recent example of Bredon, which has also inspired these remarks, shows that this change of rings is not valid without the lc^{v} hypothesis. A complete discussion of the change in rings will appear in a forthcoming book of Bredon on sheaf theory.

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ON SUB-ALGEBRAS OF A C^*-ALGEBRA

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The following noncommutative extension of the Stone-Weierstrass approximation theorem has been obtained by Glimm.

Theorem. Let $\mathscr{A}$ be a C^*-algebra with identity I, and let $\mathscr{B}$ be a C^*-sub-algebra containing I. Suppose that $\mathscr{B}$ separates the pure state space of $\mathscr{A}$. Then $\mathscr{B} = \mathscr{A}$.

In the present paper, we apply Glimm's theorem to obtain the following noncommutative generalisation of another result of Stone.

Let $\mathscr{A}$ be a C^*-algebra with identity I and pure state space $\mathscr{P}$. Let $\mathscr{B}$ be a C^*-sub-algebra of $\mathscr{A}$, and defne

$$\mathscr{N} = \{f: \ f \text{ is a pure state of } \mathscr{A} \text{ and } f(B) = 0 \ (B \in \mathscr{B})\},$$

$$\mathscr{E} = \{(g, h): \ g, h \in \mathscr{P} \text{ and } g(B) = h(B) \ (B \in \mathscr{B})\},$$

$$\mathscr{K}_{\mathscr{B}} = \{A: \ A \in \mathscr{A}, \ f(A) = 0 \ \ (f \in \mathscr{N}) \ \text{ and } \ g(A) = h(A) \ ((g, h) \in \mathscr{E})\}.$$

Then $\mathscr{B} = \mathscr{K}_{\mathscr{B}}$.

We will refer to this as Theorem 2 in the sequel. Glimm's theorem is to be found in [1]; Stone's, in [3].

Once it is known that $\mathscr{K}_{\mathscr{B}}$ is a C^*-sub-algebra of $\mathscr{A}$, Theorem 2 is an almost immediate consequence of Glimm's theorem (see § 4). It is clear that $\mathscr{K}_{\mathscr{B}}$ is a closed self-adjoint linear subspace of $\mathscr{A}$; accordingly, most of this paper is devoted to proving that $\mathscr{K}_{\mathscr{B}}$ is closed under multiplication (see § 3).

We remark that, if $\mathscr{A}$ is commutative, then $\mathscr{P}$ consists exactly of all homomorphism from $\mathscr{A}$ on to the complex plane $\boldsymbol{C}$; so in this case, it is immediate from its definition that $\mathscr{K}_{\mathscr{B}}$ is a C^*-sub-algebra. However, this seems not to be obvious in the general case. Indeed, for a *general* set $\mathscr{N}$ of pure states of $\mathscr{A}$ and a *general* subset $\mathscr{E}$ of $\mathscr{P} \times \mathscr{P}$, the class

$$\{A: \ A \in \mathscr{A}, \ f(A) = 0 \ (f \in \mathscr{N}) \text{ and } g(A) = h(A) \ ((g, h) \in \mathscr{E})\}$$

need not be a sub-algebra of $\mathscr{A}$; for example, let $\mathscr{A}$ consist of all bounded linear operators on a Hilbert space H, let $\mathscr{N}$ be void, and let $\mathscr{E}$ consist of a single pair of vector states arising from orthogonal unit vectors.

2. Notation.

Throughout, $\mathscr{A}$ is a C^*-algebra-by which we shall mean a uniformly closed self-adjoint algebra of operators acting on a (complex) Hilbert space H. We shall always assume that $\mathscr{A}$ contains

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the identity operator I on H. A *state* of $\mathscr{A}$ is a linear functional f on $\mathscr{A}$ such that $f(A^*A) \geqq 0$ $(A \in \mathscr{A})$ and $f(I) = 1$. The set of all states is convex and weak * compact; the Krein-Milman theorem ensures the existence of extreme points, and these are called *pure states*. The *pure state space* of $\mathscr{A}$, denoted by $\mathscr{P}$ (or $\mathscr{P}(\mathscr{A})$ if $\mathscr{A}$ has to be specified), is the weak * closure of the set of all pure states.

Given a state f of $\mathscr{A}$, there is a *-representation ϕ_f of $\mathscr{A}$ on a Hilbert space H_f, and a unit vector x_f in H_f, such that $\phi_f(\mathscr{A})x_f$ is dense in H_f, and

$$f(A) = \langle \phi_f(A)x_f,\ x_f\rangle \qquad (A \in \mathscr{A}) .$$

To within unitary equivalence, ϕ_f is unique. Furthermore, ϕ_f is irreducible if and only if f is a pure state (see, for example, [2] 245, 265, 266). We shall always use the symbols ϕ_f, H_f, x_f in the sense just described.

3. Some lemmas. Throughout this section we shall assume that $\mathscr{B}$ is a C^*-sub-algebra of $\mathscr{A}$, and that $I \in \mathscr{B}$. We use the notations introduced in the statement of Theorem 2; note that, since $I \in \mathscr{B}$, $\mathscr{N}$ is empty and

$$\mathscr{H}_{\mathscr{B}} = \{A: \quad A \in \mathscr{A} \text{ and } g(A) = h(A) \ \ ((g, h) \in \mathscr{E})\} .$$

For completeness, we give a proof of the following simple result.

LEMMA 1. (i) *Let* $f \in \mathscr{P}$, $S \in \mathscr{A}$ *and suppose that* $f(S^*S) = 1$. *Define* $g(A) = f(S^*AS)$ $(A \in \mathscr{A})$. *Then* $g \in \mathscr{P}$.

(ii) *Let* $f \in \mathscr{P}$, $x \in H_f$, $\|x\| = 1$, *and define* $g(A) = \langle \phi_f(A)x, x\rangle$ $(A \in \mathscr{A})$. *Then* $g \in \mathscr{P}$.

Proof. (i) Clearly g is a state. Suppose first that f is a pure state, and let $x = \phi_f(S)x_f$. Then for each $A \in \mathscr{A}$,

$$(1) \qquad \langle \phi_f(A)x, x\rangle = \langle \phi_f(S^*AS)x_f, x_f\rangle = f(S^*AS) = g(A) .$$

With $A = I$ we obtain $\|x\| = 1$; and since f is a pure state, ϕ_f is irreducible, so $\phi_f(\mathscr{A})x$ is dense in H_f. This, with (1), implies that ϕ_f and ϕ_g are unitarily equivalent. Thus ϕ_g is irreducible, so g is pure.

Now suppose only that $f \in \mathscr{P}$. There is a net (f_i) of pure states which converges to f in the weak * topology. Since $f_i(S^*S) \rightarrow f(S^*S) = 1$, we may suppose that $f_i(S^*S) > 0$ for each i. Let $k_i = [f_i(S^*S)]^{-1/2}$, $S_i = k_iS$, and define $g_i(A) = f_i(S_i^*AS_i)$ $(A \in \mathscr{A})$. Then $f_i(S_i^*S_i) = 1$, and the argument of the preceding paragraph shows that g_i is a pure state. For each $A \in \mathscr{A}$,

$$g_i(A) = \frac{f_i(S^*AS)}{f_i(S^*S)} \to f(S^*AS) = g(A) .$$

Hence (g_i) is a net of pure states which converges to g in the weak * topology, so $g \in \mathscr{P}$.

(ii) Since $\phi_f(\mathscr{A})x_f$ is dense in H_f, we may choose $S_n \in \mathscr{A}$ ($n = 1, 2, \cdots$) such that

$$\| \phi_f(S_n)x_f \| = 1 , \qquad \| \phi_f(S_n)x_f - x \| \to 0 .$$

Thus $f(S_n^* S_n) = 1$, and by part (i) of this lemma, we may define g_n in $\mathscr{P}$ by $g_n(A) = f(S_n^* AS_n)$ $(A \in \mathscr{A})$. Then for each $A \in \mathscr{A}$,

$$g_n(A) = \langle \phi_f(A)\phi_f(S_n)x_f,\ \phi_f(S_n)x_f \rangle \to \langle \phi_f(A)x, x \rangle = g(A) .$$

Thus $g \in \mathscr{P}$.

LEMMA 2. *Let $T \in \mathscr{H}_{\mathscr{B}}$, $S \in \mathscr{B}$. Then $S^*TS \in \mathscr{H}_{\mathscr{B}}$.*

Proof. Let $(f_1, f_2) \in \mathscr{E}$. We have to show that $f_1(S^*TS) = f_2(S^*TS)$. Since $S^*S \in \mathscr{B}$, we have $f_1(S^*S) = f_2(S^*S)$; and after multiplying S by a suitable scalar, we may clearly suppose that $f_1(S^*S)$ is either 0 or 1.

If $f_i(S^*S) = 0$, then S is in the left kernel of f_i ($i = 1, 2$), and $f_1(S^*TS) = f_2(S^*TS) = 0$.

If $f_i(S^*S) = 1$, define $g_i(A) = f_i(S^*AS)$ $(A \in \mathscr{A})$. By Lemma 1 (i), $g_i \in \mathscr{P}$. If $B \in \mathscr{B}$, then $S^*BS \in \mathscr{B}$, so $f_1(S^*BS) = f_2(S^*BS)$; that is, $g_1(B) = g_2(B)$. Hence $(g_1, g_2) \in \mathscr{E}$, and since $T \in \mathscr{H}_{\mathscr{B}}$, it follows that $g_1(T) = g_2(T)$; that is, $f_1(S^*TS) = f_2(S^*TS)$. This completes the proof.

LEMMA 3. *Let $T \in \mathscr{H}_{\mathscr{B}}$ and $R, S \in \mathscr{B}$. Then $R^*TS \in \mathscr{H}_{\mathscr{B}}$.*

Proof. This follows from Lemma 2 since

$$\begin{aligned} 4\,R^*TS = {} & (R + S)^*T(R + S) - (R - S)^*T(R - S) \\ & - i(R + iS)^*T(R + iS) + (R - iS)^*T(R - iS) . \end{aligned}$$

LEMMA 4. *Let $f \in \mathscr{P}$ and let M be a closed subspace of H_f which is invariant under $\phi_f(\mathscr{B})$. Then M is a invariant under $\phi_f(\mathscr{H}_{\mathscr{B}})$.*

Proof. Suppose that the lemma is false. Then we may choose $T \in \mathscr{H}_{\mathscr{B}}$ and $x \in M$ such that $\phi_f(T)x \notin M$. Let $y = (I - E)\phi_f(T)x$, where E is the projection from H_f on to M. Given t in $[0, 2\pi)$, define $y_t = x + \exp(it)y$, $z_t = ky_t$, where

$$k = [\| x \|^2 + \| y \|^2]^{-1/2} = \| y_t \|^{-1} .$$

Thus $z_t \in H_f$, $\|z_t\| = 1$, and by Lemma 1 (ii) we may define $g_t \in \mathscr{P}$ by $g_t(A) = \langle \phi_f(A)z_t, z_t\rangle$ $(A \in \mathscr{A})$. Since $\phi_f(\mathscr{B})$ leaves both M and $H_f \ominus M$ invariant, it follows that for each $B \in \mathscr{B}$,

$$\begin{aligned} g_t(B) &= k^2\langle \phi_f(B)(x + e^{it}y),\ x + e^{it}y\rangle \\ &= k^2[\langle \phi_f(B)x, x\rangle + \langle \phi_f(B)y, y\rangle]\ , \end{aligned}$$

which is independent of t. Hence, for each s, t in $[0, 2\pi)$, we have $(g_s, g_t) \in \mathscr{E}$. Since $T \in \mathscr{H}_{\mathscr{B}}$, it follows that $g_s(T) = g_t(T)$; so $g_t(T)$ is independent of $t \in [0, 2\pi)$. However,

$$\begin{aligned} g_t(T) &= k^2\langle \phi_f(T)(x + e^{it}y),\ x + e^{it}y\rangle \\ &= p + qe^{it} + re^{-it}\ , \end{aligned}$$

where p, q, r are independent of t and

$$r = k^2\langle \phi_f(T)x, y\rangle = k^2\, \|y\|^2 \neq 0\ .$$

Thus $g_t(T)$ is not independent of $t \in [0, 2\pi)$, and we have obtained a contradiction. This proves the lemma.

LEMMA 5. *$\mathscr{H}_{\mathscr{B}}$ is a C^*-sub-algebra of $\mathscr{A}$.*

Proof. Suppose that $(g, h) \in \mathscr{E}$. Let M_g be the closed subspace of H_g which is generated by $\phi_g(\mathscr{B})x_g$. It follows from Lemma 4 that M_g is invariant under $\phi_g(\mathscr{H}_{\mathscr{B}})$. When $T \in \mathscr{H}_{\mathscr{B}}$, we shall write $\phi_g(T) \mid M_g$ for the operator (from M_g into M_g) obtained by restricting $\phi_g(T)$ to M_g. Similar notations will be used with h in place of g.

Given $T \in \mathscr{H}_{\mathscr{B}}$ and $R, S \in \mathscr{B}$, we have (Lemma 3) $R^*TS \in \mathscr{H}_{\mathscr{B}}$. Since $(g, h) \in \mathscr{E}$, it follows that $g(R^*TS) = h(R^*TS)$, or equivalently that

$$(2) \qquad \langle \phi_g(T)\phi_g(S)x_g,\ \phi_g(R)x_g\rangle = \langle \phi_h(T)\phi_h(S)x_h,\ \phi_h(R)x_h\rangle\ .$$

By taking $T = I$, we deduce the existence of a unitary operator U from M_g on to M_h such that

$$(3) \qquad U\phi_g(S)x_g = \phi_h(S)x_h \qquad (S \in \mathscr{B})\ .$$

Equation (2) then implies that

$$\langle \phi_g(T)v, w\rangle = \langle \phi_h(T)Uv, Uw\rangle \qquad (T \in \mathscr{H}_{\mathscr{B}})$$

for all $v, w \in \phi_g(\mathscr{B})x_g$, hence for all $v, w \in M_g$. The last equation is equivalent to

$$(4) \qquad \phi_g(T) \mid M_g = U^*[\phi_h(T) \mid M_h]U \qquad (T \in \mathscr{H}_{\mathscr{B}})\ .$$

Now suppose that $T_1, T_2 \in \mathscr{H}_{\mathscr{B}}$. Given $(g, h) \in \mathscr{E}$, construct U as above. Since $\phi_g(T_i)$ leaves M_g invariant ($i = 1, 2$), so does $\phi_g(T_1T_2)$, and

$$\phi_g(T_1T_2) \mid M_g = [\phi_g(T_1) \mid M_g][\phi_g(T_2) \mid M_g] ;$$

similar considerations apply with h in place of g. From (4), with $T = T_1, T_2$, we deduce that

$$\phi_g(T_1T_2) \mid M_g = U^*[\phi_h(T_1T_2)\, M_h]U .$$

Since $x_g \in M_g$ and $Ux_g = x_h$, the last equation implies that

$$\langle\phi_g(T_1T_2)x_g,\ x_g\rangle = \langle\phi_h(T_1T_2)x_h,\ x_h\rangle ;$$

that is, $g(T_1T_2) = h(T_1T_2)$. This holds whenever $(g, h) \in \mathscr{E}$, so $T_1T_2 \in \mathscr{H}_{\mathscr{B}}$.

We have now shown that $\mathscr{H}_{\mathscr{B}}$ admits multiplication; since $\mathscr{H}_{\mathscr{B}}$ is clearly a closed self-adjoint linear subspace of $\mathscr{A}$, the lemma is proved.

4. Proof of Theorem 2. We shall use the notations introduced in the statement of Theorem 2. It is immediate from the definition of $\mathscr{H}_{\mathscr{B}}$ that $\mathscr{B} \subseteq \mathscr{H}_{\mathscr{B}}$.

We first consider the case in which $I \in \mathscr{B}$, so that the theory developed in § 3 applies to show that $\mathscr{H}_{\mathscr{B}}$ is a C^*-algebra. We remark that each element f of the pure state space $\mathscr{P}(\mathscr{H}_{\mathscr{B}})$ can be extended to an element $\bar{f}$ of $\mathscr{P}(\mathscr{A})$. For there is a net (f_i) of pure states of $\mathscr{H}_{\mathscr{B}}$, converging to f in the weak * topology. Each f_i can be extended to a pure state $\bar{f}_i$ of $\mathscr{A}$ (see, for example, [2] 304). Since $\mathscr{P}(\mathscr{A})$ is compact, the net $(\bar{f}_i)$ has at least one weak * limit point $\bar{f} \in \mathscr{P}(\mathscr{A})$, and $\bar{f}$ is an extension of f.

Suppose that $\mathscr{B} \neq \mathscr{H}_{\mathscr{B}}$. Then by Glimm's theorem there exist distinct $g, h \in \mathscr{P}(\mathscr{H}_{\mathscr{B}})$ such that $g(B) = h(B)$ ($B \in \mathscr{B}$). We may extend g, h to elements, $\bar{g}, \bar{h}$ respectively of $\mathscr{P}(\mathscr{A})$. Clearly $(\bar{g}, \bar{h}) \in \mathscr{E}$. Thus, by the definition of $\mathscr{H}_{\mathscr{B}}$, $\bar{g}(T) = \bar{h}(T)$ whenever $T \in \mathscr{H}_{\mathscr{B}}$; that is, $g = h$, contrary to hypothesis. This proves Theorem 2 for the case in which $I \in \mathscr{B}$.

If $I \notin \mathscr{B}$, let $\mathscr{B}_1 = \mathscr{B} + \boldsymbol{C}I$ be the C^*-algebra generated by I, $\mathscr{B}$ ($\boldsymbol{C}$ denotes the complex field). With an obvious modification of the notation introduced in Theorem 2, it is clear that $\mathscr{N}(\mathscr{B}_1)$ is empty and that $\mathscr{E}(\mathscr{B}_1) = \mathscr{E}(\mathscr{B})$. Thus $\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{H}_{\mathscr{B}_1}$; since $I \in \mathscr{B}_1$, the first part of this proof shows that $\mathscr{B}_1 = \mathscr{H}_{\mathscr{B}_1}$, so $\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{B}_1$.

Now let f be the pure state of $\mathscr{B}_1$ defined by $f(\lambda I + B) = \lambda$ ($\lambda \in \boldsymbol{C}$, $B \in \mathscr{B}$), and let g be any extension of f to a pure state of $\mathscr{A}$. Clearly $g \in \mathscr{N}(\mathscr{B})$. Hence $g(\mathscr{H}_{\mathscr{B}}) = (0)$, and

$$\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{B}_1 \cap g^{-1}(0) = f^{-1}(0) ;$$

that is, $\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{B}$. The reverse inclusion has already been noted, so $\mathscr{B} = \mathscr{H}_{\mathscr{B}}$.

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SOME TOPOLOGICAL PROPERTIES OF CERTAIN SPACES OF DIFFERENTIABLE HOMEOMORPHISMS OF DISKS AND SPHERES

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Let $D_n = \{x \in E_n : |x| \leqq 1\}$, and $S^n = \{x \in E_{n+1} : |x| = 1\}$. We denote by H_n the space of C^∞ homeomorphisms of D_n onto itself leaving a neighborhood of the boundary fixed. Let K_n be the space of C^∞ orientation preserving homeomorphisms of S^n onto itself. It is not required that maps in the two spaces have differentiable inverses. In both space the C^k topology is used.

The purpose of this paper is to establish the following two theorems:

THEOREM 1. *H_n is contractible to a point for any n.*

THEOREM 2. *K_n is arcwise connected for any n.*

NOTATION. $f(x) = (f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n))$ where $x = (x_1, \cdots, x_n)$, or simply $f(x)$ will denote mappings of E_n into E_n. The shorter form will be used where the meaning is clear.

The topological analog of Theorem 1 is established by a mapping described by Alexander (1923) [1]. Smale (1959) [4] proved the corresponding result for $n = 2$ in the space of diffeomorphisms on D_n leaving a neighborhood of the boundary fixed. Kneser (1926) [3] proved that the space of all orientation preserving homeomorphisms of S^2 onto S^2 has the rotation group as a deformation retract, while Smale gave the corresponding result for the space of orientation preserving diffeomorphisms on S^2 in the paper referred to above. Fisher's work (1960) [2] gives the analog of Theorem 2 in the topological case for $n = 3$.

II. Proof of Theorem 1. Let $m(v)$ be a mapping on I (the unit interval $[0, 1]$) with the following properties:

(a) $m(v) \in C^\infty$;

(b) $m'(v) > 0$ on $\left(0, \frac{3}{4}\right)$;

(c) $m(v) = 1$ on $\left(\frac{3}{4}, 1\right]$;

(d) $m(v) = e^{-(1/r)}$ on $\left(0, \frac{1}{4}\right)$;

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(e) $m(0) = 0$.

Now define $k(v, t) = \begin{cases} 1 - (1 - e^{-(1/t)+1})\ (1 - m(v)) & t \neq 0, \\ m(v) & t = 0, \end{cases}$

on $I \times I$. We see that:

(a′) $k(v, t) \in C^\infty$ on $I \times I$;

(b′) $k(v, t)$ is monotonic in v for each $t \in I$;

(c′) $k(v, t) = 1$ for $v \geqq \frac{3}{4}$ for all $t \in I$;

(d′) $k(v, 1) = 1$ for all $v \in I$;

(e′) $k(v, 0) = m(v)$;

(f′) $0 \leqq k(v, t) \leqq 1$ on $I \times I$.

The mapping

$$x \longrightarrow k(|x|^2, t)x \tag{1}$$

is in H_n for each t. At $t = 1$ the mapping is the identity, while at $t = 0$ the mapping has all partial derivatives of all orders zero at the origin.

The mapping given by Alexander was defined as follows:

$$f_t(x) = \begin{cases} tf\left(\dfrac{x}{t}\right), & t \neq 0 \ (f \text{ extended to be the identity outside } D_n)\,, \\ x, & t = 0. \end{cases}$$

In the C^k topology the mapping of $H_n \times I \longrightarrow H_n$ defined by $(f, t) \longrightarrow f_t$ (the Alexander map) will not be continuous for $k \geqq 1$. In general, $\lim_{t \to 0} f_t \neq f_0$ because at the origin the derivatives of f_t do not converge to the derivatives of the identity mapping. However, by composing the Alexander mapping with (1), we obtain the mapping required in Theorem 1. Thus define

$$h : H_n \times I \longrightarrow H_n$$

by

$$h(f, t) = kf_t$$

where

$$kf_t(x) = k(|f_t(x)|^2, t)\ f_t(x)\,.$$

In particular $h(f, 1) = f$ for all $f \in H_n$, while $h(f, 0)$ is the mapping given by (1). Because of the form of map (1) at the origin, all derivatives of all orders of kf_t approach zero there and the problem mentioned above is removed. The argument that h is continuous is tedious but straightforward.

III. Local straightening of mappings in E_n. The proof of Theorem 2 requires some local straightening procedures for maps in E_n which we now give. For this purpose let L be the space of C^∞ orientation preserving homeomorphisms mapping $U_r = \{x \in E_n : |x| \leqq r\}$ into E_n, leaving the origin fixed and topologized by the C^k topology. We will use $J(f)_p$ to represent the Jacobian matrix of f evaluated at $p \in U_r$, and $|J(f)_p|$ the corresponding determinant.

LEMMA 1. *Suppose $f \in L$ with $J(f)_p = (a_{ij})$, p the origin and (a_{ij}) nonsingular. Then there is a path $f_t \in L$ from f to g, where g agrees with f in a neighborhood of the boundary of U_r and is the linear map with Jacobian (a_{ij}) in a neighborhood of the origin. Also for all t, f_t agrees with f in a neighborhood of the boundary of U_r.*

Proof. Let $\sigma(v)$ be a mapping on $[0, \infty)$ with the following properties:

(a) $\sigma(v) \in C^\infty$;
(b) $\sigma(v) = 1$ on $[0, \alpha)$, $\alpha > 0$;
(c) $\sigma(v) = 0$ for $v \geqq 1$;
(d) $\sigma'(v) \leqq 0$ for $v \in [0, \infty)$.

We see that $|\sigma'(v)| < M$ for some M. Let $c < r$ be chosen so that for $x \in U_c$,

(i) $\left| a_{ij} - \dfrac{\partial f_i(x_1, \cdots, x_n)}{\partial x_j} \right| < \varepsilon, \varepsilon > 0; i = 1, \cdots, n; j = 1, 2, \cdots, n.$

Then for $x \in U_c$,

(ii) $|a_{i1}x_1 + \cdots + a_{in}x_n - f_i(x_1, \cdots, x_n)| < n\varepsilon c$ for $i = 1, 2, \cdots, n$.

Now define

$$f_t(x_1, \cdots, x_n) = \Big(f_1(x_1, \cdots, x_n) + t\sigma\left(\frac{x_1^2 + \cdots + x_n^2}{c^2}\right)(a_{11}x_1 + \cdots + a_{1n}x_n - f_1(x_1, \cdots, x_n)), \cdots, f_n(x_1, \cdots, x_n) + t\sigma\left(\frac{x_1^2 + \cdots + x_n^2}{c^2}\right)(a_{n1}x_1 + \cdots + a_{nn}x_n - f_n(x_1, \cdots, x_n))\Big).$$

At $t = 0$, $f_t = f$; at $t = 1$, f_t is linear with Jacobian (a_{ij}) inside a neighborhood of the origin; for all t, f_t agrees with f outside U_c. The element in the (i, j)th position of $J(f_t)$ differs from a_{ij} by at most

$$\left| a_{ij} - \frac{\partial f_i(x_1, \cdots, x_n)}{\partial x_j} \right|$$

$$+ \left| \frac{2x_j t}{c^2} \sigma' \left(\frac{x_1^2 + \cdots + x_n^2}{c^2} \right) (a_{i1}x_1 + \cdots + a_{in}x_n - f_i(x_1, \cdots, x_n)) \right|$$
$$+ \left| t\sigma \left(\frac{x_1^2 + \cdots + x_n^2}{c^2} \right) \left(a_{ij} - \frac{\partial f_i(x_1, \cdots, x_n)}{\partial x_j} \right) \right| .$$

On U_c, $|x_j| \leqq c$ so that the expression is bounded by $\varepsilon + (2/c) Mn \varepsilon c + \varepsilon = (2 + 2Mn)\varepsilon$. Hence by choosing ε sufficiently small, $|J(f_t)|$ will remain positive on U_c for all t so that f_t will be a homeomorphism on U_r. Continuity of the path f_t in L is immediate from the definition of f_t.

LEMMA 2. *Let* $f(x_1, \cdots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n, a_{22}x_2 + \cdots + a_{2n}x_n, \cdots, a_{nn}x_n) \in L$ *with* $a_{11} \cdots a_{nn} > 0$. *Then there is a path* $f_t \in L$ *such that* $a \leqq t \leqq b$, $f_t(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$ *at* $t = a$, $f_t(x_1, \cdots, x_n) = (a_{11}x_1, \cdots, a_{nn}x_n)$ *in a neighborhood of the origin at* $t = b$, *and* $f_t(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$ *in a neighborhood of the boundary of* U_r *for each* t.

Proof. We construct the path in $n - 1$ arcs as follows. Choose a positive c_1 less than r. Let $k_1 > 1$ be sufficiently large so that whenever

$$x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2 \leqq c_1^2 ,$$

we have $|x_i| < \varepsilon$, $i = 2, 3, \cdots, n$
Now define

$$f_t(x_1, \cdots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n - t\sigma \left(\frac{x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2}{c_1^2} \right) \cdot$$
$$(a_{12}x_2 + \cdots + a_{2n}x_n), a_{22}x_2 + \cdots + a_{2n}x_n, \cdots, a_{nn}x_n) .$$

Then $f_t = f$ when $t = 0$ and f_t at $t = 1$ is the mapping

$$(a_{11}x_1, a_{22}x_2 + \cdots + a_{2n}x_n, \cdots, a_{nn}x_n)$$

in a neighborhood of the origin. For each t, $f_t = f$ outside

$$x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2 = c_1^2$$

so that $f_t = f$ outside U_{c_1}. Also $J(f_t)$ in the $(1, 1)$ position differs from a_{11} by

$$\left| t \frac{2x_1}{c_1^2} \sigma' \left(\frac{x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2}{c_1^2} \right) (a_{12}x_2 + \cdots + a_{1n}x_n) \right| .$$

This expression is zero outside the ellipsoid $x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2 = c_1^2$. Inside this, $|x_1| < c_1$ so if $|a_{ij}| < M_1$, $j = 2, \cdots, n$, and M is a bound on the derivative of $\sigma(v)$, the expression is at most $1 \cdot (2/c_1) \cdot M(n-1) \cdot M_1 \cdot \varepsilon$. This expression is small whenever ε is small (nothing that ε

can be chosen independent of c_1). Thus by choosing ε small, $|J(f_t)|$ will remain positive inside the ellipsoid and f_t will be a homeomorphism for each t.

Thus we assume $f \in L$ and for $c_i > 0$ with $|x| \leqq c_i < r$ the mapping is given by

$$(a_{11}x_1, \cdots, a_{i-1,i-1}x_{i-1}, a_{ii}x_i + a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n, \\ a_{i+1,i+1}x_{i+1} + \cdots + a_{i+1,n}x_n, \cdots, a_{nn}x_n).$$

Let $k_i > 1$ be sufficiently large so that whenever $x_1^2 + \cdots + x_i^2 + k_i^2x_{i+1}^2 + \cdots + k_i^2x_n^2 \leqq c_i^2$, it follows that $|x_j| < \varepsilon$, $j = i+1, \cdots, n$. Define for $x \in U_r$

$$\begin{aligned} f_t(x_1, \cdots, x_n) &= \Big(f_1(x_1, \cdots, x_n), \cdots, f_{i-1}(x_1, \cdots, x_n), f_i(x_1, \cdots, x_n) \\ &\quad - t\sigma\Big(\frac{x_1^2 + \cdots + x_i^2 + k_i^2x_{i+1}^2 + \cdots + k_i^2x_n^2}{c_i^2}\Big)(a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n), \\ &\quad f_{i+1}(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)\Big). \end{aligned}$$

For the proper choice of ε, we can repeat the argument given above.

LEMMA 3. *Suppose $f(x_1, \cdots, x_n) = (a_1x_1, \cdots, a_nx_n) \in L$, $a_i > 0$ for all i. There is a path f_t in L from f to a mapping which is the identity in a neighborhood of the origin, and $f_t = f$ for all t in a neighborhood of the boundary of U_r.*

Proof. First, if $a > 0$ let $p(x)$ be a function on $(-\infty, \infty)$ with:
(a) $p(x) \in C^\infty$;
(b) $p'(x) > 0$ on $(-\infty, \infty)$;
(c) $p(x) = x$ in a neighborhood of the origin;
(d) $p(x) = ax$ outside $(-s+\alpha, s-\alpha)$, $\alpha > 0$.

We again construct the arc in segments. Choose $s_1 < r$ and define

$$f_t(x_1, \cdots, x_n) = \Big(a_1x_1 + t\sigma\Big(\frac{x_2^2 + \cdots + x_n^2}{s_1^2}\Big)(p_1(x_1) - a_1x_1), a_2x_2, \cdots, a_nx_n\Big),$$

where σ is defined in Lemma 1 and $p_1(x_1)$ satisfies properties $(a)-(d)$ above for $s = s_1$. At $t = 0$, $f_t = f$; at $t = 1$ in a neighborhood of the origin f_t is the mapping $(x_1, a_2x_2, \cdots, a_nx_n)$. Also for all $t \in I$, $f_t = f$ outside the cylinder $x_2^2 + \cdots + x_n^2 \leqq s_1^2$, $-s_1 \leqq x_1 \leqq s_1$. $J(f_t)$ in the (1, 1) position is

$$\left[1 - t\sigma\left(\frac{x_2^2 + \cdots + x_n^2}{s_1^2}\right)\right] a_1 + t\sigma\left(\frac{x_2^2 + \cdots + x_n^2}{s_1^2}\right) p_1'(x_1) ,$$

which is positive for all t on the cylinder given above. Hence f_t is a homeomorphism for each t.

Now there is an s_2 with $0 < s_2 < s_1$ so that on the cylinder $x_1^2 + x_3^2 + \cdots + x_n^2 \leqq s_2^2, -s_2 \leqq x_2 \leqq s_2$ the mapping is given by $(x_1 a_2 x_2, \cdots, a_n x_n)$. On this cylinder define

$$f_t(x_1, \cdots, x_n) = \Big(x_1, a_2 x_2 + t\sigma\left(\frac{x_1^2 + x_3^2 + \cdots + x_n^2}{s_2^2}\right) \times (p_2(x_2) - a_2 x_2), a_3 x_3, \cdots, a_n x_n\Big) .$$

Here $p_2(x_2)$ satisfies conditions (a)-(d) given above with $s = s_2$. Repeating the process we complete the desired path.

IV. **Proof of Theorem 2.** The proof now consists of fitting together properly the mappings already constructed.

Let $f \in K_n$. Then there is a point p on S^n so that f has nonsingular Jacobian at that point. Let $(0_1, P_1)$ be a coordinate neighborhood where $0_1 = S^n - p_1(p_1$ antipodal to $p)$ and P_1 an associated stereographic projection. Now there is a path e_t, $t \in I$, in the rotation group on S^n so that e_0 is the identity map, $e_1 f = g$ leaves p fixed and $P_1(e_1 f)P_1^{-1} = P_1 g P_1^{-1}$ has a triangular Jacobian with positive diagonal elements at the origin. Let C be a closed disk on S^n so that for some $r > 0$, $U_r \subset P_1(C)$. Applying Lemmas $1 - 3$ there is a path $(P_1 g P_1^{-1})_t$, $t \in I$, in the space of mappings on U_r from $P_1 g P_1^{-1}$ to a mapping which is the identity in a neighborhood of the origin. Furthermore, for all t, $(P_1 g P_1^{-1})_t$ agrees with $P_1 g P_1^{-1}$ for all $x \in P_1(C)$ except on an interior set of U_r. Define $g_t \in K_n$ by

$$g_t = \begin{cases} P_1^{-1}(P_1 g P_1^{-1})_t P_1 & \text{on } C \\ g & \text{outside } C. \end{cases}$$

Then $g_0 = g$ and g_1 is the identity in a neighborhood of p.

Next let C_1 and C_2 be two closed sets covering S^n where C_1 is a circular disk on S^n with p the center of the disk, and so that C_1 is in an open set left pointwise fixed by g_1. We further assume $p \notin C_2$. Let $(0_2, P_2)$ be a coordinate neighborhood with $C_2 \subset 0_2 = S^n - p$ and P_2 an associated stereographic projection. Then except for a trivial dilation $P_2 g_1 P_2^{-1}$ is an element of the space H_n. By Theorem 1 there is a path $(P_2 g_1 P_2^{-1})_t$ from $P_2 g_1 P_2^{-1}$ to the identity map on $P_2(C_2)$. We now define $h_t \in K_n'$ by

$$h_t = \begin{cases} P_2^{-1}(P_2 g_1 P_2^{-1})_t P_2 & \text{on } C_2 \\ g_1 & \text{outside } C_2. \end{cases}$$

The path from f to the identity map is now complete and Theorem 2 is established.

The spaces H_n and K_n are intermediate spaces to the topological spaces of Alexander and Kneser, and the diffeomorphism spaces treated by Smale. It is interesting to note that methods used in this paper are related to methods used in the larger nondifferentiable spaces and the smaller differomorphism spaces. Alexander's mapping is altered to give Theorem 1, while Theorem 2 parallels Smale's work.

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SOME RESULTS IN THE LOCATION OF ZEROS OF POLYNOMIALS

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Three out of the four theorems proved in this paper deal with the location of the zeros of a polynomial $P(z)$ whose zeros $z_i, i = 1, 2, \cdots, n$ satisfy the conditions $|z_i| \leqq 1$, and $\sum_{i=1}^{n} z_i^p = 0$ for $p = 1, 2, \cdots, l$. One of those estimates is

$$\left| \frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} \right| < \frac{l+1}{|z|(|z|^{l+1} - 1)}$$

for $|z| > 1$.

The fourth result is of a different nature. It refines, in particular, a theorem due to Eneström and Kakeya. It is shown that no zero of the polynomial $h(z) = \sum_{k=0}^{n} b_k z^k$ lies in the disk

$$\left| z - \frac{\beta e^{-i\theta}}{(\beta + 1)} \right| < \frac{1}{\beta + 1},$$

where $\beta = \max_{|z|=1} |h'(z)| / \max_{|z|=1} |h(z)|$, and $\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$.

We generalize and strengthen certain well-known results due to Biernacki [1], Dieudonné [3, 5], and Kakeya [8].

We use repeatedly a recent result due to Walsh which is a generalized form of an earlier theorem of his [10]. It concerns the case in which all the zeros of a polynomial lie within a certain distance of their centroid.

THEOREM 1. *Let $h(z) = \sum_{k=0}^{n} b_k z^k (b_k$ complex$)$,*

$$\beta = \frac{\max_{|z|=1} |h'(z)|}{\max_{|z|=1} |h(z)|},$$

$\max_{|z|=1} |h(z)| = |h(e^{i\theta})|$, and let C_β be the disc $|z - \beta e^{-i\theta}/(\beta + 1)| < 1/(\beta + 1)$, then no zero of h lies in C_β.

Proof. Consider the function $F(z) = e^{-i\varphi} h(ze^{i\theta})/m$, where $h(e^{i\theta}) = me^{i\varphi}$. Then F satisfies the conditions, $|F(z)| < 1$ in $|z| < 1$, $F(1) = 1$. Let $x_n \to 1$ as $n \to \infty$, $0 < x_n < 1$, and let $\alpha = \lim_{n\to\infty} [(1 - |F(x_n)|)/(1 - x_n)]$. Then $\alpha \leqq |F'(1)|$. It follows readily (see [2] p. 57) that

$$\lim_{n\to\infty} [(1 - |F(x_n)|)/(1 - x_n)] = F'(1) = e^{i(\theta - \varphi)} h'(e^{i\theta})/m = |h'(e^{i\theta})|/m .$$

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We apply now the following result due to Julia [2]: If a function f is regular in the unit disc and $|f(z)| < 1$ for $|z| < 1$, and there exists a sequence of number $z_1, \cdots, z_n, \cdots$ such that $\lim_{n\to\infty} z_n = 1$, $\lim_{n\to\infty} f(z_n) = 1$, $\lim_{n\to\infty} [(1 - |f(z_n)|)/(1 - |z_n|)] = \alpha$ then

$$\frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leqq \alpha \frac{|1 - z|^2}{1 - |z|^2} \qquad \text{for } |z| < 1 . \tag{1}$$

In (1), set $f(z) = F(z)$, $\alpha = |h'(e^{i\theta})|/m$. If $F(z_0) = 0$ and $|z_0| < 1$, then $(1 - |z_0|^2)/|1 - z_0|^2 \leqq \alpha$, which is equivalent to $e^{-i\theta} z_0 \notin C_\alpha$. Since $\alpha \leqq \beta$, it follows that $C_\beta \subset C_\alpha$; hence $e^{-i\theta} z_0 \notin C_\beta$, which concludes the proof.

COROLLARY 1. *Let* $h(z) = \sum_{k=0}^n b_k z^k$, $b_k > 0$. *Then* $\beta = \sum_{k=1}^n k b_k / \sum_{k=0}^n b_k$, *and no zero is in the disc*

$$\left| z - \frac{\sum_{k=0}^n k b_k}{\sum_{k=0}^n (k+1) b_k} \right| < \frac{\sum_{k=0}^n b_k}{\sum_{k=0}^n (k+1) b_k} .$$

In particular, if b_k *is a strictly increasing sequence, then all the zeros of* $h(z)$ *lie in the complement of* C_β *with respect to the unit disc. This makes more precise the theorem of Eneström and Kakeya* [8].

In a recent paper, Tchakaloff [9] (see also [7]) has proved that if all the zeros of the polynomials

$$P_k(z) = a_n^{(k)} z^n + \cdots + a_0^{(k)} (a_n^{(k)} > 0, k = 1, \cdots, m) \tag{2}$$

lie in the unit disc and if $A_k > 0 (k = 1, \cdots, m)$, then all the zeros of the polynomial $\sum_{k=1}^m A_k P_k(z)$ lie in the disc $|z| \leqq 1/\sin(\pi/2n)$, and that this is the best possible result. We prove a more precise result in the case where there is more information about the zeros of $P_k(z)$.

THEOREM 2. *Let the polynomials* $P_k(z)(k = 1, \cdots, m)$ *of the form* (2) *have all their zeros* $z_{ik}(i = 1, \cdots, n; k = 1, \cdots, m)$ *in the unit disc and let* $A_k > 0(k = 1, \cdots, m)$. *Suppose that* $\sum_{i=1}^n z_{ik}^p = 0$ *for* $p = 1, \cdots, l(k = 1, \cdots, m)$. *Then all the zeros of the polynomial* $\sum_{k=1}^m A_k P_k(z)$ *lie in the disc* $|z| \leqq (\sin \pi/2n)^{-1/(l+1)}$. *For values of the form* $n = (l+1)r$, *the exact bound does not exceed* $(\sin(\pi(l+1))/2n))^{-1/(l+1)}$.

Proof. Without loss of generality we may assume that $a_n^{(k)} = 1$. By a recent result due to Walsh [11] the polynomials P_k satisfy the equality $P_k(z) = (z - \varphi_k(z))^n$, where $|\varphi_k(z)| < |z|^{-l}$ for $|z| > 1$. Let ζ be a point outside the unit disc at which the circle $|z| = |\zeta|^{-l}$

subtends an angle Ψ. On the circle $|z| = |\zeta|^{-l}$ there exists a point a, such that $0 \leqq \arg((\zeta - \varphi_k)/(\zeta - a)) \leqq \Psi$, and

$$\sum_{k=1}^{m} A_k P_k(\zeta) = (\zeta - a)^n \sum_{k=1}^{m} A_k \left(\frac{\zeta - \varphi_k}{\zeta - a}\right)^n . \tag{3}$$

One deduces from equation (3) that

$$\sum_{k=1}^{m} A_k P_k(\zeta) \neq 0 \text{ if } \Psi < \frac{\pi}{n} .$$

For $\Psi = \pi/n$, $\sin(\pi/2n) = |\zeta|^{-(l+1)}$. This proves the first part of the theorem. The example $A_1 = A_2 = 1$, $m = 2$, $P_1(z) = (z^{l+1} + \mu)^r$, $P_2(z) = (z^{l+1} + \bar{\mu})^r$, where $\mu = i\exp(i\pi/2n)$, proves the second part of the theorem, since in this case the polynomial $P_1(z) + P_2(z)$ has the zero

$$z = \left[\sin\frac{\pi(l+1)}{2n}\right]^{-1/(l+1)} .$$

Dieudonné has proved [3], (for a different proof see [4]), that if the polynomial P has all its zeros in the closed unit disc, then

$$\left|\frac{P'(z)}{P(z)} - \frac{P''(z)}{P'(z)}\right| \leqq \frac{1}{|z| - 1} , \qquad \text{for } |z| > 1 . \tag{4}$$

We give a short proof of (4), which at the same time yields a stronger inequality in the case where the centroid of the zeros of P is at the origin.

THEOREM 3. *If all the zeros $z_i (i = 1, \cdots, n)$ of the polynomial $P(z)$ lie in the closed unit disc and if $\sum_{i=1}^{n} z_i^k = 0 (k = 1, \cdots, l)$, then for $|z| > 1$ the following sharp estimate holds*

$$\left|\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z}\right| \leqq \frac{l+1}{|z|(|z|^{l+1} - 1)} . \tag{5}$$

Inequality (5) holds also for $l = 0$, in which case the second condition imposed on the z_i is to be omitted.

Proof. By a recent result due to Walsh [12], there exists a function $\varphi(z)$, $|\varphi(z)| < |z|^{-l}$, such that for $|z| > 1$

$$\frac{P'(z)}{P(z)} = \frac{n}{z - \varphi(z)} . \tag{6}$$

An estimate due to Goluzin [6], applied to φ yields the inequality

$$|\varphi'(z)| \leqq \frac{l|z|^{l-1}}{|z|^{2l} - 1}(1 - |\varphi(z)|^2) , \tag{7}$$

for $|z| > 1$. Since by (6)

$$\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z} = \frac{\varphi(z) - z\varphi'(z)}{z(z - \varphi(z))} \tag{8}$$

is follows, using (7), that

$$\left|\frac{P''(z)}{P'(z)} - \frac{P'(z)}{P(z)} - \frac{1}{z}\right| \leqq \frac{1}{|z|}\left[\frac{|\varphi(z)|}{|z| - |\varphi(z)|} + \frac{l|z|^l}{|z|^{2l} - 1}\frac{1 - |\varphi(z)|^2}{|z| - |\varphi(z)|}\right]$$

It remains to prove the inequality

$$\frac{x}{a - x} + \frac{la^l}{a^{2l} - 1}\frac{1 - x^2}{a - x} \leqq \frac{l + 1}{a^{l+1} - 1} \tag{9}$$

for all $0 \leqq x \leqq a^{-l}$, and $a > 1$.

If we denote the left hand side of (9) by $f(x)$, then $f(a^{-l}) = (l + 1)/(a^{l+1} - 1)$, and $f'(x) \geqq 0$ provided the function $g(x) = a^{2l+1} - a + la^l(x^2 - 2ax + 1)$ is nonnegative. Since $g'(x) \leqq 0$ it is enough to show that $h(a) = g(a^{-l})$ is nonnegative. Indeed one verifies that $h(1) = 0$ and $h'(a) > 0$ for all $a > 1$.

The particular case $P(z) = z^n - 1$, $l = n - 1$, shows that the bound (5) cannot, in general, be improved.

The result due to Dieudonné follows from (7) and (8).

Finally, we discuss a problem raised by Biernacki [1], which was also treated by Dieudonné [5], namely that of determining a region containing all but, possibly, one zero of the polynomial $aP(z) + P'(z)$ for all complex a. Each of the above authors has proved that if all the zeros of P lie in the unit disc, then the concentric disc of radius $2^{1/2}$ is the smallest concentric disc that has the above mentioned proporty. Assuming additional information about the zeros of P, we obtain a smaller disc for all but possibly $l + 1$ zeros of the polynomial $z^lP(z) + aP'(z)$.

THEOREM 4. *If all the zeros $z_i (i = 1, \cdots, n)$ of the polynomial $P(z)$ lie in the closed unit disc and if $\sum_{i=1}^n z_i^k = 0 (k = 1, \cdots, l)$, then for all complex a at least $n - 1$ zeros of the polynomial $z^lP(z) + aP'(z)$ lie in the disc $|z| \leqq 2^{1/(2(l+1))}$.*

Proof. Proceeding as in the proof of Theorem 3, we have

$$\frac{P'(z)}{P(z)} = -\frac{z^l}{a} = \frac{n}{z - \varphi(z)},$$

satisfied by any zero of the polynomial $z^l P + aP'$ which exceeds 1 in modulus. Set $g(z) = z^{-l}\varphi(1/z)$, $w = z^{l+1}$ and $h(w) = g(z)$. Then $|g(z)| < 1$ if $|z| < 1$ and

$$g(z) = \frac{1}{z^{l+1}} + an \tag{10}$$

$$h(w) = \frac{1}{w} + an \,. \tag{11}$$

If for some a the polynomial $z^l P + aP'$ has at most $n - 2$ zeros in the disc $|z| \leqq 2^{1/(2(l+1))}$, then equation (10) has at least $l + 2$ roots in the disc $|z| < 2^{-1/(2(l+1))}$, and hence equation (11) has at least two roots in the disc $|w| < 2^{-1/2}$. This was proved to be impossible in [5]

Theorem 4 is sharp for all l and n of the form $n = 2k(l + 1)$, $k = 1, 2, \cdots$. The upper limit is attained by the zeros of the polynomial

$$P(z) = (z^{2l+2} - 2^{1/2}z^{l+1} + 1)^{n/(2(l+1))} \,.$$

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ON SIMPLE ALGEBRAS OBTAINED FROM HOMOGENEOUS GENERAL LIE TRIPLE SYSTEMS

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We continue the investigation of the simple anti-commutative algebras obtained from a homogeneous general L.t.s. In particular we consider the algebra which satisfies

$$(1) \qquad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) .$$

The usual process of analyzing a nonassociative algebra is to decompose it relative to elements whose right and left multiplications are diagonalizable linear transformations e.g. idempotents or Cartan subalgebras. In this paper we show that such a process yields only Lie algebras and indicates the difficulty in finding any non-Lie multiplication table for a simple anticommutative algebra satisfying (1).

A general Lie triple system [2] is an extension of a Lie triple system used in differential geometry and Jordan algebras. A general L.t.s. may be regarded as an anti-commutative algebra A with a trilinear operation $[x, y, z]$ so that the mappings $D(x, y): z \rightarrow [x, y, z]$ are derivations of A which generate a Lie algebra, $I(A)$, under commutation satisfying certain natural identities. A homogeneous general L.t.s. is a general L.t.s. for which the operation $[x, y, z]$ is a homogeneous expression in the products of x, y and z; that is, using anti-commutativity, $[x, y, z] = \alpha xy \cdot z + \beta yz \cdot x + \gamma zx \cdot y$ for some fixed α, β, γ in the base field. From [1] we see that if A is a homogeneous general L.t.s. over a field of characteristic zero which is either an irreducible general L.t.s. or $I(A)$-irreducible or a simple algebra, then A is a Lie or Malcev algebra or satisfies

$$(1) \qquad J(x, y, z)w = J(w, x, yz) + J(w, y, zx) + J(w, z, xy)$$

where $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$. The main result of this paper is the following theorem.

THEOREM. *If A is a simple finite dimensional anti-commutative algebra over a field F of characteristic zero which satisfies* (1) *and if A contains a nonzero element u so that right multiplication by u, R_u, is a diagonalizable linear transformation, then A is a Lie algebra.*

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2. Proof of theorem. For any anti-commutative algebra we have the identity

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned}$$

But using (1) we also have

$$\begin{aligned} wJ(x, y, z) - xJ(y, z, w) + yJ(z, w, x) - zJ(w, x, y) \\ = -2[J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \\ + J(wx, y, z) + J(wy, z, x) + J(wz, x, y)] . \end{aligned}$$

Thus using the two preceding identities we have

$$\begin{aligned} J(w, xy, z) + J(w, yz, x) + J(w, zx, y) \\ = J(wx, y, z) + J(wy, z, x) + J(wz, x, y) . \end{aligned} \tag{2}$$

Now let $u \neq 0$ be an element of A so that $R_u: x \rightarrow xu$ is a diagonalizable linear transformation. Then $R_u \neq 0$, for this implies that the one dimensional subspace uF is an ideal of A and therefore equals A. Thus $A^2 = 0$, a contradiction to the simplicity of A. Since R_u acts diagonally in A we may write

$$A = A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$$

where

$$A_\lambda = \{x \in A : x(R_u - \lambda I) = 0\} .$$

We shall now prove

$$A_\alpha A_\beta \subset A_{\alpha+\beta} . \tag{3}$$

For let $x \in A_\alpha$, $y \in A_\beta$, then from (1)

$$\begin{aligned} J(u, x, y)R_u &= J(u, u, xy) + J(u, x, yu) + J(u, y, ux) \\ &= \beta J(u, x, y) - \alpha J(u, y, x) \\ &= (\alpha + \beta)J(u, x, y) . \end{aligned}$$

Thus $J(u, x, y) \in A_{\alpha+\beta}$ and therefore

$$xy(R_u - (\alpha + \beta)I) = xy \cdot u + yu \cdot x + ux \cdot y \in A_{\alpha+\beta} .$$

From this $xy(R_u - (\alpha + \beta)I)^2 = 0$ and setting $xy = \Sigma z_\gamma \in A_0 \oplus \sum_{\alpha \neq 0} A_\alpha$ we see by the diagonal action of R_u that $xy \in A_{\alpha+\beta}$. In particular (3) shows A_0 is a subalgebra of A.

Next we shall show

$$(4)\qquad J(A_\alpha, A_\beta, A_\gamma) = 0 \quad\text{or}\quad \alpha + \beta + \gamma = 0$$

for any characteristic roots α, β, γ of R_u. Let $x \in A_\alpha, y \in A_\beta, z \in A_\gamma$, then from (3) $J(x, y, z) \in A_{\alpha+\beta+\gamma}$ and therefore

$$\begin{aligned}(\alpha + \beta + \gamma)J(x, y, z) &= J(x, y, z)R_u \\ &= J(u, x, yz) + J(u, y, zx) + J(u, z, xy) \\ &= -\alpha x \cdot yz + (\alpha + \beta + \gamma)x \cdot yz + (\beta + \gamma)yz \cdot x \\ &\quad - \beta y \cdot zx + (\alpha + \beta + \gamma)y \cdot zx + (\alpha + \gamma)zx \cdot y \\ &\quad - \gamma z \cdot xy + (\alpha + \beta + \gamma)z \cdot xy + (\alpha + \beta)xy \cdot z \\ &= 0 .\end{aligned}$$

and this equation proves (4).

from (1) and (3) we have

$$J(A_0, A_0, A_0)A_0 \subset J(A_0, A_0, A_0)$$

and for $\alpha \neq 0$ we have from (1), (3) and (4),

$$\begin{aligned}J(A_0, A_0, A_0)A_\alpha &\subset J(A_\alpha, A_0, A_0) \\ &= 0 .\end{aligned}$$

Thus $J(A_0, A_0, A_0)A \subset J(A_0, A_0, A_0)$ and therefore $J(A_0, A_0, A_0)$ is an ideal of A whicn is contained in $A_0 \neq A$. Since A is a simple algebra this yields

$$(5)\qquad J(A_0, A_0, A_0) = 0 .$$

Next we shall prove that if α is a nonzero characteristic root so that $-\alpha$ is also a characteristic root, then

$$(6)\qquad J(A_\alpha, A_{-\alpha}, A_0) = 0 .$$

For using (1), (3) and (5) we obtain

$$J(A_\alpha, A_{-\alpha}, A_0)A_0 \subset J(A_\alpha, A_{-\alpha}, A_0)$$

and for any $\beta \neq 0$ we also obtain

$$\begin{aligned}J(A_\alpha, A_{-\alpha}, A_0)A_\beta &\subset J(A_\beta, A_\alpha, A_{-\alpha}A_0) \\ &\quad + J(A_\beta, A_{-\alpha}, A_0A_\alpha) \\ &\quad + J(A_\beta, A_0, A_\alpha A_{-\alpha}) \\ &\subset J(A_\beta, A_\alpha, A_{-\alpha}) + J(A_\beta, A_0, A_0) \\ &= 0 ,\end{aligned}$$

also using (4). Thus as in the proof of (5), $J(A_\alpha, A_{-\alpha}, A_0)$ is an ideal of A which must be zero. Adopting the usual convention that if α is a characteristic root but $-\alpha$ is not, then $A_{-\alpha} = 0$ we see that (6) holds for any characteristic root α.

Next let

$$B = \sum_{\alpha\neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha\neq 0} A_\alpha ,$$

then if $\beta \neq 0$ we see from (3) that $BA_\beta \subset B$. If $\beta = 0$, then from (6) we obtain $(A_\alpha A_{-\alpha})A_0 \subset A_\alpha A_{-\alpha}$ and therefore $BA_0 \subset B$. Thus B is an ideal of A and therefore $B = 0$ or $B = A$. If $B = 0$, then $R_u = 0$, a contradiction. Therefore we have

$$(7) \qquad A = \sum_{\alpha\neq 0} A_\alpha A_{-\alpha} \oplus \sum_{\alpha\neq 0} A_\alpha .$$

Now from (4) and (6) we have for any characteristic roots β and $\alpha \neq 0$, $J(A_\alpha, A_{-\alpha}, A_\beta) = 0$ and therefore

$$(8) \qquad J(A_\alpha, A_{-\alpha}, A) = 0 \qquad (\alpha \neq 0) .$$

We shall use (7) and (8) together with the following lemma to prove A is a Lie algebra.

LEMMA. *Let* $N = \{x \in A : J(x, A, A) = 0\}$, *then*
(i) $J(a, b, A) = 0$ *implies* $ab \in N$;
(ii) N *is an ideal of* A *which is a Lie algebra.*

Proof. Clearly (ii) follows from (i). So let $a, b \in A$ be such that $J(a, b, A) = 0$ and let $w, z \in A$. Then from (1) and (2) we have

$$(9) \qquad \begin{aligned} 0 &= wJ(a, b, z) \\ &= J(w, ab, z) + J(w, bz, a) + J(w, za, b) \\ &= J(wa, b, z) + J(wb, z, a), \text{ using (2)} . \end{aligned}$$

Now interchanging z and w in this last equation we obtain $0 = J(za, b, w) + J(zb, w, a) = J(w, bz, a) + J(w, za, b)$ and using this in (9) yields $J(ab, w, z) = 0$; that is, $ab \in N$.

To show that A is a Lie algebra, suppose it is not. Then from the lemma $N = 0$ and from (8) $A_\alpha A_{-\alpha} \subset N = 0$. Thus from (7) $A = \sum_{\alpha\neq 0} A_\alpha$ and therefore $A_0 = 0$; this contradicts $0 \neq u \in A_0$.

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ON SMALL MAPS OF MANIFOLDS

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A result announced by R. F. Brown in 1963, and completed by Brown and Fadell, generalizing classical results of H. Hopf for differentiable manifolds, is the following:

THEOREM: Let M be a compact connected topological manifold; then

(a) M admits arbitrarily small maps with a single fixed point;

(b) If the Euler characteristic χ_M of M is zero, then M admits arbitrarily small maps without fixed points (and conversely). Here a map is small if it is close to the identity map. We propose to give a short proof of this theorem.

We will use the recent result of J. Kister (also Mazur and Stallings) that any microbundle over a complex is a bundle [4]. We note that according to [2] the result (b) holds also for manifolds with boundary.

2. Characteristic class. We consider the tangent microbundle $\tau_M: M \xrightarrow{d} M \times M \xrightarrow{p_1}$; here d is the diagonal map, and p_1 the first projection (cf. [5]). Attached to τ_M is the Thom class u, a well-defined element of $H^n(M \times M, M \times M - d(M))$ (here $n = \dim M$); the coefficients used are the integers **Z**, if M is orientable, and twisted integers, determined by the orientations of the horizontal factor M at the points of $M \times M$, in the nonorientable case. (Cf. [6] for details in the orientable case.) We write $\tilde{u}$ for the image of u in the absolute group $H^n(M \times M)$; the Euler class e_M is the image of $\tilde{u}$ in $H^n(M)$ under the diagonal map d^* (twisted coefficients in the nonorientable case). Furthermore, M has a fundamental cycle μ (again twisted coefficients for nonorientable M). It is a well-known fact that the value $\langle e_M, \mu\rangle$ of e_M on μ equals the Euler-Poincaré characteristic χ_M of M.

[Since this is not easy to find in the literature, we sketch a proof: First assume M orientable. Let $\{x_i\}$ be a basis for $H^*(M)$ modulo torsion, and let $\{\alpha_i\}$ be the basis of $H_*(M)$ modulo torsion, dual to $\{x_i\}$ under $\langle\ ,\ \rangle$; put $r_i = \dim \alpha_i$. Define $\{x'_j\}$ by $\mathfrak{d}x'_i = \alpha_i$, where $\mathfrak{d}$ is the Poincaré duality operator $\mathfrak{d}x = x \cap \mu$; then $\{x'_j\}$ is again a basis for $H^*(M)$ modulo torsion. Finally let $\{\alpha'_j\}$ be dual to $\{x'_j\}$ under $\langle\ ,\ \rangle$. One verifies that $d_*\mu = \Sigma\alpha_i \times \alpha'_i$ modulo torsion (use $\langle x \times y, d_*\mu\rangle = \langle x \cup y, \mu\rangle$). Now $\tilde{u}$ satisfies the relation $\langle x, \alpha\rangle = (-1)^{n-r}\langle\tilde{u}, \mathfrak{d}x \times \alpha\rangle$ for $x \in H^r(M)$ (cf. [6]). Therefore we have $\langle e_M, \mu\rangle = \langle\tilde{u}, d_*\mu\rangle =$

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$\langle \tilde{u}, \Sigma\alpha_i \times \alpha_i' \rangle = \Sigma(-1)^{r_i}\langle x_i', \alpha_i' \rangle = \Sigma(-1)^{r_i} = \chi_M$. For nonorientable M let $\hat{M}$ be the orientable double covering, and use the facts that the Thom class is preserved under the covering map, that the fundamental cycle of $\hat{M}$ maps onto twice the (twisted) fundamental cycle of M, and that $\chi_{\hat{M}} = 2\chi_M$ (as one can see, e.g., from the Smith sequence).]

In particular, if $\chi_M = 0$, then also the Euler class e_M vanishes. Furthermore, in all this discussion we may, by Kister's result, replace the tangent microbundle by an actual bundle (in the local product sense) whose fibre is $\mathbf{R}^n$ with a well-defined origin and which therefore has a well-defined 0-section. We denote this bundle by $\bar{\tau}_M$.

3. **Proof of theorem.** We begin with part (b); thus assume $\chi_M = 0$. Embed M in a number space $\mathbf{R}^k$ with $k \geqq 2n + 1$, and let V be a (closed) polyhedral neighborhood of M that retracts onto M, via the map r. We consider the bundle $r^*\bar{\tau}_M$, induced from the bundle $\bar{\tau}_M$ (see end of § 2) by r. By naturality the Euler class of $r^*\bar{\tau}_M$ vanishes. Therefore, if K is any polyhedron of dimension $\leqq n$ contained in V, the restriction of $r^*\bar{\tau}_M$ to K admits a nonvanishing section (i.e., one that does not meet the 0-section of $r^*\bar{\tau}_M$); to prove this one uses the interpretation of the Euler class as obstruction. Let $\mathscr{A}$ be a finite, open covering of M, of dimension n, such that (a) the nerve $N_{\mathscr{A}}$ can be realized in V and (b) an associated barycentric map $f: M \rightarrow N_{\mathscr{A}}$ (cf. [3], p. 69) is homotopic to the identity 1_M of M in V; this exists of course. Let s be a nonvanishing section of $r^*\bar{\tau}_M \mid N_{\mathscr{A}}$. Applying the covering homotopy theorem to the map $s \circ f$ of M into the bundle formed by the complement of the 0-section of $r^*\bar{\tau}_M$ and to the homotopy between f and 1_M, one gets a nonvanishing section of $r^*\bar{\tau}_M \mid M$, i.e. of $\bar{\tau}_M$. This section amounts of course to a fixed-point-free map of M into itself. Again according to Kister, $\bar{\tau}_M$ can be assumed to lie in any preassigned neighborhood of the diagonal of $M \times M$, which means that the map can be constructed as close to the identity as one pleases.

The converse is classical (Lefschetz fixed point theorem).

4. **Proof of theorem continued.** We come to part (a). As before we imbed M in a Euclidean space $\mathbf{R}^k$, and r is a retraction of some neighborhood of M onto M. Let A be a coordinate system in M (i.e., an open subset homeomorphic to $\boldsymbol{R}^n$), and let B, respectively C, be the subsets of A corresponding to the set of points in $\boldsymbol{R}^n$ of norm <1, respectively $<\frac{1}{2}$. There exists a polyhedral neighborhood W of $M - B$ in $\boldsymbol{R}^k$, whose r-image lies in $M - C$. Since $H^n(M - C)$ (twisted coefficients if needed) vanishes ($M - C$ being a manifold with

nonempty boundary), the characteristic class of $r^*\bar{\tau}_M \mid W$ is zero. By the same argument as before, the bundle $\bar{\tau}_M \mid M - B$ has a nonvanishing section, which can be interpreted as a map f of $M - B$ into M, without fixed points. We may assume that the f-image of the bounday of $M - B$ lies in A (by taking $\bar{\tau}_M$ small enough), and it is then clear, using $A \approx \boldsymbol{R}^n$, how to extend f to a map of M into itself whose only fixed point is the point of A corresponding to the origin of $\boldsymbol{R}^n$.

If f is homotopic to the identity map of M (as it will be if it is small enough: apply r to the linear homotopy in $\boldsymbol{R}^k$), then the index of the fixed point is χ_M: the index equals $\pm$ the intersection number of the graph of f in $M \times M$ and the diagonal, and it is well known that this is χ_M under the present circumstances. In fact, this last remark yields another version of the proof of (a): if $\chi_M = 0$, one can extend f over B without any fixed point.

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$|\varepsilon(z)|$-CLOSENESS OF APPROXIMATION

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For a given function $F(Q)$ defined for $Q \in S$, the connection between these questions is investigated: (1) For arbitrary $\varepsilon > 0$ (or possibly $\{\varepsilon_i\}$, where ε_i corresponds to a compoment S_i of S), does there exist a function f of a specified class $\mathscr{F}$ such that $\sup_{Q \in S} |F(Q) - f(Q)| < \varepsilon$ on S (or ε_i on S_i)?; (2) Given an admissible function $\varepsilon(Q)$, does there exist a function $f \in \mathscr{F}$ such that $|F(Q) - f(Q)| \leqq |\varepsilon(Q)|$ on S? A continuous *function* $\varepsilon(Q)$ defined on S is *admissible* if for each zero Q_β there is a positive integer n_β such that $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero in a deleted neighborhood of Q_β. A typical result is: Corresponding to any $F(z)$ analytic on a closed bounded set S and to any admissible $\varepsilon(z)$, there exists a rational function $r(z)$ with its poles on a certain preassigned set such that $|F(z) - r(z)| \leqq |\varepsilon(z)|$ on S.

When the sup-topology is used in approximating a given function F defined on a set S by a function f in a certain class $\mathscr{F}$, it is required that, for arbitrary $\varepsilon > 0$, there exists $f \in \mathscr{F}$ such that

$$\sup |F(X) - f(X)| < \varepsilon \text{ for } X \in S .$$

In this paper the connection is investigated between existence of such an approximating function and existence of an approximating $g \in \mathscr{F}$ when for any admissible function $\varepsilon(X)$ it is required $|F(X) - g(X)| \leqq |\varepsilon(X)|$ when $X \in S$.

The latter formulation has the advantage of automatically specifying that, at any zero X_0 of $\varepsilon(X)$ on S, $g(X_0) = F(X_0)$ and at multiple zeros corresponding derivatives of F and g agree, provided F has derivatives at these points. One interesting application, in case F is continuous and is well-behaved near zeros, is that in which

$$|F(X) - f(X)| \leqq p\,|F(X)|$$

is required, where p denotes a preassigned per cent.

Approximation in the real case in which a neighborhood N_{ξ_1,ξ_2} of F consists of those f such that $\xi_1(x) \leqq F(x) - f(x) \leqq \xi_2(x)$ has been suggested by P.C. Hammer.[1] If $[\xi_2(x) - \xi_1(x)]/2$ is an "admissible" $\varepsilon(x)$, the problem reduces to the $|\varepsilon(x)|$-closeness of approximation

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[1] S.I.A.M. Conference on Approximation Theory, Gatlinburg, Tennessee, 1963.

considered in this paper. For $\xi_1(x) \leqq F(x) - f(x) \leqq \xi_2(x)$ if and only if

$$- [\xi_2(x) - \xi_1(x)]/2 \leqq F(x) - [\xi_1(x) + \xi_2(x)]/2$$
$$- f(x) \leqq [\xi_2((x) - \xi_1(x)]/2$$

This paper is perhaps of most interest in connection with approximation in the complex plane. However, as the Weierstrass-factor Theorem, Mittag-Leffler Theorem, and Runge Theorem [2] upon which the results depend, hold also on the open Riemann surface, the theorems are stated in abstract form for the open Riemann surface: then certain specializations to the complex plane are given in the corollaries.

As is customary, "open" Riemann surface denotes a noncompact Riemann surface [1]. A point on a Riemann surface is denoted by Q, a point in the complex plane, in particular, by z, and a point on the real axis by x. For the sake of clarity the notation $f(Q)$ is frequently used to denote the function f.

When it is specified a function has *poles coinciding with* those of another function, it is to be understood that they have identical principal parts; likewise, if a function has *zeros coinciding with* those of a second function, the order of the respective zeros is the same.

For reference we state:

HYPOTHESIS H. *Suppose that S is a closed set on the open Riemann surface $\mathfrak{R}$, Let B^* consist of precisely one point of each of those components of $\mathfrak{R} - S$ whose closure is compact.*

Theorem 1 includes the case that S is compact with no interior points. For example, if $\mathfrak{R}$ is the finite complex plane, S may be a bounded closed interval on the real axis; in fact, S may be any closed bounded set with or without interior points.

THEOREM 1. *Assume Hypothesis H and suppose a function $\varepsilon(Q)$ ($\neq 0$) defined on S. Let R be an open set (which may be $\mathfrak{R}$) such that $S \subset R \subset \mathfrak{R}$ and suppose $\mathscr{S}$ is a collection of functions meromorphic on R, analytic on $R - B^*$. Then these approximation requirements* (1) *and* (2) *are equivalent.*

(1) *Corresponding to any function $M(Q)$ analytic on S^0 (the interior of S) and continuous on S, there exists $k \in \mathscr{S}$ such that $| M(Q) - k(Q) | \leqq | \varepsilon(Q) |$ when $Q \in S$.*

(2) *Corresponding to any function $m(Q)$ meromorphic on S^0 and continuous on S except at poles, there exists $f = h + k$, where $k \in \mathscr{S}$ and h is meromorphic on $\mathfrak{R}$ with its only poles coinciding with those of m on S, such that $| m(Q) - f(Q) | \leqq | \varepsilon(Q) |$ on S.*

Proof. Clearly, (2) includes (1). We proceed to prove (1) implies (2).

The set of points at which m has poles on S is an isolated set on $\mathfrak{R}$. Hence, according to the Mittag-Leffler partial fractions theorem [2, p. 591; 7] there exists a function h meromorphic on $\mathfrak{R}$ whose only poles coincide with those of m on S and have the same principal parts. (We note that, if m has only a finite number of poles on S and if $\mathfrak{R}$ is the finite complex plane, then h may be required to be a rational function.)

The function $m - h$ is analytic on S^0 and continuous on S. Hence, by the conclusion in (1), there is a function $k \in \mathscr{S}$, such that

$$|[m(Q) - h(Q)] - k(Q)| \leqq |\varepsilon(Q)|$$

when $Q \in S$, that is,

$$|m(Q) - [h(Q) + k(Q)]| \leqq |\varepsilon(Q)|$$

on S.

Thus, $h + k$, which is meromorphic on R and analytic on $R - B^*$ except for poles on S coinciding with those of m, is a function f as required.

COROLLARY 1.1. *The theorem is true if in*
(1) *$M(Q)$ is assumed analytic on S and in*
(2) *$m(Q)$ is assumed meromorphic on S.*

COROLLARY 1.2. *For $\mathfrak{R}$ the finite complex plane and S a compact set on $\mathfrak{R}$, the theorem is true if in*
(1) *k is required to be a rational function and in*
(2) *f is required to be a rational function.*

H. J. Landau [5] proved: If on the complex plane, S is a closed bounded set with no interior and if there exist cutting sets of S whose closures have arbitrarily small measure, then any function continuous on S may be uniformly approximated on S by a rational function whose poles lie in $B^* \cup \infty$. It follows from Corollary 1.2 that, if m is continuous on such a set S except for a finite number of poles, $m(z)$ can be uniformly approximated by a rational function whose poles lie in $B^* \cup \infty$ and at the poles of m on S.

By the Carleman approximation theorem [3; 4] if $w(x)$ is continuous on the real axis, then corresponding to any $\{\varepsilon_i\}$, there exists an entire function f such that $|w(x) - f(x)| < \varepsilon_i$ when $i - 1 < |x| \leqq i$, $i = 1, 2, \cdots$. Hence, Theorem 1 implies that, if $w(x)$ is continuous on the finite real axis except for a finite or a denumerable number of poles with limit point at ∞, then $w(x)$ can be approximated in the above sense by a meromorphic function f whose poles lie on the real axis and coincide with those of w. According to an extension by the author [8, Theorem 3] of the Carleman Theorem, if S consists of the union of closed circular disks S_i tangent externally on the real axis and extending to infinity and if w is analytic at interior points of S, continuous on S, then, corresponding to any $\{\varepsilon_i\}$, there exists an entire function f such that $| w(z) = f(z) | < \varepsilon_i$ on $S_i, i = 1, 2, \cdots$. By Theorem 1, w may be allowed poles on S^0 provided the approximating function f is allowed coincident poles.

An analogue of the type of generalization given in Theorem 1 for a Q-set has previously been used by the author [8; 9].

A *sequential limit point* of a set S is a limit point of a set of points chosen one from each component of S. A set S in the extended complex plane whose components $S_1, S_2, \cdots$, are compact and whose set of sequential limit points $B \subset \mathscr{C}(S)$ is called a Q-set [9]. We require, in addition, that a *Q-set on* an *open Riemann surface* $\mathfrak{R}$ be a closed set, that is, $\mathfrak{R}$ contains no sequential limit point of S. When in the complex domain $\mathfrak{R}$ is chosen as the extended plane minus B, the set of sequential limit points of S, a Q-set is closed.

A *function* $\varepsilon(Q)$ defined for $Q \in S$ is *admissible on* S if

(1) It is continuous on S;

(2) Corresponding to each of its zeros Q_β on S, there is a positive integer n_β such that $\varepsilon(Q)/(Q - Q_\beta)^{m_\beta}$ is bounded from zero in a neighborhood $N_{Q_\beta} \subset S$. The smallest positive integer n_β satisfying the condition in (2) is called the *order of* the *zero* of $\varepsilon(Q)$ at Q_β.

THEOREM 2. *Assume Hypothesis H with $S = \cup S_n$, where the S_n are compact and disjoint. Let R be an open set such that $S \subset R \subset \mathfrak{R}$. Suppose M is any function which is analytic on S^0, continuous on S. Then* (1) *below implies* (2); *also, if S is a Q-set or a compact set,* (2) *implies* (1), *and if K is any isolated interior subset of S, $f(z) = M(z)$ can be required on K.*

(1) *Corresponding to any $\{\varepsilon_n\}$ (ε if S is compact), there exists f analytic on $R - B^*$, meromorphic on R, such that $|M(Q) - f(Q)| \leqq \varepsilon_n$ when $Q \in S_n$, $n = 1, 2, \cdots$ (or ε when $Q \in S$).*

(2) *Corresponding to any $\varepsilon(Q)$ which is admissible on S, there exists F analytic on $R - B^*$ and meromorphic on R such that*

$$| M(Q) - F(Q) | \leqq | \varepsilon(Q) |$$

on S. If f in (1) *can be required to be a rational function and if S is compact, then F can be required to be a rational fuction.*

Proof. We first show (1) implies (2). Admissibility requirement (2) for $\varepsilon(Q)$ implies the zeros of ε on S are isolated. Hence, by the

Weierstrass-factor Theorem [2, p. 591] there exists g analytic on $\mathfrak{R}$ whose only zeros are the zeros Q_β of $\varepsilon(Q)$ and are of the respective orders n_β. Let $\varepsilon_n = \inf|\varepsilon(Q)/g(Q)|$ for Q on S_n (or $\varepsilon = \inf|\varepsilon(Q)/g(Q)|$ for Q on S). Now, by Theorem 1 with $\varepsilon(Q) = \varepsilon_n$ on S_n (or ε on S) and (1) above, there exists a function k meromorphic on R, analytic in $R - B^*$ except at zeros of g on S, such that $|M(Q)/g(Q) - k(Q)| \leqq \varepsilon_n$ (or ε on S) where defined. Then on each S_n (or S)

$$|M(Q) - g(Q)k(Q)| \leqq |g(Q)|\varepsilon_n$$

(or $|g(Q)|\varepsilon$). Now $g\cdot k$, which has removable singularities at the Q_β, satisfies the requirements for F.

Next we consider the converse, giving the proof for the case S is a Q-set. Since $\{\varepsilon_n\}$ defines an admissible $\varepsilon(Q)$, (1) is a special case of (2). We are to verify also that interpolation conditions can be assigned. The Weierstrass-factor theorem yields existence of a function g analytic on $\mathfrak{R}$ such that g has zeros on K of the same orders as the interpolation conditions. For $\varepsilon_n(Q) = \varepsilon_n[g(Q)/\max|g(Q)|]$ when $Q \in S_n$, and $\varepsilon(Q)$ defined by $\varepsilon_n(Q)$ on S_n, $\varepsilon(Q)$ is admissible on S. By hypothesis (2), there is F analytic on $R - B^*$, meromorphic on R, such that

$$|M(Q) - F(Q)| \leqq |\varepsilon(Q)|$$

on S. Since $|\varepsilon(Q)| \leqq \varepsilon_n$ on S_n and $\varepsilon(Q)$ vanishes on K, F satisfies the interpolation conditions, in addition to the requirements for f in the conclusion of (1).

COROLLARY 2.1. *If M is analytic on the closed bounded set S in the finite complex plane, then, corresponding to any admissible $\varepsilon(z)$, there exists a rational function r having its poles on B^* such that $|M(z) - r(z)| \leqq |\varepsilon(z)|$ when $z \in S$.*

Proof. This follows from the Walsh formulation of the Runge Theorem [10, p. 15] and Theorem 2 with $n = 1$ and $R = \mathfrak{R}$ defined as the finite complex plane.

The next corollary is obtained by applying a result of Mergelyan [6; 10, p. 367].

COROLLARY 2.2. *If in the complex plane M is continuous on the closed bounded set S, analytic on S^0, and if S does not separate the plane, then, corresponding to any admissible $\varepsilon(z)$, there exists a polynomial $p(z)$ such that $|M(z) - p(z)| \leqq |\varepsilon(z)|$ on S.*

COROLLARY 2.3. *Suppose S is a Q-set ($= \cup S_n$) and $\varepsilon(z)$ is admissible on $S \subset \mathfrak{R}$, the extended plane minus the set of sequential limit points of S. Then, if M is analytic on S, there exists a function*

f analytic on $\mathfrak{R} - B^*$, meromorphic on $\mathfrak{R}$, such that $|M(z) - f(z)| \leqq |\varepsilon(z)|$ everywhere M is defined on S.

If M is meromorphic on S, there exists f analytic on $R - B^*$, except at poles of M on S, and meromorphic on R such that $|M(z) - f(z)| \leqq |\varepsilon(z)|$ everywhere M is defined on S.

Proof. The first part is an immediate consequence of Theorem 2 and a previous theorem of the author [9, Theorem 3]. The latter part then follows from Corollary 1.1.

For $\varepsilon(Q)$ continuous on S, in order that (2) of Theorem 2 hold, the admissibility restriction (2) on ε is necessary at any interior zero of ε at which M is analytic. For, if $|M(Q) - F(Q)| \leqq |\varepsilon(Q)|$ on S, then, at a zero Q_β of ε, $M(Q_\beta) = F(Q_\beta)$. If (as is the case if M is analytic at Q_β and $F(Q) \not\equiv M(Q)$) $M(Q) - F(Q) = (Q - Q_\beta)^{n_\beta} g(Q)$, where, in some neighborhood $N_{Q_\beta} \subset S$, g is bounded from zero, then

$$|M(Q) - F(Q)| \leqq |\varepsilon(Q)|$$

on S implies $|(Q - Q_\beta)^{n_\beta}/\varepsilon(Q)| \, |g(Q)| \leqq 1$ on N_{Q_β}, where defined. The last inequality is possible only if the first factor is bounded on N_{Q_β}, that is, $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero on N_{Q_β}. At an interior point of S, M is necessarily analytic if Hypothesis (1) of Theorem 2 is satisfied; hence, if the conclusion of Theorem 2 is to hold, continuous $\varepsilon(Q)$ must satisfy admissibility requirement (2) at any interior zero of ε.

An example is next given to illustrate an application of Theorem 2 for the case $n = 1$. Let $R = \mathfrak{R} = \{z/|z| < \infty\}$; $M(z) = z \sin 1/z$ for $z \neq 0$, $M(0) = 0$; $\varepsilon(z) = (z - 1)^5(z - 3/4)(z - \frac{1}{2})g(z)$, where g is any function continuous and nonvanishing on S; $S = \{x/0 \leqq x \leqq 1\} \cup_{j=1}^{3} \gamma_j$ where the γ_j are nonintersecting closed disks with centers at the zeros of $\varepsilon(z)$. Now, by a Walsh approximation theorem [10, p. 47], $M(z)$ can be uniformly approximated by a polynomial, that is, (1) in Theorem 2 is satisfied with $f(z)$ a polynomial in z. Hence, Theorem 2 implies that for any admissible $\varepsilon(z)$, in particular as defined above, there is a polynomial $F(z)$ such that $|M(z) - F(z)| \leqq |\varepsilon(z)|$ on S.

The next theorem yields degree of convergence in the $O(\varepsilon_n(Q))$-sense by setting $S = S_1 = S_2 = \cdots$, also other special results as stated in the corollaries.

Corresponding to given $\{\varepsilon_n\}$, $\{\varepsilon_n(Q)\}$ with $\varepsilon_n(Q)$, defined on S_n and nonvanishing on ∂S_n, $n = 1, 2, \cdots$, will be called ε_n-admissible on $S = \cup S_n$ if there exists $g(Q)$ analytic on $\mathfrak{R}$ such that, for each n, $\varepsilon_n(Q) = g(Q)\phi_n(Q)$ and $\varepsilon_n \leqq \inf |\phi_n(Q)|$, $n = 1, 2, \cdots$, for $Q \in S_n$.

THEOREM 3. *Assume Hypothesis H, with $S = \bigcup_{n=1}^{\infty} S_n$, where the S_n are compact, but not necessarily disjoint. Let $\mathcal{S}_n$ be a collection* *of functions each meromorphic on an open set* R_n *and analytic on* $R_n - B^*$, *where* $S_n \subset R_n \subset \mathfrak{R}$. ($R_n$ *may be* $\mathfrak{R}$.) *Suppose a certain sequence of positive constants* $\{\varepsilon_n\}$ *assigned. Then* (1) *below implies* (2).

(1) *Corresponding to any* $\{m_n\}$, *with* m_n *analytic on* S_n^0, *continuous on* S_n, *and such that* $m_n(Q) = m_j(Q)$ *on* $S_n \cap S_j$ (*if this is not the null set*), *there exists* $f_n, f_n \in \mathscr{S}_n$, *and* M (*independent of* n) *such that* $|m_n(Q) - f_n(Q)| < M\varepsilon_n$ *on* S_n.

(2) *Corresponding to any* ε_n-*admissible* $\{\varepsilon_n(Q)\}(\varepsilon_n(Q) = g(Q)\phi_n(Q))$ *and to* $\{m_n\}$ *defined as in* (1), *there exists* h *meromorphic on* $\mathfrak{R}$ *whose only poles lie on* B^* *or coincide with those of* $m_n(Q)/g(Q)$ *on* S *and there exists* $f_n \in \mathscr{S}_n$ *such that*

$$|m_n(Q) - g(Q)[h(Q) + f_n(Q)]| \leqq M_1 |\varepsilon_n(Q)|$$

on S_n, $n = 1, 2, \cdots$. *If in* (1) *the* f_n *can be chosen as the same function for all* n, *the same is true for the* f_n *in* (2). *If, in* (1), M *is independent of* $\{m_n(Q)\}$, *then, in* (2), $M_1 = M$.

Proof. By the Mittag-Leffler theorem there exists h meromorphic on $\mathfrak{R}$ whose only poles coincide with those of m_n/g on S_n, $n = 1, 2, \cdots$. Now $(m_n(z)/g(z)) - h(z)$ is analytic on S_n^0, continuous on S_n. Hence, by hypothesis (1), there exists $f_n \in \mathscr{S}_n$ such that on S_n

$$|[m_n(Q)/g(Q) - h(Q)] - f_n(Q)| < M_1\varepsilon_n \leqq M_1 |\phi_n(Q)| .$$

This yields the required result.

If in both (1) and (2) the m_n are assumed analytic on S_n, the theorem remains true.

COROLLARY 3.1. *Let* m *be analytic on the bounded closed set* S *which does not separate the complex plane. Suppose* $\{\varepsilon_n\}$ *is a certain sequence of positive constants such that there exist polynomials* $\{p_n(z)\}$ *of respective degrees* n *and some* M *such that* $|m(z) - p_n(z) < M\varepsilon_n$ *on* S. *Then, for* ε_n-*admissible* $\{\varepsilon_n(z)\}$ *with* $\varepsilon_n(z) = P_N(z)\phi_n(z)$, *where* $P_N(z)$ *is a polynomial of degree* N, *there exist polynomials* $P_{N+n}(z)$ *of degrees* $N + n$ *such that* $|m(z) - P_{n+N}(z)| \leqq M_1 |\varepsilon_n(z)|$ *on* S.

Proof. In the theorem set $S = S_1 = S_2 = \cdots$ and $m(z) = m_1(z) = m_2(z) = \cdots$, and let $\mathscr{S}_n$ denote the set of all polynomials of degree n. Since, by the hypothesis, (1) is satisfied, the conclusion of the theorem yields the result when it is noted that h can be chosen as an appropriate rational function.

EXAMPLE. If $m(z)$ is analytic on S, $|z| \leqq 1$, m is analytic in a larger region D_ρ: $|z| < \rho$ [10, p. 79]. Fix R, $1 < R < \rho$, and set $\varepsilon_n = 1/R^n$. Let ϕ be any function which is continuous and nonvanishing on

S and let $P_N(z)$ be a polynomial of degree N, nonvanishing on ∂S. Then K can be chosen so that, for $\varepsilon_n(z)$ defined as $KP_N(z)\phi(z)/(z^n + R^n)$, and $\phi_n(z) = K\phi(z)/(z^n + R^n)$, $\{\varepsilon_n(z)\}$ is ε_n-admissible on S. There are known to be polynomials p_n of respective degrees n such that, for some M, $| m(z) - p_n(z) | < M/R^n$ on S [10, p. 79], whence, by Corollary 3.1, there exist polynomials q_{n+N} of degrees $n + N$ such that

$$| m(z) - q_{n+N}(z) | \leqq M_1 | \varepsilon_n(z) |$$

on S, for some M_1 independent of n.

The polynomials p_{n+N} in Corollary 3.1 cannot be required to be of degree less than $n + N$. For m analytic on S defined as in the Example, choose $P_N(z)$ as a polynomial whose only zeros coincide with those of $m(z)$ on S, and define $\varepsilon_n(z) = (K/R^n)P_N(z)$, $1 < R < \rho$. Suppose there exist polynomials $p_k(z)$ of degree k such that

$$| m(z) - p_k(z) | \leqq M_1K | P_n(z) |/R^n$$

on S. Without loss of generality it can be supposed the zeros of p_k coincide with those of m on S [10, p. 310]. Now $N = m/P_N$ is analytic on S, except for removable singularities, and

$$| N(z) - p_K(z)/p_N(z) | \leqq M_2/R^n$$

on S. Since $p_k(z)/p_N(z)$ is a polynomial of degree $k - N$, this would yield a degree of convergence stronger than maximal convergence if $k - N < n$ [10, p. 79].

The result stated in Corollary 2.3, which is a direct consequence of Theorem 2, is essentially that of Corollary 3.2.

COROLLARY 3.2. *Suppose $m(z)$ is analytic on $S = \cup S_n$, a Q-set with components S_n, and let B denote its set of sequential limit points. Let $\mathfrak{R}$ be the extended complex plane minus B and define B^* as in Hypothesis H. Then, corresponding to any $\varepsilon(z) = g(z)\phi(z)$ with g analytic on $\mathfrak{R}$ and ϕ bounded from zero on each S_n, there exists f analytic on $\mathfrak{R}$–B^*, meromorphic on $\mathfrak{R}$, such that*

$$| m(z) - f(z) | \leqq | \varepsilon(z) | \text{ on } S .$$

Proof. In the theorem, let $R_n = \mathfrak{R}$, $\mathscr{S} = \mathscr{S}_1 = \mathscr{S}_2 = \cdots$ be the set of functions analytic on $\mathfrak{R}$–B^*, meromorphic on $\mathfrak{R}$, and define $m_n(z) = m(z)$ on S_n, $\varepsilon_n(z) = \varepsilon(z)$ on S_n, $\phi_n(z) = \phi(z)$ on S_n, $\varepsilon_n = \inf | \phi_n(z) |$ for $z \in S_n$. We note $\{\varepsilon_n(z)\}$ is ε_n-admissible. By a theorem of the author [9], M (1) of the theorem is satisfied, with $n = 1$ and $f_1(z) = f_2(z) = \cdots$, whence the theorem implies (2), yielding the required result.

COROLLARY 3.3. *Let* $S = \bigcup_{n=1}^{\infty} S_n$, *where the* S_n *are closed circular disks of radii one-half tangent externally along the positive real axis and ordered by increasing distance from the origin. Suppose* m *is analytic on each* S_n^0, *continuous on* S. *Then, for* $\varepsilon(z) = g(z)\phi(z)$, *where* g *is an entire function (nonvanishing on* ∂S*) and* ϕ *is bounded from zero on each* S_n, *there exists an entire function* F *such that* $|m(z) - F(z)| \leqq |\varepsilon(z)|$ *on* S.

Proof. Let $R = \mathfrak{R}$ be the finite complex plane, B^* the null set, and $\mathscr{S} = \mathscr{S}_1 = \mathscr{S}_2 = \cdots$ the class of entire functions. Define $m_n(z) = m(z)$ on S_n, $n = 1, 2, \cdots$, and set $\varepsilon_n(z) = \varepsilon(z)$ on S_n. Then define $\phi_n(z) = \phi(z)$ on S_n and $\varepsilon_n = \inf |\phi_n(z)|$ for $z \in S_n$. By a previous result [8, Theorem 3], corresponding to any $\{\varepsilon_n\}$, there exists $f(z) = f_1(z) = f_2(z) = \cdots$, $f \in \mathscr{S}$, such that $|m(z) - f(z)| < \varepsilon_n$ on S_n. Then (2) of the theorem with $F(z) = g(z)[h(z) + f(z)]$ yields the required result.

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ISOMETRIC IMMERSIONS OF MANIFOLDS OF NONNEGATIVE CONSTANT SECTIONAL CURVATURE

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Let M^d denote a C^∞ Riemannian manifold which is d-dimensional and complete. Our first result states that an isometric immersion of a flat M^d into $(d+k)$-dimensional Euclidean space, $k < d$, is n-cylindrical if the relative nullity of the immersion has constant value n. This result was obtained by O'Neill with the additional hypothesis of vanishing relative curvature. We next consider the case in which M^d and $\bar{M}^{d+k}$, $k < d$, are manifolds of the same constant positive sectional curvature. In this case we show that an isometric immersion of M^d into $\bar{M}^{d+k}$ is totally geodesic if the relative curvature of the immersion is zero on a certain subset of M^d.

Let M^d and $\bar{M}^{d+k}$ be C^∞ Riemannian manifolds of the same constant sectional curvature C, M^d being assumed complete and $k < d$. Let $\psi: M^d \to \bar{M}^{d+k}$ be an isometric immersion. The character of such immersions has been studied in [4] and [5] in terms of what Chern and Kuiper call the *index of relative nullity* of ψ [2]. This function, ν, assigns to each $m \in M$ the dimension of $\mathscr{N}(m)$, the subspace of vectors x in the tangent space M_m such that $T_x = 0$. The linear *difference operators* T_x act on $\bar{M}_{\psi(m)}$ and contain the same information as the classical second fundamental form operators S_z where z is a tangent vector to $\bar{M}$ orthogonal to $d\psi(M_m)$ [1]. In fact T_x is characterized by its skew-symmetry and the equation $T_x(z) = d\psi(S_z(x))$. Our first theorem concerns the case in which M^d is flat and $\bar{M}^{d+k} = R^{d+k}$, $d+k$ dimensional Euclidean space. It states that when ν is constant on M^d the immersion ψ is 'cylindrical'. We next investigate the corresponding situation for $C > 0$.

We use essentially the notation in [4]. In particular we identify M^d with $\psi(M^d)$ when it seems safe to do so. Let N denote the bundle of normal k-frames of M relative to ψ; that is

$$N = \{(m, E) \mid m \in M \text{ and } E \text{ is a } k\text{-frame (orthonormal set of } k \text{ vectors) of } \bar{M}_{\psi(m)} \text{ orthogonal to } d\psi(M_m)\} .$$

The Riemannian connection of $\bar{M}^{d+k}$ induces a natural connection on N. The curvature form of this connection is called the *relative curvature* of ψ. We say that $\psi: M^d \to R^{d+k}$ is *n-cylindrical* provided

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M and ψ can be expressed as Riemannian products $M^d = B^{d-n} \times R^n$ and $\psi = \bar{\psi} \times 1$ where $\bar{\psi}$ is an isometric immersion of B^{d-n} in R^{d+k-n} and 1 is the identity map of R^n. We can now state our first theorem precisely. This result was obtained by O'Neill as Theorem 2 of [4] but with an additional hypothesis, namely, the assumption of zero relative curvature. We shall use a similar assumption in our Theorem 3.

THEOREM 1. *Let M^d be a complete, flat, C^∞ Riemannian manifold. An isometric immersion $\psi\colon M^d \to R^{d+k}$ is n-cylindrical if the relative nullity has constant value n.*

We summarize some results applicable to an isometric immersion between two manifolds of constant curvature C. Let $\mathscr{N}^\perp(m)$ be the orthogonal complement of $\mathscr{N}(m)$ in M_m. From [5] we have: *If n denotes the minimum value of ν, then $n \geqq d - k$ and G, the open subset of M^d on which $\nu = n$, is foliated by complete totally geodesic subspaces (the leaves of $\mathscr{N}$) which are also totally geodesic relative to ψ. Also there exists for any $m \in G$ an $x \in \mathscr{N}^\perp(m)$ such that T_x is injective on $\mathscr{N}^\perp(m)$.* The two cases of interest to us are:

Case 1. $G = M^d$ (i.e., ν is constant), $\bar{M}^{d+k} = R^{d+k}$ $(C = 0)$ and $a = \infty$ (see below).

Case 2. $C > 0$ and $0 < a < \pi/4\sqrt{C}$.

The parameter a appears in the following lemma. *Let $\gamma\colon (-a, a) \to L$ be a unit speed geodesic in a leaf L of $\mathscr{N}$ in G. Then there exists a frame field $E = (E_1, \cdots, E_{d+k})$ on a neighborhood or γ in G such that:*

1. *The geodesic γ is an integral curve of E_1;*
2. *Each integral curve of E_1 is a geodesic of M;*
3. *The vector fields $E_1, \cdots, E_n$ are contained in $\mathscr{N}$, $E_{n+1}, \cdots, E_d$ in $\mathscr{N}^\perp$, and $E_{d+1}, \cdots, E_{d+k}$ are contained in the orthogonal complement of $\psi(M_m)$ in $\bar{M}_{\psi(m)}$;*
4. *The frame E is parallel on γ.* The construction for this lemma is contained in Lemma 1 of [5], except we use the additional fact that the leaves of $\mathscr{N}$ are R^n planes in Case 1 for $a = \infty$. We pull the connection form $\bar{\phi}$ of the frame bundle of $\bar{M}^{d+k}$ down to G by way of the frame field E. Using the following index convention,

$$1 \leqq a, b \leqq n\,; \qquad n + 1 \leqq q, r, s \leqq d\,;$$
$$1 \leqq i, j \leqq d\,; \qquad d + 1 \leqq \alpha, \beta \leqq d + k\,,$$

we get

$$\phi_{ij} = \bar{\phi}_{ij} \circ dE \quad \text{(connection forms of } M\text{)},$$
$$\tau_{i\alpha} = \bar{\phi}_{i\alpha} \circ dE \quad \text{(Codazzi forms)},$$
$$\theta_{\alpha\beta} = \bar{\phi}_{\alpha\beta} \circ dE \quad \text{(normal connection forms)}.$$

A set of linear operators on $\mathcal{N}^{\perp}$ dependent on the frame field E can be defined by

$$P_{E_a}(E_s) = \Sigma_r \phi_{ra}(E_s) E_r \, .$$

From the second structural equation and the properties of the frame field E one can show that *the matrix $P(t)$ of $P_{\gamma'(t)}$ satisfies the differential equation $P' = -P^2 - CI$ on $(-a, a)$* where I denotes the $(d-n) \times (d-n)$ identity matrix. See Lemma 3 of [5]. Our proof of Theorem 1 hinges on the central result from [4] which states that *if for all $m \in M^d$ and $x \in \mathcal{N}^{(m)}$ we have that $P_x = 0$ then the immersion is n-cylindrical.* Theorem 1 can now be easily proved with the help of the following lemma which is applicable in both Case 1 and Case 2.

LEMMA 1. *Let $m \in L$. If $x \in \mathcal{N}(m)$ and $y \in \mathcal{N}^{\perp}(m)$ then $T_{P_x(y)} = T_y \circ P_x$ on $\mathcal{N}^{\perp}(m)$.*

Proof. Since L is complete there exists a geodesic $\gamma: (-a, a) \to L$ with $\gamma(0) = m$ and a frame field E as defined above in a neighborhood of γ. From $T_{E_i}(E_j) = \Sigma_\alpha \tau_{\alpha j}(E_i) E_\alpha$ and the definition of $\mathcal{N}$ we get that $\tau_{\alpha a} = 0$. Using this fact with the Codazzi equation for $\tau_{\alpha a}$ we have

$$0 = d\tau_{a\alpha} = -\Sigma_i \phi_{ai} \wedge \tau_{i\alpha} - \Sigma_\beta \tau_{a\beta} \wedge \theta_{\beta\alpha} = \Sigma_q \phi_{aq} \wedge \tau_{q\alpha} \, .$$

This implies that

$$\Sigma_{\alpha,q} \phi_{qa}(E_s) \tau_{\alpha q}(E_r) E_\alpha = \Sigma_{\alpha,q} \phi_{qa}(E_r) \tau_{\alpha q}(E_s) E_\alpha$$

or that

$$T_{E_r}(P_{E_a}(E_s)) = T_{E_s}(P_{E_a}(E_r)) \, .$$

Hence for $x \in \mathcal{N}(m)$ and $y, z \in \mathcal{N}^{\perp}(m)$ we have

$$T_y(P_x(z)) = T_z(P_x(y)) = T_{P_x(y)}(z) \, ,$$

the last equality above following from the symmetry of the second fundamental form operators.

2. Proof of Theorem 1. We shall show that $P_x = 0$ for $x \in \mathcal{N}(m)$, $m \in M^d$. We may assume x is a unit vector and γ is a unit speed complete geodesic of the leaf through m with $\gamma'(0) = x$. By a previous remark we may pick $y \in \mathscr{N}^{\perp}(m)$ such that T_y is injective on $\mathscr{N}^{\perp}(m)$. Then $\mathscr{N}^{\perp} + T_y(\mathscr{N}^{\perp})$ is invariant under both T_y and $T_{P_x(y)}$. Hence the $2(d-n) \times 2(d-n)$ matrix of $T_y \mid (\mathscr{N}^{\perp} + T_y(\mathscr{N}^{\perp}))$ can be represented by a $(d-n) \times (d-n)$ matrix A in the upper right hand corner, $-A^t$ in the lower left hand corner and zeros elsewhere. If B is the analogous block for $T_{P_x(y)}$ then $Q = -AB^t$ will be the matrix of $T_y \circ T_{P_x(y)} \mid \mathscr{N}^{\perp}$. The difference operators T_y and $T_{P_x(y)}$ commute on M_m since M is flat and hence we have $AB^t = BA^t$. By Lemma 1, $P_x = T_y^{-1} \circ T_{P_x(y)} \mid \mathscr{N}^{\perp}$ and hence $P(0) = (A^{-1})^t B^t$. Let

$$R = -A^{-1}Q(A^{-1})^t = B^t(A^{-1})^t .$$

Since Q is symmetric so is R and therefore $P(0)$ has the same (real) eigenvalues as R. These eigenvalues satisfy $\lambda'_k = -\lambda_k^2$ on the real line (since P satisfies this equation by a result stated above) and hence each $\lambda_k = 0$. Thus $R = 0$ and this implies $P(0) = 0$ which is the desired result.

3. Positive curvature case. For completeness we include Corollary 1 of [5] as

THEOREM 2. *Let M^d and $\bar{M}^{d+k}$ be C^∞ manifolds with the same constant positive curvature C, M^d being assumed complete. Let ψ: $M^d \to \bar{M}^{d+k}$ be an isometric immersion with $2k \leqq d$. Then ψ is totally geodesic.*

As above let n denote the minimum value of ν and let G consist of the $m \in M^d$ for which $\nu(m) = n$.

THEOREM 3. *Let M^d and $\bar{M}^{d+k}$ be C^∞ manifolds with the same constant positive curvature C, M^d being assumed complete. Let ψ: $M^d \to \bar{M}^{d+k}$ be an isometric immersion with $k < d$. Then ψ is totally geodesic if the relative curvature of ψ is zero on G.*

Proof. The proof is by contradiction. If ψ is not totally geodesic then $n < d$. Let L be a leaf in G and let $m \in L$. We first show that for any $x \in \mathscr{N}(m)$, P_x is a symmetric operator and is independent of the frame field used in its definition. Let $y \in \mathscr{N}^{\perp}(m)$ such that T_y is injective on $\mathscr{N}^{\perp}$. Using a geodesic $\gamma: (-a, a) \to L$ with $\gamma'(0) = x$ and Lemma 1 we have as in the proof of Theorem 1 that $P(0) = (A^{-1})^t B^t$. Since the relative curvature of ψ is zero we get from the Ricci equation of the immersion that the Codazzi forms satisfy the relation $\Sigma_i \tau_{\alpha i} \wedge \tau_{i\beta} = 0$. From this we conclude that T_y and $T_{P_x(y)}$

commute on $(d\psi(M_m))^\perp$ or $A^tB = B^tA$. This equation implies that $P(0)$ is symmetric. From the first structural equation we have that

$$[E_r, E_s] = \Sigma_i(\phi_{ri}(E_s) - \phi_{si}(E_r))E_i$$

which together with the symmetry of P_x implies $[E_r, E_s] \in \mathscr{N}^\perp$; thus $\mathscr{N}^\perp$ is integrable. For $x \in \mathscr{N}$, P_x is actually a second fundamental form operator of the leaf through $\mathscr{N}^\perp$ and thus P_x is independent of the choice of frame field used in its definition.

From the completeness of L it follows that we can find a unit speed geodesic γ in L defined on the real line. Since M is of constant positive curvature, γ is a compact immersion and $P_{\gamma'}$ is a periodic function on the real line. Let λ be one of the $d - n$ real eigenvalue functions determined by the symmetric operator $P_{\gamma'}$. We may assume λ attains a maximum at $m = \gamma(0)$. Let E be a frame field as above. Then λ must satisfy $\lambda'(0) = -\lambda^2(0) - C = 0$ since P satisfies $P' = -P^2 - CI$ on an interval containing 0. This implies $\lambda(0)$ is not real, which is the desired contradiction. Hence $n \geqq d$ or ψ is totally geodesic on M.

As a Corollary we get a result of O'Neill's from [3]. Let $S^{d+1}(C)$ denote the sphere of curvature C.

COROLLARY 1. *Let M^d and $\bar{M}^{d+1}$ be C^∞ manifolds with the same constant positive curvature C, M^d being assumed complete. Then any isometric immersion $\psi: M^d \to \bar{M}^{d+1}$ is totally geodesic. In particular if $\bar{M}^{d+1} = S^{d+1}(C)$ then any such immersion is an imbedding onto a great sphere.*

Proof. The vanishing of the relative curvature of ψ is trivial in the hypersurface case. In case $\bar{M}^{d+1} = S^{d+1}(C)$ we have that $\psi(M) = S^d(C) \subset S^{d+1}(C)$. Letting $\bar{S}^d(C)$ denote the universal covering manifold of M^d and π the natural projection, we have that $\psi \circ \pi$ is a local isometry onto $\psi(M)$. Hence $\psi \circ \pi$ and therefore ψ is injective. Thus ψ is an imbedding onto $S^d(C)$.

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INVARIANT SPLITTING IN JORDAN AND ALTERNATIVE ALGEBRAS

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Let A be a finite-dimensional Jordan or alternative algebra over a field F of characteristic 0. Let N denote the radical of A. Then A possesses maximal semisimple subalgebras isomorphic to A/N, [5], [6], any two of which are strictly conjugate, [2], [9]. If G is a finite group of automorphisms and antiautomorphisms of A, then A possesses G-invariant maximal semisimple subalgebras, [10]. We investigate here the uniqueness question for such G-invariant maximal semisimple subalgebras. The result is that the strict conjugacy can be chosen to commute pointwise with G and to be in the enveloping associative algebra generated by the right and left multiplications in A.

Similar results have been obtained for associative algebras, [11], and Lie algebras, [12]. However, in the associative case, the conjugacy can be obtained in terms of adjoints of G-symmetric elements, i.e., elements left fixed by the automorphisms in G and sent into their negatives by the antiautomorphisms in G. In the Lie algebra case, one needs only to consider automorphisms, and the conjugacy is obtained in terms of adjoints of fixed points of G. In each case, the conjugacy is in the enveloping associative algebra of A. In both the Jordan and alternative cases, the automorphisms which occur would commute pointwise with G if the elements of A which occur in their formulation in terms of right and left multiplications were to be fixed points of G. However, we have not obtained the conjugacies in this form, and it seems to be an open question whether or not it is always possible to do so.

If G is assumed fully reducible, instead of finite, then A will also possess G-invariant maximal semisimple subalgebras. This is noted in the Jordan case in [4] when G contains only automorphisms, and the same proof can be extended to cover the alternative case, even if G also contains antiautomorphisms. We have answered the uniqueness question for the similar situation in the associative and Lie cases, [13]. For the Jordan and alternative case, the problem seems more complicated. We note here that it is easily answered if $N^2 = 0$, with the strict conjugacy commuting pointwise with G. However, the general question remains open.

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2. Preliminaries. If $a \in A$, we let R_a and L_a stand for right and left multiplication by a, i.e., $xR_a = xa$, $xL_a = ax$. The following two lemmas are easily proved by straightforward calculation.

LEMMA 1. *Let g be an automorphism of A. Then $g^{-1}R_a g = R_{ag}$ and $g^{-1}L_a g = L_{ag}$.*

LEMMA 2. *Let g be an antiautomorphism of A. Then $g^{-1}R_a g = L_{ag}$, $g^{-1}L_a g = R_{ag}$.*

A derivation of A will be called inner if it is in the enveloping Lie algebra generated by the right and left multiplications in A, [7]. We will have occasion to use the following types of inner derivations. If A is Jordan, and $x, s \in A$, then $[R_x, R_s] = R_x R_s - R_s R_x$ is an inner derivation of A which, for $x \in N$, will be a nilpotent element of the radical of the enveloping associative algebra generated by multiplications in A by elements of A, [1], [2], [8]. If A is alternative, and $s, x \in A$, then $D_{s,x} = [R_s, R_x] + [L_s, R_x] + [L_s, L_x]$ is an inner derivation of A which, for $x \in N$, will be a nilpotent element of the radical of the enveloping associative algebra generated by the left and right multiplications of A, [7], [9].

LEMMA 3. *If A is alternative, $a, b \in A$, then $[R_a, L_b] = [L_a, R_b]$, and $D_{a,b} = -D_{b,a}$.*

Proof. $x[R_a, L_b] = b(xa) - (bx)a = -(b, x, a)$, where $(b, x, a) = (bx)a - b(xa)$ is the associator of b, x, and a. Also $x[L_a, R_b] = (ax)b - a(xb) = (a, x, b)$. The first part of Lemma 3 follows from the skew-symmetry of the associator function. Hence

$$\begin{aligned} D_{b,a} &= [R_b, R_a] + [L_b, R_a] + [L_b, L_a] \\ &= -[R_a, R_b] - [R_a, L_b] - [L_a, L_b] \\ &= -[R_a, R_b] - [L_a, R_b] - [L_a, L_b] = -D_{a,b} . \end{aligned}$$

LEMMA 4. *Let A be Jordan, and g an automorphism of A. Then $g^{-1}[R_a, R_b]g = [R_{ag}, R_{bg}]$.*

This is immediate from Lemma 1.

LEMMA 5. *Let A be alternative, and g an automorphism or antiautomorphism of A. Then $g^{-1}D_{a,b}g = D_{ag,bg}$.*

Proof. This is clear from Lemma 1 if g is an automorphism. Let g be an antiautomorphism. Then, using Lemma 2, $g^{-1}D_{a,b}g = [L_{ag}, L_{bg}] + [R_{ag}, L_{bg}] + [R_{ag}, R_{bg}] = D_{ag,bg}$ by Lemma 3.

If D is a nilpotent derivation of A, then $\exp D = I + D + (D^2/2!) + \cdots$ is an automorphism of A. We assume familiarity with the Campbell-Hausdorff formula, [3], $(\exp D_1)(\exp D_2) = \exp D_3$, where D_3 is in the Lie algebra generated by D_1 and D_2.

3. The Jordan case.

THEOREM 1. *Let A be a finite-dimensional Jordan algebra over a field F of characteristic 0. Let G be a finite group of automorphisms of A. Let S be a G-invariant maximal semisimple subalgebra of A. Let T be a G-invariant semisimple subalgebra of A. Then there exists an automorphism $U = \exp D$ of A such that*

(1) *U maps T into S,*

(2) *D (and hence U) commutes pointwise with G,*

(3) *D is a nilpotent inner derivation of A which is in the radical of the enveloping associative algebra of A.*

Proof. Let N denote the radical of A. Let s and n denote the projections of the vector space $A = S \oplus N$ onto S and N respectively. Then s and n are linear mappings such that

(i) $s(t_1t_2) = s(t_1)s(t_2)$

(ii) $n(t_1t_2) = s(t_1)n(t_2) + n(t_1)s(t_2) + n(t_1)n(t_2)$

(iii) $s(tg) = s(t)g, \qquad n(tg) = n(t)g$

for $t_1, t_2, t \in T$, $g \in G$.

(i) and (ii) follow since N is an ideal. (iii) follows from the invariance of T, S and N under G.

Now set $N_1 = N$, $N_i = N_{i-1}^2 + AN_{i-1}^2$. By [5], the N_i form a nonincreasing sequence of ideals terminating in 0. Now $T_1 = T \subseteqq A = S + N_1$. Suppose that we have found automorphisms $U_0 = \exp 0$, $U_1 = \exp D_1, \cdots, U_{i-1} = \exp(D_{i-1})$ of A satisfying (2) and (3) of Theorem 1 such that $T_i = TU_0U_1 \cdots U_{i-1} \subseteqq S + N_i$. Then we will show that there exists an automorphism U_i of A satisfying (2) and (3) of Theorem 1 such that $T_iU_i \subseteqq S + N_{i+1}$. Hence if $N_k = 0$, then $U = U_0U_1 \cdots U_{k-1}$ will be the desired automorphism by the Campbell-Hausdorff formula.

Now T_i is a G-invariant semisimple subalgebra of A, so that (i), (ii), (iii) hold for $t_1, t_2, t \in T_i$. Consider the space $N_i \mid N_{i+1}$. We consider this as a T_i-module by defining $t \cdot \bar{n} = \bar{n} \cdot t = \overline{ns(t)}$ for $n \in N_i$, $t \in T_i$. Then by (ii), we have

(iv) $\overline{n(t_1t_2)} = \overline{n(t_1)} \cdot t_2 + t_1 \cdot \overline{n(t_2)}$.

(iv) says that the map $t \to \overline{n(t)}$ is a derivation of T_i into the module $N_i \mid N_{i+1}$. Hence, by [2], there exist elements $x_1, \cdots, x_p$ in N_i, $t_1, \cdots, t_p \in T_i$ such that

(v) $\overline{n(t)} = \sum_{j=1}^{p} ((\bar{x}_j \cdot t) \cdot t_j - \bar{x}_j \cdot (tt_j))$ for $t \in T_i$ i.e.,

$$\overline{n(t)} = \sum_{j=1}^{p} \overline{(x_j s(t))s(t_j)} - \overline{x_j s(tt_j)} \,.$$

Using (i), we have

(vi) $n(t) \equiv s(t) \sum_{j=1}^{p} [R_{x_j}, R_{s(t_j)}] \pmod{N_{i+1}}$ for $t \in T_i$.

Let $g \in G$. Then

$$[R_{x_j g}, R_{s(t_j)g}] = g^{-1}[R_{x_j}, R_{s(t_j)}]g$$

by Lemma 4. Hence

$$\begin{aligned} s(t) \sum_{j=1}^{p} [R_{x_j g}, R_{s(t_j)g}] &= s(t)g^{-1}\Big(\sum_{j=1}^{p} [R_{x_j}, R_{s(t_j)}]\Big)g \\ &= s(tg^{-1})\Big(\sum_{j=1}^{p} [R_{x_j}, R_{s(t_j)}]\Big)g \equiv n(tg^{-1})g = n(t) \pmod{N_{i+1}} \,. \end{aligned}$$

It follows that if we set $D_i = -(1/m) \sum_{g \in G} (\sum_{j=1}^{p} [R_{x_j g}, R_{s(t_j)g}])$, where m is the order of G, then

(vii) $n(t) \equiv -s(t)D_i \pmod{N_{i+1}}$ for $t \in T_i$.

Now D_i clearly satisfies (3) of the Theorem, since the $x_j g \in N$. To see that D_i satisfies (2) of the Theorem, we fix a value of j. Then $\sum_{g \in G} [R_{x_j g}, R_{s(t_j)g}] = \sum_{g \in G} g^{-1}[R_{x_j}, R_{s(t_j)}]g$ clearly commutes pointwise with G. Hence so does D_i, which is a linear combination of such mappings.

Finally, set $U_i = \exp D_i$. If $t \in T_i$, then $tU_i = t + tD_i + (t/2)D_i^2 + \cdots = s(t) + n(t) + s(t)D_i + n(t)D_i + (t/2)D_i^2 + \cdots$.

Now $n(t) \in N_i$, so that $n(t)D_i \in N_{i+1}$. Also, since the $x_1, \cdots, x_p \in N_i$, we have that $(t/2)D_i^2 + \cdots \in N_{i+1}$. Therefore

$$\begin{aligned} tU_i &\equiv s(t) + n(t) + s(t)D_i \pmod{N_{i+1}} \\ &\equiv s(t) \pmod{N_{i+1}} \text{ by (vii).} \end{aligned}$$

Hence $T_i U_i \subseteqq S + N_{i+1}$. This completes the proof of the Theorem.

COROLLARY 1. *Let A be a finite-dimensional Jordan algebra over a field of characteristic* 0. *Let G be a finite group of automorphisms of A. Let S and T be G-invariant maximal semisimple subalgebras of A. Then S and T are strictly conjugate via an automorphism of A of the type described in Theorem* 1.

COROLLARY 2. *Let A and G be as in Corollary* 1. *Let T be any G-invariant semisimple subalgebra of A. Then T is contained in a G-invariant maximal semisimple subalgebra of A.*

Corollary 1 is an immediate consequence of Theorem 1. Corollary 2 follows from the existence of a G-invariant maximal semisimple subalgebra S of A. For then if U is an automorphism of A which maps T into S, and which commutes with G pointwise, it follows that SU^{-1} is a G-invariant maximal semisimple subalgebra of A which contains T.

4. The alternative case.

THEOREM 2. *Let A be a finite-dimensional alternative algebra over a field F of characteristic* 0. *Let G be a finite group of automorphisms and antiautomorphism of A. Let S be a G-invariant maximal semisimple subalgebra of A. Let T be a semisimple subalgebra of A. Then there exists an automorphism $U = \exp D$ of A such that*

(1) *U maps T into S,*

(2) *D (and hence U) commutes pointwise with G,*

(3) *D is a nilpotent inner derivation of A which is in the radical of the enveloping associative algebra of A.*

Proof. The proof is similar to Theorem 1. We define s and n as in Theorem 1, but use $N_i = N^i$ instead. We consider $N^i \mid N^{i+1}$ as a two-sided T_i-module by $t \cdot \bar{n} = \overline{s(t)n}$ and $\bar{n} \cdot t = \overline{ns(t)}$. Then (i), (ii), (iii) and (iv) are valid. Hence, by [9], there exist elements $x_1, \cdots, x_p \in N^i$ and $t_1, \cdots, t_p \in T_i$ such that

(v) $\overline{n(t)} = t \sum_{j=1}^{p} D_{t_j, \bar{x}_j}$ for $t \in T_i$

where $D_{t_j \bar{x}_j}$ is the inner derivation $[R_{t_j}, R_{\bar{x}_j}] + [L_{t_j}, R_{\bar{x}_j}] + [L_{t_j}, L_{\bar{x}_j}]$ of T_i into its two-sided module $N^i \mid N^{i+1}$. As in Theorem 1, we obtain

(vi) $n(t) \equiv s(t) \sum_{j=1}^{p} D_{s(t_j), x_j} (\text{mod } N^{i+1})$ for $t \in T_i$,

where $D_{s(t_j), x_j}$ is the inner derivation $[R_{s(t_j)}, R_{x_j}] + [L_{s(t_j)}, R_{x_j}] + [L_{s(t_j)}, L_{x_j}]$ of A.

Now let $g \in G$. Then by Lemma 5, we have $g^{-1}(D_{s(t_j), x_j})g = D_{s(t_j)g, x_j g}$. Hence, for any $g \in G$, $s(t) \sum_{j=1}^{p} D_{s(t_j)g, x_j g} = s(t)g^{-1}(\sum_{j=1}^{p} D_{s(t_j), x_j})g = s(tg^{-1})(\sum_{j=1}^{p} D_{s(t_j), x_j})g \equiv n(t)(\text{mod } N^{i+1})$ by (iii) and (v).

Now set $D_i = -(1/m) \sum_{g \in G} (\sum_{j=1}^{p} D_{s(t_j)g, x_j g})$, where m is the order of G. Then we have

(vii) $n(t) \equiv -s(t)D_i (\text{mod } N^{i+1})$ for $t \in T_i$.

D_i satisfies (3) of the Theorem since the $x_j g \in N$. To see that D_i satisfies (2) of the Theorem, we fix a value of j. Then $\sum_{g \in G} D_{s(t_j)g, x_j g} = \sum_{g \in G} g^{-1} D_{s(t_j), x_j} g$ commutes pointwise with G. Hence so does D_i, which is a linear combination of such mappings.

Now we set $U_i = \exp D_i$, and get that $T_i U_i \subseteqq S + N^{i+1}$ as in Theorem 1. Finally, we put $U = U_0 U_1 \cdots U_{k-1}$, where $N^k = 0$, and use the Campbell-Hausdorff formula to complete the proof of the

Theorem.

As in the Jordan case, we have the following two corollaries of Theorem 2.

COROLLARY 1. *Let A be a finite-dimensional alternative algebra over a field of characteristic* 0. *Let G be a finite group of automorphisms and antiautomorphisms of A. Let S and T be G-invariant maximal semisimple subalgebras of A. Then S and T are strictly conjugate via an automorphism of A of the type described in Theorem* 2.

COROLLARY 2. *Let A and G be an in Corollary* 1. *Let T be any G-invariant semisimple subalgebra of A. Then T is contained in a G-invariant maximal semisimple subalgebra of A.*

5. **The fully reducible case.** Let A be a finite-dimensional Jordan or alternative algebra over a field of characteristic zero. If G is a fully reducible group of automorphisms and antiautomorphisms of A, then it follows from [4] that G will leave invariant a maximal semisimple subalgebra of A. The analogue of Corollaries 1 has not been answered as yet for this case. However, if $N^2 = 0$, then any automorphism of the form described in the proofs of Theorems 1 and 2 which carries a G-invariant maximal semisimple subalgebra T onto another one, S, is unique, and hence will commute pointwise with G.

For let $U_1 = \exp D_1$, $U_2 = \exp D_2$ be of this form and both map T onto S. Then $D_1^2 = D_2^2 = 0$, so that $U_1 = I + D_1$, $U_2 = I + D_2$. If $t \in T$, then $tU_1 = t + tD_1 \in S$ and $tU_2 = t + tD_2 \in S$. Hence their difference $tD_1 - tD_2 \in S \cap N = 0$, since D_1 and D_2 have range in N. Hence $D_1 = D_2$ on T. Also D_1 and D_2 are both 0 on N since $N^2 = 0$. Hence $D_1 = D_2$ since $A = T + N$.

Now let $g \in G$. Then $g^{-1}U_1g = I + g^{-1}D_1g$ will map T onto S and $g^{-1}D_1g$ is a derivation of square zero having range in N. Hence, by the above, $g^{-1}D_1g = D_1$, that is, D_1, and hence U_1, commutes pointwise with G.

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ON A CONJECTURE OF R. J. KOCH

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Dedicated to Professor Alexander Doniphan Wallace on the occasion of his sixtieth birthday

R. J. Koch proved that if X is a compact, continuously partially ordered space and if W is an open subset of X which has no local minima, then each point of W is the supremum of an order arc which meets $X - W$. More recently he extended this result to quasi ordered spaces in which the sets $E(x) = \{y: x \leqq y \leqq x\}$ are assumed to be totally disconnected and W is a chain. He conjectured that the latter hypothesis is superfluous, and we show here that Koch's conjecture is correct.

As a corollary it follows that if X is a compact, continuously quasi ordered space with zero (i.e., a unique minimal element), if each set $E(x)$ is totally disconnected, and if each set $L(x) = \{y: y \leqq x\}$ is connected, then X is arcwise connected.

We begin by recalling a few definitions (see [1], [2], [3] and [4]). We say that $X = (X, \Gamma)$ is a continuously quasi ordered space provided X is a Hausdorff space, Γ is a quasi order (= reflexive, transitive relation) on X and the graph of Γ is a closed subset of $X \times X$. We identify Γ with its graph and regard the symbols $x \leqq y$, and $x \Gamma y$ and $(x, y) \in \Gamma$ as synonyms.

A *chain* of a quasi ordered space X is a subset C of X such that $a \leqq b$ or $b \leqq a$ holds for each a and b in C. We also define

$$L(a, \Gamma) = \{x \in X: (x, a) \in \Gamma\} ,$$
$$M(a, \Gamma) = \{x \in X: (a, x) \in \Gamma\} ,$$
$$E(a, \Gamma) = L(a, \Gamma) \cap M(a, \Gamma) ,$$

for each $a \in X$. It is also convenient to define

$$I(a, b, \Gamma) = M(a, \Gamma) \cap L(b, \Gamma) ,$$

the closed "interval" from a to b. Where there is no ambiguity we shall write ($L(a)$ (resp., $M(a)$, $E(a)$, $I(a, b)$) for $L(a, \Gamma)$, (resp., $M(a, \Gamma)$, $E(a, \Gamma)$, $I(a, b, \Gamma)$). It is well known [3] that if X is a continuously quasi ordered space then the sets $L(a)$, $M(a)$, $E(a)$ and $I(a, b)$ are closed and, if X is compact, then X contains a minimal element, that is, an element m such that $L(m) - E(m)$ is empty.

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A subset Y of the quasi ordered space (X, Γ) is said to *have no local Γ-minima* if, for each $x \in Y$ and each neighborhood U of x, the set

$$Y \cap U \cap L(x, \Gamma) - E(x, \Gamma)$$

is nonempty. This definition is due to Koch [2].

In case the relation Γ is a partial order, it is known that a connected chain joining two distinct points is an arc. (Here we use the term *arc* to describe a continuum with precisely two non-cutpoints.) An arc which is also a chain is termed an *order arc.*

The following two lemmas will be of later use.

LEMMA 1. *Let X be a compact, continuously quasi ordered space, let a and b be members of X, and let K be a closed subset of X such that $I(a, b) \cap K = 0$. Then there exist open sets U and V such that $a \in U$, $b \in V$ and for each $a' \in U$ and $b' \in V$ it follows that $I(a', b') \cap K = 0$.*

Proof. Suppose, on the contrary, that for all neighborhoods U and V of a and b, respectively, there exists $a' \in U$ and $b' \in V$ such that $I(a' b') \cap K \neq 0$. Then

$$\Gamma \cap (\bar{U} \times K) \cap (K \times \bar{V}) \neq 0.$$

These sets form a family of nonempty closed sets with the finite intersection property and hence their intersection is nonempty:

$$\Gamma \cap (\{a\} \times K) \cap (K \times \{b\}) \neq 0 ,$$

that is to say, $I(a, b) \cap K \neq 0$, contrary to the hypothesis.

LEMMA 2. *If R is an open subset of the compact, continuously quasi ordered space X, then the set*

$$F = \{(a, b) \in X \times X\colon I(a, b) - R \neq 0\}$$

is closed.

Proof. If $(a, b) \notin F$ then $I(a, b) \cap (X - R) = 0$. By Lemma 1, there are open sets U and V with $a \in U$ and $b \in V$ such that for each $a' \in U$ and $b' \in V$ it follows that $I(a', b') \subset R$, and hence $(U \times V) \cap F = 0$. Therefore, F is closed.

2. Koch's theorem for quasi ordered spaces. The crux of our proof is embodied in the following theorem.

THEOREM. *Let $X = (X, \Gamma)$ be a compact, continuously quasi* *ordered space and let W be an open subset of X. If*

(i) *$E(x, \Gamma)$ is totally disconnected for each $x \in X$,*

(ii) *W has no local Γ-minima, then X admits a minimal quasi order which has a closed graph and satisfies* (i) *and* (ii). *Moreover, this minimal quasi order is a partial order.*

Proof. Let $\{\Gamma_\alpha\}$ be a maximal nest of quasi orders on X such that each Γ_α has a closed graph and satisfies (i) and (ii), and let $\Gamma = \cap \{\Gamma_\alpha\}$. Clearly (X, Γ) is a continuously quasi ordered space and $E(x, \Gamma)$ is totally disconnected. We will show that W has no local Γ-minima.

Let $x \in W$ and let U be a neighborhood of x; since W is open and $E(x, \Gamma)$ is totally disconnected, we may assume that $U \subset W$ and that $E(x, \Gamma) \cap U$ is closed. Since X is normal there exist open sets V and R such that

$$E(x, \Gamma) \cap U \subset V \subset \bar{V} \subset U ,$$
$$X - U \subset R \subset \bar{R} \subset X - \bar{V} .$$

For each α, the compact set $L(x, \Gamma_\alpha) \cap \bar{V}$ has a Γ_α-minimal element which we denote x_α. And since W has no local Γ_α-minima there exists

$$y_\alpha \in (X - \bar{R}) \cap L(x_\alpha, \Gamma_\alpha) - E(x_\alpha, \Gamma_\alpha) .$$

It follows that

$$y_\alpha \in L(x, \Gamma_\alpha) - \bar{R} \cup \bar{V}$$

so that the sets $L(x, \Gamma_\alpha) - R \cup V$ are compact, nonempty and nested. Consequently there exists

$$y \in L(x, \Gamma) - R \cup V$$

and it is clear that $y \notin E(x, \Gamma)$. That is, W has no local Γ-minima.

Now suppose that Γ is not a partial order; then there exists a nondegenerate set $E(x, \Gamma)$. Since $E(x, \Gamma)$ is compact and totally disconnected, there exist nonempty, closed and disjoint sets A and B whose union is $E(x, \Gamma)$. Since X is normal there exist disjoint open sets P and Q such that $A \subset P$ and $B \subset Q$. Let

$$F = \{(a, b) : I(a, b) - P \cup Q \neq 0\} .$$

By Lemma 2, F is a closed subset of $X \times X$ and hence

$$\Delta = \Gamma - ((P \times Q) - F)$$

is also closed. Since P and Q are disjoint, Δ is a reflexive relation on X.

We claim that Δ is a quasi order. For suppose $p \Delta q$ and $q \Delta r$ but $(p, r) \in (X \times X) - \Delta$. Now $(p, r) \in \Gamma$ so that $(p, r) \in (P \times Q) - F$ and hence $q \in P$ or $q \in Q$. If $q \in P$ then, since $r \in Q$ and $(q, r) \in \Delta$ we infer that $(q, r) \in F$ and thus $I(q, r) - P \cup Q \neq 0$. But $I(q, r) \subset I(p, r)$ and hence $I(p, r) - P \cup Q \neq 0$, contrary to the fact that $(p, r) \in (P \times Q) - F$. A similar contradiction ensues if $q \in Q$, and thus Δ is a quasi order.

Since $\Delta \subset \Gamma$ it is obvious that each set $E(x, \Delta)$ is totally disconnected. Now suppose $z \in W$ and that O is a neighborhood of z, $O \subset W$. If $z \in W - Q$ then

$$L(z, \Delta) = L(z, \Gamma)$$

and hence there exists

$$y \in O \cap L(z, \Delta) - E(z, \Delta) .$$

And if $z \in Q$, the fact that W has no local Γ-minima insures the existence of

$$y \in O \cap Q \cap L(z, \Gamma) - E(z, \Gamma) .$$

But $y \notin P$ implies $y \in L(z, \Delta)$, so that in any event W has no local Δ-minima.

Finally we note that Δ contradicts the minimality of Γ, for if $a \in A$ and $b \in B$ then $(a, b) \in \Gamma - \Delta$. Therefore Γ is a partial order.

COROLLARY 1. *Let X be a compact, continuously quasi ordered space and let W be an open subset of X. If conditions* (i) *and* (ii) *of the theorem are satisfied, then each point of W is the supremum of an order arc which meets $X - W$.*

Proof. By the preceeding theorem we may assume that the quasi order is a partial order. Thus Koch's theorem for partially ordered spaces applies.

An element 0 of the quasi ordered space X is a *zero* of X provided

$$0 = E(0) = \cap \{L(x) \colon x \in X\} .$$

COROLLARY 2. *If X is a compact, continuously quasi ordered space with zero, if each set $E(x)$ is totally disconnected and if each set $L(x)$ is connected, then X is arcwise connected.*

Proof. Let $W = X - \{0\}$; the connectedness of the sets $L(x)$ guarantees that W has no local minima and therefore each point of W lies in arc containing 0.

Following Koch we say that a subset C of the quasi ordered space X is *biconnected* if C is connected and if each of the sets $E(x) \cap C$ is connected.

COROLLARY 3. *Let X be a compact, continuously quasi ordered space and suppose there exists $a \in X$ such that*

$$E(a) = \cap \{L(x): x \in X\} .$$

If $X - E(a)$ has no local minima then each element of X can be joined to $E(a)$ by a biconnected chain.

Proof. Let Z denote the compact, continuously partially ordered space which is obtained when $E(x)$ is identified with a point, for each $x \in X$. Let $\phi(X) = Z$ be the canonical quotient map and let

$$X \xrightarrow{m} Y \xrightarrow{l} Z$$

be the monotone-light factorization of ϕ. It is easy to see that Y inherits a quasi order from Z which has a closed graph and is such that $E(y)$ is totally disconnected, for each $y \in Y$. Moreover, Y-$m(E(\mathrm{a}))$ has no local minima and hence, by the theorem, there are order arcs joining points of Y to $m(E(a))$. Since m is monotone, the corollary follows at once.

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DEVELOPMENT OF THE MAPPING FUNCTION AT A CORNER

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Let D be a domain in the plane which is partially bounded by two curves Γ_1 and Γ_2 which meet at the origin and form there an interior angle $\pi\tau > 0$. Let N be an integer $\geqq 2$ and let α be a real number such that $0 < \alpha < 1$. Suppose that for $i = 1, 2, \Gamma_i$ admits a parametrization $x = x_i(t), y = y_i(t), 0 \leqq t \leqq 1$, where x_i and y_i have Nth derivatives which are uniformly α-Hölder continuous, and $|x_i'(t)| + |y_i'(t)| > 0$. Let $F(z)$ map the upper half plane conformally onto D in such a way that $F(0) = 0$. Then if τ is irrational $F(z)$ has an asymptotic expansion in powers of z and z^τ, with error term $o(z^{N\tau-\varepsilon})$. If $\tau = p/q$, a reduced fraction, then $F(z)$ has an asymptotic expansion in powers of z, z^τ, and $z^p \log z$, with error term $o(z^{N\tau-\varepsilon})$. In both cases ε is an arbitrarily small positive number. Furthermore expansions for derivatives of $F(z)$ of order $\leqq N$ may be obtained by differentiating formally.

The behavior of such conformal maps at corners was first investigated by Lichtenstein [9]. Let $F^{-1}(z)$ be the function inverse to $F(z)$ which maps D onto the upper half plane. Lichtenstein showed that if Γ_1 and Γ_2 are analytic then

$$\frac{d}{dz}F^{-1}(z) = z^{1/\tau-1}\varphi(z) \tag{1.1}$$

where $\varphi(z)$ is continuous in $\bar{D}$ and $\varphi(0) \neq 0$. This result was later generalized in two ways. One was to weaken the requirements on Γ_1 and Γ_2. It follows from the work of Kellogg [4] and Warschawski [10] that with very modest conditions imposed on Γ_1 and Γ_2 one has

$$F^{-1}(z) = z^{1/\tau}\varphi(z)$$

where again $\varphi(z)$ is continuous in $\bar{D}$ and $\varphi(0) \neq 0$. In particular this follows if one assumes that Γ_1 and Γ_2 have continuously turning tangents in a neighborhood of the origin (though weaker conditions will suffice).

The other generalization of Lichtenstein's theorem was an improvement of the result (1.1), maintaining the analyticity requirement. For the case $\tau = 1$ Lewy [8] showed that $F(z)$ has an asymptotic expansion

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in powers of z and $\log z$. Later Lehman [6] showed that expansions of the kind mentioned in the first paragraph are valid for all angles $\pi\tau > 0$, provided Γ_1 and Γ_2 are analytic. Thus in this paper we dovetail the results of the two developments. Furthermore we shall indicate some applications to the behavior at corners of solutions of elliptic partial differential equations; see [3], [5], [7], [8], [11] and [12].

2. Principal results. Let N be an integer $\geqq 2$ and let α be a real number such that $0 < \alpha < 1$. Assume that for $i = 1, 2, \Gamma_i$ admits a parametrization $x = x_i(t)$, $y = y_i(t)$ where $x_i(t)$ and $y_i(t)$ are uniformly $C^{N+\alpha}$ for $0 \leqq t \leqq 1$,[1] and assume that there exists a $\delta > 0$ such that $|x_i'(t)| + |y_i'(t)| \geqq \delta$ for $0 \leqq t \leqq 1$. Let $F(z)$ map the upper half plane conformally onto D. Then $G(z) \equiv F(z^{1/\tau})$ maps the sector $0 < \arg z < \pi\tau$ onto D and we have the following theorems.

THEOREM 1. *If τ is irrational then there exists a polynomial $P(z, z^\tau)$ such that as $z \to 0$, $0 \leqq \arg z \leqq \pi$,*

$$F(z) = z^\tau P(z, z^\tau) + o(z^{N\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0,0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^\tau, z^p \log z)$ such that as $z \to 0$, $0 \leqq \arg z \leqq \pi$,

$$F(z) = z^\tau P(z, z^\tau, z^p \log z) + o(z^{N\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leqq N$ may be obtained by differentiating formally.

THEOREM 2. *If τ is irrational then there exists a polynomial $P(z, z^{1/\tau})$ such that as $z \to 0$, $0 \leqq \arg z \leqq \pi\tau$,*

$$G(z) = zP(z, z^{1/\tau}) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0,0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^{1/\tau}, z^q \log z)$ such that as $z \to 0$, $0 \leqq \arg z \leqq \pi\tau$,

$$G(z) = zP(z, z^{1/\tau}, z^q \log z) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leqq N$ may be obtained by differentiating formally.

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[1] This means there exists a constant K such that for $0 \leqq s < t \leqq 1$ and $0 \leqq n \leqq N$

$$\left|\frac{d^n}{dt^n}x_i(s) - \frac{d^n}{dt^n}x_i(t)\right| + \left|\frac{d^n}{dt^n}y_i(s) - \frac{d^n}{dt^n}y_i(t)\right| \leqq K|s-t|^\alpha .$$

From Theorems 1 and 2 one can obtain an asymptotic expansion for the inverse function $F^{-1}(z)$ which maps D onto the upper half plane. The method is an iterative one, starting with $F(z) = o(z^{\tau-\varepsilon})$ and increasing the exponent of the error term; see, for instance, Wasow [11], pp. 49–50.

THEOREM 3. *If τ is irrational then there exists a polynomial $P(z, z^{1/\tau})$ such that as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$F^{-1}(z) = z^{1/\tau}P(z, z^{1/\tau}) + o(z^{N-1+1/\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0,0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^{1/\tau}, z^q \log z)$ such that as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,

$$F^{-1}(z) = z^{1/\tau}P(z, z^{1/\tau}, z^q \log z) + o(z^{N-1+1/\tau-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leqq N$ may be obtained by differentiating formally,

Since $G^{-1}(z) = (F(z))^\tau$, we have, by the binomial theorem.

THEOREM 4. *If τ is irrational there exists a polynomial $P(z, z^{1/\tau})$ such that as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$G^{-1}(z) = zP(z, z^{1/\tau}) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0,0) \neq 0$. If $\tau = p/q$, a reduced fraction, then there exists a polynomial $P(z, z^{1/\tau}, z^q \log z)$ such that as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,

$$G^{-1}(z) = zP(z, z^{1/\tau}, z^q \log z) + o(z^{N-\varepsilon})$$

where ε is an arbitrarily small positive number and $P(0, 0, 0) \neq 0$. Furthermore expansions for derivatives of order $\leqq N$ may be obtained by differentiating formally.

3. Applications to partial differential equations. The expansions of Theorems 2 and 4 have immediate applications to a previous paper of the author [12]. In particular § 4 and 5 of [12] need only be modified suitably to obtain the following theorems.

Let $U(x, y)$ be a solution in D of the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + KU = F$$

where K and F are $(N-1)$-times continuously differentiable in $D \cup \Gamma_1 \cup \Gamma_2 \cup \{0\}$, U is twice continuously differentiable in D, and U_x

and U_y are α-Hölder continuous in every compact subset of $D \cup \Gamma_1 \cup \Gamma_2$. We also assume that for $i = 1, 2, U$ satisfies on Γ_i a boundary condition

$$\delta_i \frac{\partial U}{\partial n} + A_i U = B_i$$

where $\delta_i = 0$ or 1, $\partial/\partial n$ represents the outgoing normal derivative, and A_i and B_i are $(N-1)$-times continuously differentiable as functions of arc length, defined on $\Gamma_i \cup \{0\}$, and $A_i(0) \neq 0$, if $\delta_i = 0$. Finally, we assume that as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,

$$U(z) = o(z^{\mu})$$

where $\mu > \max(-1, -1/\tau)$ if $\delta_1 = \delta_2$ and $\mu > \max(-1, -1/2\tau)$ if $\delta_1 \neq \delta_2$. Then

THEOREM 5. *If $\delta_1 = \delta_2 = 0$ then as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$U(z) = \log z P_1 + \log \bar{z} P_2 + P_3 + o(z^{N-1-\varepsilon})$$

where P_1, P_2 and P_3 are polynomials in $z, \bar{z}, z^{1/\tau}$ and $\bar{z}^{1/\tau}$ if τ is irrational and in $z, \bar{z}, z^{1/\tau}, \bar{z}^{1/\tau}, z^q \log z$ and $\bar{z}^q \log \bar{z}$ if $\tau = p/q$; and ε is an arbitrarily small positive number. If $B_1(0)A_2(0) = B_2(0)A_1(0)$, P_1 and P_2 vanish identically. Furthermore expansions for derivatives of $U(z)$ of order $\leqq N-2$ may be obtained by differentiating formally.

THEOREM 6. *If $\delta_1 = 0$ and $\delta_2 = 1$ (or $\delta_1 = 1$ and $\delta_2 = 0$) then as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$U(z) = P + o(z^{N-1-\varepsilon})$$

where P is a polynomial in $z, \bar{z}, z^{1/2\tau}$ and $\bar{z}^{1/2\tau}$ if τ is irrational; P is a polynomial in $z, \bar{z}, z^{1/2\tau}, \bar{z}^{1/2\tau}, z^q \log z$ and $\bar{z}^q \log \bar{z}$ if $\tau = p/q$ and q is odd; P is a polynomial in $z, \bar{z}, z^{1/2\tau}, \bar{z}^{1/2\tau}, z^{q/2} \log z$ and $\bar{z}^{q/2} \log \bar{z}$ if $\alpha = p/q$ and q is even; and ε is an arbitrarily small positive number. Furthermore expansions for derivatives of $U(z)$ of order $\leqq N-2$ may be obtained by differentiating formally.

THEOREM 7. *If $\delta_1 = \delta_2 = 1$ then as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$,*

$$U(z) = \log z P_1 + \log \bar{z} P_2 + P_3 + o(z^{N-\varepsilon})$$

where P_1, P_2 and P_3 are polynomials in $z, \bar{z}, z^{1/\tau}$ and $\bar{z}^{1/\tau}$ if τ is irrational and in $z, \bar{z}, z^{1/\tau}, \bar{z}^{1/\tau}, z^q \log z$ and $\bar{z}^q \log \bar{z}$ if $\tau = p/q$; and ε is an arbitrarily small positive number. If $U(z)$ is bounded at the origin then P_1 and P_2 vanish identically. Furthermore derivatives of $U(z)$ of order $\leqq N-1$ may be obtained by differentiating formally.

4. Some Lemmas. Later we shall need some properties of functions which are Hölder continuous in a set, but whose Hölder constants diverge in a certain way near a boundary point of the set. Let S be a subset of the plane which does not contain the origin, but of which the origin is a cluster point. Let μ and β be real numbers, $0 \leqq \beta < 1$, and let M be a nonnegative integer. Let $f(x, y)$ be a real or complex valued function such that $f(x, y) \in C^{M+\beta}$ for $(x, y) \in S$, and suppose that for $0 \leqq n \leqq M$

(i) $D^n f(x, y) = O(z^{\mu-n})$

as $z \longrightarrow 0$, $z \in S$, where D^n ranges over all nth order partial derivatives, and

(ii) there exists a constant K such that

$$\sup \frac{| D^n f(z) - D^n f(\zeta) |}{| z - \zeta |^\beta} | z |^{n+\beta-\mu} \leqq K$$

where the supremum is taken over all derivatives D^n, and over all points $z, \zeta \in S$ such that $0 < | z - \zeta | < \delta | z |, \delta | \zeta |$; δ is assumed to be some positive number < 1. The totality of such functions we designate by $W_\mu^{M+\beta}(S)$. If S is the sector $\delta_1 \leqq \arg z \leqq \delta_2$, $0 < | z | < | z_0 |$, we write $W_\mu^{M+\beta}([\delta_1, \delta_2])$. We omit the dependence on z_0 because we are only concerned with properties (i) and (ii) in some neighborhood of the origin. If S is a segment $0 < x < A$ we write $W_\mu^{M+\beta}$; properties (i) and (ii) should then be modified properly for a function of one variable. We observe that if $\beta = 0$ property (ii) follows from property (i) and the condition $| z - \zeta | < \delta | z |, \delta | \zeta |$.

We now list some properties of the W-spaces. We state them for the complex case, though with suitable modifications the properties hold for the real case. Thus we assume $0 < | z - \zeta | < \delta | z |, \delta | \zeta |$, and $z, \zeta \in S$.

1. $1 - \delta < | z/\zeta | < 1 + \delta$.

2. Let $\mu \leqq -1$. If $(\partial/\partial x)f(z), (\partial/\partial y)f(z) \in W_\mu^0(S)$ then $f(z)$ differs by a constant from a function in $W_{\mu+1}^1(S)$. The proof is contained in Bourbaki [2].

3. If $(\partial/\partial x)f(z), (\partial/\partial y)f(z) \in W_\mu^0(S)$ then f differs by a constant from a function in $W_{\mu+1}^\beta(S)$, $0 \leqq \beta < 1$. The proof follows from property 2 above and the mean value theorem for functions of two variables.

4. There exists a constant K depending only on μ, β and δ such that

$$| z^{\mu+\beta} - \zeta^{\mu+\beta} | \leqq K | z |^\mu | z^\beta - \zeta^\beta | \leqq K | z |^\mu | z - \zeta |^\beta .$$

Here we assume that S is so chosen that $z^{\mu+\beta}$ and z^β are single valued functions.

5. Let $f(z) \equiv z^\mu$, and assume z^μ is single valued for $z \in S$. Then

for all integers $M \geqq 0$ and any β such that $0 \leqq \beta \leqq 1, f(z) \in W_\mu^{M+\beta}(S)$.

6. Let M and N be integers $\geqq 0$, let α and β satisfy $0 < \alpha, \beta < 1$, and let μ and ν be real numbers. Let $f(z) \in W_\mu^{M+\alpha}(S)$ and $g(z) \in W_\nu^{N+\beta}(S)$. Let $\alpha' = \min(\alpha, \beta)$, $M' = \min(M, N)$ and $\mu' = \min(\mu, \nu)$. Then

$$f(z) + g(z) \in W_{\mu'}^{M'+\alpha'}(S)$$

and

$$f(z)g(z) \in W_{\mu\nu}^{M'+\alpha'}(S) .$$

Proof. The first statement follows from the fact that $W_{\mu'}^{M'+\alpha'}(S) \supseteqq W_\mu^{M+\alpha}(S) \cap W_\nu^{N+\beta}(S)$, and because the W-spaces are linear. For the second statement we observe first that $f(z)g(z) \in C^{M'+\alpha'}(S)$. Then

$$f(z)g(z) = O(z^\mu)O(z^\nu) = O(z^{\mu+\nu}) ,$$

and

$$\begin{aligned} |f(z)g(z) - f(\zeta)g(\zeta)| &\leqq |f(z)|\,|g(z) - g(\zeta)| + |g(\zeta)|\,|f(z) - f(\zeta)| \\ &\leqq K_1 |z|^{\mu+\nu-\beta} |z - \zeta|^\beta + K_2 |z|^{\mu+\nu-\alpha} |z - \zeta|^\alpha \\ &\leqq K_3 |z|^{\mu+\nu-\alpha'} |z - \zeta|^{\alpha'} \end{aligned}$$

since

$$|z - \zeta|^\beta = |z - \zeta|^{\beta-\alpha'} |z - \zeta|^{\alpha'} \leqq \delta^{\beta-\alpha'} |z|^{\beta-\alpha'} |z - \zeta|^{\alpha'}$$

and

$$|z - \zeta|^\alpha \leqq \delta^{\alpha-\alpha'} |z|^{\alpha-\alpha'} |z - \zeta|^{\alpha'} .$$

The proof then follows easily from induction.

We now state three lemmas. The analogous theorems for the real case follow without difficulty.

LEMMA 1. *Let $\mu > 0$ and let $f(z) \in W_\mu^{M+\alpha}(S)$. Suppose also that $|f(z)| \geqq \delta_1 |z|^\mu$, $z \in S$, for some $\delta_1 > 0$. Let S' be the range of S and suppose $g(z) \in W_\nu^{N+\beta}(S)$. Then*

$$h(z) = g(f(z)) \in W_{\mu\nu}^{M'+\alpha\beta}(S)$$

where $M' = \min(M, N)$.

Proof. It is clear that $h(z) = O(z^{\mu\nu})$. Next, for $|z - \zeta| < \delta|z|, \delta|\zeta|$

$$\begin{aligned} |h(z) - h(\zeta)| &= |g(f(z)) - g(f(\zeta))| \\ &\leqq K_1 \max(|f(z)|^{\nu-\beta}, |f(\zeta)|^{\nu-\beta}) |f(z) - f(\zeta)|^\beta \\ &\leqq K_2 \max(|z|^{\mu(\nu-\beta)}, |\zeta|^{\mu(\nu-\beta)}) |z|^{(\mu-\alpha)\beta} |z - \zeta|^{\alpha\beta} \\ &\leqq K_3 |z|^{\mu\nu-\alpha\beta} |z - \zeta|^{\alpha\beta} \end{aligned}$$

provided $|f(z) - f(\zeta)| \leqq \delta |f(z)|, \delta |f(\zeta)|$. In the contrary case, however, suppose $|f(z)| \leqq |f(\zeta)|$ and $|f(z) - f(\zeta)| > \delta |f(z)|$. Then

$$\begin{aligned}\frac{|g(f(z)) - g(f(\zeta))|}{|z-\zeta|^{\alpha\beta}} &\leqq K_4 |g(f(z)) - g(f(\zeta))| \, |z|^{(\mu-\alpha)\beta} |f(z) - f(\zeta)|^{-\beta} \\ &\leqq K_5 \max(|f(z)|^\nu, |f(\zeta)|^\nu) |f(z)|^{-\beta} |z|^{(\mu-\alpha)\beta} \\ &\leqq K_6 \max(|z|^{\mu\nu}, |\zeta|^{\mu\nu}) |z|^{-\alpha\beta} \\ &\leqq K_7 |z|^{\mu\nu-\alpha\beta}\end{aligned}$$

and thus $h(z) \in W^{\alpha\beta}_{\mu\nu}(S)$.

Writing $f = \varphi + i\psi$, φ, ψ real, we have

$$\frac{\partial}{\partial x} h(z) = g_\varphi(f(z))\varphi_x(z) + g_\psi(f(z))\psi_x(z) .$$

Now by definition $g_\varphi \in W^{N-1+\beta}_{\nu-1}(S')$, and thus $g_\varphi(f(z)) \in W^{M'-1+\alpha\beta}_{(\nu-1)\mu}(S)$ as well as $\varphi_x, \psi_x \in W^{M-1+\alpha}_{\mu-1}(S)$, and thus, by Proposition 6,

$$\frac{\partial}{\partial x} h(z) \in W^{M'-1+\alpha\beta}_{\mu\nu-1}(S) .$$

The lemma follows by similar arguments.

LEMMA 2. *Let $f(z)$ map S onto a set S' in such a way that $f(z)$ is conformal on the interior of S, and suppose $f(z) \in W^{M+\alpha}_{\mu}(S)$, $\mu > 0$. Assume also that $|f'(z)| \geqq \delta_1 |z|^{\mu-1}$, $z \in S$, for some $\delta_1 > 0$. Let $g(\zeta)$ be the function inverse to $f(z)$ which maps the interior of S' into S, and assume that $g(\zeta) \in C^{M+\alpha}(S')$ (this is the case if S and S' have sufficiently smooth boundaries).*

Then

$$g(\zeta) \in W^{M+\alpha}_{1/\mu}(S') .$$

Proof. Let z_0 be fixed. Then for $|z|$ sufficiently small we have $|f(z)| \leqq |f(z_0)|$. Thus

$$\begin{aligned}|f(z_0)| - |f(z)| &= \big| |f(z_0)| - |f(z)| \big| \leqq |f(z_0) - f(z)| \\ &= \left| \int_z^{z_0} f'(w) dw \right| \\ &\leqq \int_z^{z_0} K_1 |w|^{\mu-1} |dw| \leqq K_2(1 + |z_0|^\mu - |z|^\mu)\end{aligned}$$

where the path of integration is taken to be a union of paths $\arg w =$ const. and $|w| =$ const. Thus

$$|f(z)| \geqq K_3 + K_2 |z|^\mu \geqq K_2 |z|^\mu .$$

Since $|f(z)| \leqq K_4 |z|^\mu$ we have

$$| g'(\zeta) | = |f'(z)|^{-1} \leqq \delta_1^{-1} | z |^{1-\mu} \leqq K_5 | \zeta |^{1/\mu-1} .$$

By Propositions 2 and 3 we have $g(\zeta) \in W^{\alpha}_{1/\mu}(S')$.

Next,

$$\frac{| g'(\zeta_1) - g'(\zeta_2) |}{| \zeta_1 - \zeta_2 |^{\alpha}} = \frac{| f'(z_2) - f'(z_1) |}{| f'(z_1) | | f'(z_2) | | z_1 - z_2 |^{\alpha}} \frac{| z_1 - z_2 |^{\alpha}}{| \zeta_1 - \zeta_2 |^{\alpha}}$$
$$\leqq K_6 | z_1 |^{1-\mu} | z_2 |^{1-\mu} | z_1 |^{\mu-1-\alpha} | f'(z) |^{-\alpha}$$

where z lies between z_1 and z_2. Since $1 - \delta \leqq | z_1/z_2 | \leqq 1 + \delta$ we have

$$\frac{| g'(\zeta_1) - g'(\zeta_2) |}{| \zeta_1 - \zeta_2 |^{\alpha}} \leqq K_7 | z_1 |^{1-\mu-\alpha-(\mu-1)\alpha} \leqq K_8 | \zeta_1 |^{(1/\mu)-1-\alpha} .$$

Thus $g(z) \in W^{1+\alpha}_{1/\mu}(S')$. The proof follows by induction.

LEMMA 3. *Let $f(z) \in W^{N+\alpha}_{\mu}(S)$ and let $P(z)$ be a polynomial of degree $< \mu$ with $P(0) = 1$. Let γ be a positive real number. Then there exists a function $f_1(z) \in W^{N+\alpha}_{\mu}(S)$ and a polynomial $P_1(z)$ of degree $< \mu$ such that*

$$(P(z) + f(z))^{\gamma} = P_1(z) + f_1(z) .$$

Proof. The proof follows easily from the binomial theorem.

In obtaining the asymptotic expansions we shall come across certain integrals which were studied in [8], [5], and [12]. To estimate these integrals we use the following lemmas. The first was proved in [8] and [5]. The second is a generalization of a theorem in [8], [5], and [12] and will be proved in § 9. The integrals are Lebesgue integrals extended over positive values of t. The variable z lies on the logarithmic Riemann surface with branch point at the origin. The kernel of the integrands is the function $\log (t - z)$ which we define in the following way. For fixed t we make cuts along the Riemann surface from $te^{2\pi ik}$ to $\infty e^{2\pi ik}$, $k = 0, \pm 1, \pm 2, \cdots$. The logarithm is uniquely defined, except for z lying on these cuts, as the analytic continuation of the logarithm which is real for $0 < | z | < t$, $\arg z = 0$.

LEMMA 4. *Let A be a positive number, μ a real number > -1, and n a nonnegative integer. For $0 < \arg z < 2\pi$, let*

$$f(z) = \int_0^A t^{\mu}(\log t)^n \log (t - z) dt .$$

Then there exists a polynomial $P(\log z)$ and a power series $p(z)$ which converges for $| z | < A$, such that

$$f(z) = z^{\mu+1}P(\log z) + p(z) .$$

If μ is an integer the polynomial P is of degree $n+1$; otherwise it is of degree n.

LEMMA 5. *Let μ be a real number > -1 which is not an integer, and let $\beta(t) \in W_{\mu}^{N-1+\alpha}$ for $0 < t \leqq A$. For $0 < \arg z < \pi$, let*

$$g(z) = \int_0^A \beta(t) \log (t - z) dt .$$

Then there exists a polynomial $q(z)$ of degree $< \mu + 1$ such that

$$\varphi(z) \equiv g(z) - q(z) \in W_{\mu+1}^{N+\alpha}((0, \pi)) .$$

A similar result obtains for $-\pi < \arg z < 0$, with the same polynomial $q(z)$.

5. **Preliminary results.** It follows from Warschawski [10] that $F^{-1}(z)$, which maps D onto the upper half plane, satisfies the relation

$$F^{-1}(z) = z^{1/\tau}\varphi(z) \tag{5.1}$$

where $\varphi(z)$ is continuous in $D \cup \Gamma_1 \cup \Gamma_2 \cup \{0\}$ and $\varphi(0) \neq 0$. We shall show in this section that $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D \cup \Gamma_1 \cup \Gamma_2)$.

It follows easily from the Cauchy integral theorem that

$$F^{-1}(z) \in W_{1/\tau}^{N+\alpha}([\lambda, \pi\tau - \lambda])$$

where λ is a small positive number: one simply examines the integral

$$\frac{d^n}{dz^n} F^{-1}(z) = \frac{n!}{2\pi i} \oint \frac{F^{-1}(\zeta) d\zeta}{(\zeta - z)^{n+1}}$$

taken over a circle about z of radius $\delta |z|$, bearing in mind that $F^{-1}(z) = 0(z^{1/\tau})$ as $z \to 0, z \in D \cup \Gamma_1 \cup \Gamma_2$. Thus it will suffice to show that $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$ and $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D'')$ where $D' = D \cap \{z: \arg z \geqq \pi\tau - 2\lambda\}$ and $D'' = D \cap \{z: \arg z \leqq 2\lambda\}$. Because of the symmetry between Γ_1 and Γ_2 we need only show that $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$.

Next, if we have $V(z) = Im\, F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$, then, by Warschawski's result above and the Cauchy-Riemann equations, we have $Re\, F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$, and thus $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D')$. Thus we shall show $V(z) \in W_{1/\tau}^{N+\alpha}(D')$.

Now we make a transformation which has the effect of straightening out Γ_2. Let $y = \beta(x)$ be a parametrization of Γ_2 (if $\tau = 1/2$ or $3/2$ this is impossible; but a small rotation about the origin would take care of this difficulty). Then it can be shown that $\beta(x) \in C^{N+\alpha}$ for $0 \leqq x \leqq A$, where A is a small positive number; furthermore, by the hypotheses of § 2, for $0 \leqq n \leqq N$, $0 \leqq x_1, x_2 \leqq x_0$

$$\sup_{x_1 \neq x_2} \frac{|\beta^{(n)}(x_1) - \beta^{(n)}(x_2)|}{|x_1 - x_2|^\alpha} < \infty .$$

We make the transformation $\xi = x, \eta = y - \beta(x)$, and set $v(\xi, \eta) = V(x, y)$. Then v is defined (at least) for $0 < \xi^2 + \eta^2 < A_1$, $-\xi \tan \delta \leqq \eta < 0$, provided A_1 and δ are chosen small enough. The points (ξ, η) are images of a subset of the points (x, y) such that $(\pi\tau - \delta_1)x \leqq y \leqq \beta(x)$, where δ_1 is a small positive number. Since $\beta(x) = O(x)$, we find that $\delta_2 \leqq y/x \leqq 1/\delta_2$ for some $\delta_2 > 0$, and thus, since

$$\xi^2 + \eta^2 = x^2 + y^2 - 2y\beta(x) + (\beta(x))^2 ,$$

$$\delta_3 \leqq \frac{x^2 + y^2}{\xi^2 + \eta^2} \leqq 1/\delta_3$$

for some $\delta_3 > 0$. Since $V(x, y) = O(z^{1/\tau})$, we have $v(\xi, \eta) = O(\zeta^{1/\tau})$, where $\zeta = \xi + i\eta$.

We now state a lemma which is a special case of a theorem of Agmon, Douglis and Nirenberg ([1], pp. 657–660). Let $0 < R < 1$ and let S be the semicircle $\xi^2 + \eta^2 < R, \eta \leqq 0$. For $\zeta \in S$ let d_ζ denote the distance from ζ to the circular part of the boundary of S.

LEMMA 6. *Let $u(\xi, \eta)$ be a solution of a uniformly elliptic partial differential equation*

$$Lu = au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + fu = 0 ,$$

whose coefficients are $C^{N-2+\alpha}$ in S with uniform α-Hölder constants. Let $u(\xi, 0) = 0$ for $-R < \xi < R$. If $u \in C^{2+\alpha}(S)$ then $u \in C^{N+\alpha}(S)$, and there exists a constant K, independent of u and R, such that

$$|u|_{N+\alpha} \leqq K \sup_{\zeta \in S} |u(\zeta)|$$

where

$$|u|_{N+\alpha} = \sup_{4|\zeta_1 - \zeta_2| < d_{\zeta_1}, d_{\zeta_2}} d_{\zeta_1}^{N+\alpha} \frac{|D^N u(\zeta_1) - D^N u(\zeta_2)|}{|\zeta_1 - \zeta_2|^\alpha} + \sum_{k=0}^{N} \sup_{\zeta \in S} d_\zeta^k |D^k u(\zeta)| ;$$

the suprema are taken over all kth and Nth order derivatives of u.

Since $V(x, y)$ is harmonic, we have

$$Lv \equiv v_{\xi\xi} + (1 + \beta'(\xi)^2)v_{\eta\eta} - 2\beta'(\xi)v_{\xi\eta} - \beta''(\xi)v_\eta = 0$$

for $0 < \xi^2 + \eta^2 < A_1$, $-\xi \tan \delta \leqq \eta \leqq 0$. Also

$$v(\xi, 0) = V(x, \beta(x)) = 0 .$$

We now apply the lemma to v and the semicircles

$$(\xi - \xi_0)^2 + \eta^2 \leqq \xi_0^2 \sin^2 \delta, \eta \leqq 0$$

where $0 < \xi_0 < (1/2)A_1$; these semicircles are tangent to the rays $\eta = 0$, and $\eta = -\xi \tan \delta$. In each semicircle we have, for some $K_1 > 0$,

$$\sup | v(\zeta) | \leqq K_1 | \zeta |^{1/\tau} .$$

In the semicircle $(\xi - \xi_0)^2 + \eta^2 \leqq ((1/2)\xi_0 \sin \delta)^2, \eta \leqq 0$, we have $d_\zeta \geqq (1/2)\xi_0 \sin \delta$. Thus for $(\xi - \xi_0)^2 + \eta^2 \leqq ((1/2)\xi_0 \sin \delta)^2$, $\eta \leqq 0$, we have

$$\begin{aligned} \zeta^k | D^k v(\zeta) | &\leqq \left(\xi_0\left(1 + \frac{1}{2} \sin \delta\right)\right)^k | D^k v(\zeta) | \\ &\leqq \left(\frac{2}{\sin \delta}\left(1 + \frac{1}{2} \sin \delta\right) d_\zeta\right)^k | D^k v(\zeta) | \leqq K_2 | \zeta |^{1/\tau} \end{aligned}$$

for $0 \leqq k \leqq N$. Thus $v(\zeta) \in W^N_{1/\tau}([-\delta_4, 0])$ where δ_4 is small. By the mean value theorem $v(\zeta) \in W^{N-1+\alpha}_{1/\tau}([-\delta_4, 0])$. To estimate $| D^N v(\zeta_1) - D^N v(\zeta_2) |$ we use the lemma again; the details are similar to those above. Thus we can conclude that $v(\zeta) \in W^{N+\alpha}_{1/\tau}([-\delta_4, 0])$. Since

$$\delta_3 \leqq \frac{x^2 + y^2}{\xi^2 + \eta^2} \leqq \frac{1}{\delta_3} ,$$

it follows, by easy calculations, that for some small positive λ,

$$V(z) \in W^{N+\alpha}_{1/\tau}([\pi\tau - \lambda, \pi\tau]) .$$

Thus we conclude that $F^{-1}(z) \in W^{N+\alpha}_{1/\tau}(D \cup \Gamma_1 \cup \Gamma_2)$.

6. A preliminary transformation. From now on for the sake of definiteness we will assume that Γ_1 is tangent to the positive x-axis at the origin and that Γ_2 is tangent to the ray $\arg z = \pi\tau$ at the origin.

We set $H(z) = (F(z))^{1/\tau}$. Then $H(z)$ maps the upper half plane conformally onto a domain D' which is the image of D under the transformation $z \to z^{1/\tau}$. D' is partially bounded by curves Γ_1' and Γ_2' which have horizontal tangents at the origin. From the binomial theorem it is clear that theorem 1 is equivalent to an asymptotic expansion

$$(6.1) \qquad \begin{aligned} &H(z) = zP(z, z^\tau) + o(z^{(N-1)\tau-\varepsilon}) && (\alpha \text{ irrational}) \\ &H(z) = zP(z, z^\tau, z^p \log z) + o(z^{(N-1)\tau-\varepsilon}) && (\alpha = p/q) \end{aligned}$$

as $z \to 0$, $0 \leqq \arg z \leqq \pi$, where $\varepsilon > 0$ can be chosen arbitrarily small and the polynomial P has a nonvanishing constant term; furthermore we must show that we can differentiate (6.1) N times. Since Theorems 2, 3 and 4 follow directly from Theorem 1, we need only prove (6.1).

By Lemma 1, and since $F^{-1}(z) \in W_{1/\tau}^{N+\alpha}(D \cup \Gamma_1 \cup \Gamma_2)$, we have

$$H^{-1}(z) = F^{-1}(z^\tau) \in W_1^{N+\alpha^2}(D' \cup \Gamma_1' \cup \Gamma_2') .$$

By Lemma 2, $H(z) \in W_1^{N+\alpha^2}([0, \pi])$.

7. An integral representation. We will now construct an integral representation for $H(z)$ based on the equations for Γ_1 and Γ_2. Let $F(z) = \xi + i\eta$. Then we have

$$\eta = \sum_{n=1}^{N-1} c_n \xi^n + \varphi_1(\xi)$$

where $\varphi_1 \in W_N^{N+\alpha}$; this is merely the Taylor series for Γ_1, and is valid for $0 \leqq \xi \leqq \xi_0$.

We will now adopt the convention of dropping subscripts on coefficients whose value is unimportant; then we have

$$\eta = \sum_{n=1}^{N-1} c\xi^n + \varphi_1(\xi) .$$

With $w = H(z) = u + iv$, we have

$$\begin{aligned} w = (\xi + i\eta)^{1/\tau} &= \xi^{1/\tau}\left(1 + i \sum_{n=1}^{N} c\xi^{n-1} + \frac{\varphi_1(\xi)}{\xi}\right)^{1/\tau} \\ &= \xi^{1/\tau}\left(a + \sum_{n=1}^{N} c\xi^n + \varphi_2(\xi)\right) , \end{aligned}$$

and by Lemma 3, $\varphi_2 \in W_{N-1}^{N+\alpha}$. It is readily seen that $Re\, a \neq 0$. Then we have, by separating real and imaginary parts,

$$u = a_1\xi^{1/\tau}(1 + c\xi + c\xi^2 + \cdots + c\xi^{N-2} + \varphi_3(\xi)) \tag{7.1}$$

$$v = \xi^{1/\tau}(c + c\xi + c\xi^2 + \cdots + c\xi^{N-2} + \varphi_4(\xi)) \tag{7.2}$$

with $\varphi_3, \varphi_4 \in W_{N-1}^{N+\alpha}$. Next,

$$u^\tau = a_1^\tau \xi(1 + c\xi + c\xi^2 + \cdots + c\xi^{N-2} + \varphi_5(\xi))$$

with $\varphi_5 \in W_{N-1}^{N+\alpha}$. As $a_1^\tau \neq 0$, we have, by the inverse function theorem,

$$\xi = u^\tau(c + cu^\tau + cu^{2\tau} + \cdots + cu^{(N-2)\tau} + \varphi_6(u^\tau)) \tag{7.3}$$

where φ_6, considered as a function of u^τ, belongs to $W_{N-1}^{N+\alpha}$. Thus by Lemma 1, $\varphi_7(u) = \varphi_6(u^\tau) \in W_{(N-1)\tau}^{N+\alpha^2}$. Substituting (7.3) in the right side of (7.2), we obtain

$$(7.4)\qquad \begin{aligned} v = u\Big(\sum_{j=0}^{N-2} cu^{j\tau} + \varphi_7(u)\Big)^{1/\tau} \\ \times \Big(\sum_{j=0}^{N-2} cu^{j\tau}\Big(\sum_{k=0}^{N-2} cu^{k\tau} + \varphi_7(u)\Big)^j + \varphi_4(\xi)\Big) . \end{aligned}$$

We set

$$\varphi_8(u) = \varphi_4(\xi) = \varphi_4\Big(u^\tau\Big(\sum_{j=0}^{N-2} cu^{j\tau} + \varphi_7(u)\Big)\Big) .$$

It is easily checked that $\xi(u) \in W_\tau^{N+\alpha^2}$ as a function of u, and thus $\varphi_8(u) \in W_{(N-1)\tau}^{N+\alpha 2}$. Thus, expanding the right side of (7.4), it follows that

$$v = u(c + cu^\tau + \cdots + cu^{(N-1)\tau} + \varphi_9(u))$$

with $\varphi_9 \in W_{(N-1)\tau}^{N+\alpha 2}$. Finally, $(dv/du)\,|_{u=0} = 0$, and thus

$$(7.5)\qquad v = u(cu^\tau + cu^{2\tau} + \cdots + cu^{(N-1)\tau} + \varphi_9(u)) .$$

This equation is valid for v and u defined on the segment $y = 0$, $0 \leqq x \leqq A$, provided A is chosen small enough.

If $0 < \tau < 1/2$ or $3/2 < \tau \leqq 2$ we can repeat the same argument on Γ_2: note that we never used the fact that Γ_1 has a horizontal tangent, but only that Γ_1' (and Γ_2') has a horizontal tangent at the origin. If $1/2 < \tau < 3/2$, we replace ξ by $|\,\xi\,|$; and for $0 < \tau \leqq 2$, we replace u by $|\,u\,|$.

Finally, if $\tau = 1/2$ or $3/2$ we begin with the equation

$$\xi = \sum_{n-1}^{N} c\eta^n + \varphi_{10}(\eta)$$

and carry through with the roles of ξ and η reversed. Thus we have, for $-A \leqq x \leqq 0, y = 0$,

$$(7.6)\qquad v = u(cu^\tau + cu^{2\tau} + \cdots + cu^{(N-1)\tau} + \varphi_{11}(u))$$

with $\varphi_{11} \in W_{(N-1)\tau}^{N+\alpha 2}$.

We now consider the Green's function for the upper half plane

$$(7.7)\qquad G(t, z) = -\frac{1}{2\pi}\{\log|\,t - z\,| + \log|\,t - \bar{z}\,|\} ,$$

where $t = x_t + iy_t$. It is easily seen that $(\partial/\partial y_t)G(x_t, z) = 0$. We apply Green's theorem to the functions $G(t, z)$ and $u(t) = Re\, H(t)$ on the semi-circle $0 < |\,t\,| < A$, $y_t > 0$, and obtain

$$u(z) = \int_{-A}^{A} G(t, z)\frac{\partial}{\partial y_t}u(t)dt + \int_{\substack{y_t>0 \\ |t|=A}} (uG_{n_t} - Gu_{n_t})ds_t$$

where s_t represents arc length and n_t the outward normal. By (7.7) we have

$$\int_{\substack{y_t>0\\|t|=A}} (uG_{n_t} - Gu_{n_t})ds_t = p(z) + p(\bar{z})$$

where $p(z)$ is a power series which converges for $|z| < A$. Also, for $y_t = 0$,

$$\begin{aligned} G(t, z) &= -\frac{1}{2\pi}\{\log|t - z| + \log|t - \bar{z}|\} \\ &= -\frac{1}{2\pi}\log|t - z|^2 \\ &\equiv -\frac{1}{2\pi}\{\log(t - z) + \log(t - \bar{z})\}\,. \end{aligned}$$

Here we define $\log(t - z)$ as the analytic continuation of the logarithm which is real for $0 < |z| < t$, $\arg z = 0$. The congruence holds modulo $2\pi i$; however, each of the logarithms on the right side has imaginary part $>-\pi$ and $<\pi$. Thus we may replace the congruence by equality. With these observations in mind, we obtain

$$\begin{aligned} u(z) = -\frac{1}{2\pi}\int_{-A}^{A}\frac{\partial}{\partial y_t}u(t) \\ \times\{\log(t - z) + \log(t - \bar{z})\}dt + p(z) + p(\bar{z})\,. \end{aligned} \tag{7.8}$$

Since $u(z) = Re\,H(z)$ and $p(z)$ has real coefficients, we replace (7.8) by the equation of which it is the real part, namely

$$H(z) = -\frac{1}{\pi}\int_{-A}^{A}\frac{\partial}{\partial y_t}u(t)\log(t - z)dt + p(z) + \text{const.}\,,$$

where the constant takes care of the nonuniqueness of the conjugate harmonic function of $u(z)$. We now drop this constant, changing $p(z)$ if necessary, and use (7.5) and (7.6), together with

$$\frac{\partial}{\partial y_t}u(t) = -\frac{\partial}{\partial x_t}v(t)\,,$$

to obtain

$$\begin{aligned} H(z) &= \frac{1}{\pi}\int_{-A}^{0}u_t(t, 0)\left\{\sum_{j=1}^{N-1}cu^{j\tau} + \varphi(u)\right\}\log(t - z)dt \\ &\quad + \frac{1}{\pi}\int_{0}^{A}u_t(t, 0)\left\{\sum_{j=1}^{N-1}cu^{j\tau} + \psi(u)\right\}\log(t - z)dt + p(z)\,. \end{aligned} \tag{7.9}$$

Here

$$\varphi(u) = \frac{d}{du}(u\varphi_{11}(u)) \in W^{N-1+\alpha^2}_{(N-1)\tau}$$

and

$$\psi(u) = \frac{d}{du}(u\varphi_9(u)) \in W^{N-1+\alpha^2}_{(N-1)\tau} .$$

Furthermore, (7.9) is valid for $0 \leqq \arg z \leqq \pi$, $0 < |z| < A$.

8. Obtaining the asymptotic expansions. We have, for $-A \leqq t < 0$, $H(t) \in W_1^{N+\alpha^2}$ and thus $u(t) \in W_1^{N+\alpha^2}$. Hence

$$(u(t))^{n\tau} \in W^{N+\alpha^3}_{n\tau}$$

$$u_t \in W_0^{N-1+\alpha^2}$$

and thus

$$(8.1) \qquad u_t(t, 0)\left\{\sum_{j=1}^{N-1} cu^{j\tau} + \varphi(u)\right\} \in W_\tau^{N-1+\alpha^3}$$

as a function of t, $-A \leqq t < 0$. Similarly

$$(8.2) \qquad u_t(t, 0)\left\{\sum_{j=1}^{N-1} cu^{j\tau} + \psi(u)\right\} \in W_\tau^{N-1+\alpha^3}$$

for $0 < t \leqq A$. Thus by Lemma 5, if $\tau \neq 1, 2$,

$$H(z) = az + bz^2 + \chi_1(z)$$

where $\chi_1(z) \in W^{N+\alpha^3}_{1+\tau}((0, \pi))$. As $H(z)$ has α-Hölder continuous Nth derivatives for $0 \leqq \arg z \leqq \pi$, we must have $\chi_1(z) \in W^{N+\alpha^3}_{1+\tau}([0, \pi])$.

If $\tau = 1$ or 2 Lemma 5 will not apply. However, if ε is any small positive number we can replace the $W_\tau^{N-1+\alpha^3}$ of (8.1) and (8.2) with $W_{\tau-\varepsilon}^{N-1+\alpha^3}$, and thus we can always write

$$(8.3) \qquad H(z) = az + bz^2 + \chi_1(z)$$

where $\chi_1(z) \in W^{N+\alpha^3}_{1+\tau-\varepsilon}([0, \pi])$.

We now prove Theorem 1 by induction. In the future we shall use the symbol α to represent any number between 0 and 1, and ε to represent an arbitrarily small positive number such that $n\tau - \varepsilon$ is not an integer for $0 \leqq n \leqq N$. In particular we write $\chi_1(z) \in W^{N+\alpha}_{1+\tau-\varepsilon}([0, \pi])$.

First let α be irrational. Assume that for some m, with $0 < m < N-1$, that

$$H(z) = zP_m(z, z^\tau) + \chi_m(z)$$

where $P_m(z, z^\tau)$ is a polynomial in its arguments such that $P_m(0, 0) \neq 0$ and $\chi_m(z) \in W^{N+\alpha}_{1+m\tau-\varepsilon}([0, \pi])$. That this is the case for $m = 1$ follows from the fact that the constant a of (8.3) is not equal to zero; this follows from (5.1) and the definition of $H(z)$.

Then by the inductive hypothesis we have, for $-A \leqq t < 0$,

$$u(t, 0) = t\Big(\sum_{k+l\tau<m\tau} c_{kl}t^{k+l\tau} + \varphi_{12}(t)\Big)$$

with $\varphi_{12}(t) \in W^{N+\alpha}_{m\tau-\varepsilon}$ and $c_{00} \neq 0$. A similar equation holds for $0 < t \leqq A$. Then

$$u^{n\tau} = t^{n\tau}\Big\{\sum_{k+l\tau<m\tau} ct^{k+l\tau} + \varphi^{n}_{13}(t)\Big\}$$

with $\varphi^n_{13} \in W^{N+\alpha}_{m\tau-\varepsilon}$, and

$$u_t(t, 0) = \sum_{k+l\tau<m\tau} c(k + 1 + l\tau)t^{k+l\tau} + \varphi_{14}(t)$$

with $\varphi_{14} \in W^{N-1+\alpha}_{m\tau-\varepsilon}$. Also, since $\varphi(u) \in W^{N+\alpha}_{(N-1)\tau}$ as a function of u, $\varphi(u(t)) \in W^{N-1+\alpha}_{(N-1)\tau}$ as a function of t. Thus, cross-multiplying, collecting terms, and using Lemmas 1, 2 and 3, we obtain

$$u_t(t, 0)\Big\{\sum_{j=1}^{N-1} cu^{j\tau} + \varphi(u)\Big\} = \sum_{k+l\tau<m\tau} ct^{k+l\tau} + \varphi_{14}(t) + \varphi_{15}(t)$$

with $\varphi_{14}(t) \in W^{N-1+\alpha}_{(m+1)\tau-\varepsilon}$ and $\varphi_{15} \in W^{N-1+\alpha}_{(N-1)\tau-\varepsilon}$. By the inductive hypothesis $m + 1 \leqq N - 1$ and we may write $\varphi_{16} = \varphi_{14} + \varphi_{15} \in W^{N-1+\alpha}_{(m+1)\tau-\varepsilon}$. Clearly a similar equation holds for $0 < t \leqq A$, and, applying Lemmas 4 and 5 we obtain

$$H(z) = \sum c_{kl}z^{k+l\tau} + \chi_{m+1}(z)$$

with $\chi_{m+1}(z) \in W^{N+\alpha}_{(m+1)\tau+1-\varepsilon}((0, \pi))$. As H has continuous Nth derivatives, $\chi_{m+1} \in W^{N+\alpha}_{(m+1)\tau+1-\varepsilon}([0, \pi])$. By Warschawski's results $c_{10} \neq 0$. Finally, setting $m = N - 2$, and $\chi_{N-1}(z) = o(z^{(N-1)\tau+1-2\varepsilon})$, we have, with 2ε replaced by ε,

$$H(z) = zP_{N-1}(z, z^\tau) + o(z^{(N-1)\tau+1-\varepsilon})$$

as $z \to 0$, $0 \leqq \arg z \leqq \pi$, and, for $0 \leqq n \leqq N$

$$\frac{d^n}{dz^n}(H(z) - zP_{N-1}(z, z^\tau)) = o(z^{(N-1)\tau+1-n-\varepsilon})$$

as $z \to 0$, $0 \leqq \arg z \leqq \pi$.

Now let $\tau = p/q$, a reduced fraction. For $0 < m < N - 1$ we assume that

$$H(z) = zP_m(z, z^\tau, z^p \log z) + \chi_m(z)$$

with $\chi_m(z) \in W^{N+\alpha}_{1+m\tau-\varepsilon}([0, \pi])$, and $P_m(0, 0, 0) \neq 0$. Then, for $-A \leqq t \leqq 0$,

$$u(t, 0) = t\sum ct^{j+k\tau}(t^p \log t)^l + \varphi_{17}(t)$$

$$u_t(t, 0) = \sum ct^{j+k\tau}(t^p \log t)^l + \varphi_{18}(t)$$

$$u^{n\tau} = t^{n\tau}\{\sum ct^{j+k\tau}(t^p \log t)^l + \varphi_{19}(t)\}$$

where $\varphi_{17}, \varphi_{19} \in W^{N+\alpha}_{1+m\tau-\varepsilon}$ and $\varphi_{18} \in W^{N-1+\alpha}_{m\tau-\varepsilon}$. Thus

$$u_t(t, 0)\left\{\sum_{j=1}^{N-1} cu^{j\tau} + \varphi(u(t))\right\} = \sum ct^{j+k\tau}(\log t)^l + \varphi_{20}(t)$$

where $j \geqq 0, 1 \leqq k \leqq q, 0 \leqq l \leqq j/p, j + k\tau < (m + 1)\tau$ and $\varphi_{20} \in W^{N-1+\alpha}_{(m+1)\tau}$. A similar equation obtains for $0 < t \leqq A$. Applying Lemmas 4 and 5 we obtain

$$H(z) = \sum a_{jkl} z^{j+1+k\tau}(\log z)^{l'} + \chi_{m+1}(z)$$

with $\chi_{m+1} \in W^{N+\alpha}_{(m+1)\tau+1-\varepsilon}([0, \pi])$. Terms of the form $t^{j+k\tau}(\log t)^l$, with $k < q$, contribute terms of the form $z^{j+1+k\tau}(\log z)^{l'}$ with $l' \leqq l < l + 1$. With $k = q$, however, higher powers of the logarithm appear, and we must then show $j + 1 + k\tau \geqq l' + 1$, where $l' \leqq l + 1$. But then

$$\begin{aligned} j + 1 + k\tau &= j + 1 + p \\ &\geqq pl + 1 + p \geqq p(l + 1) + 1 \geqq l + 2 \geqq l' + 1 . \end{aligned}$$

Thus we can write

$$H(z) = zP_{m+1}(z, z^\tau, z^p \log z) + \chi_{m+1}(z) ,$$

and, for $m = N - 2$ and $0 \leqq n \leqq N$,

$$\frac{d^n}{dz^n} H(z) = \frac{d^n}{dz^n}(zP_{N-1}(z, z^\tau, z^p \log z)) + o(z^{(N-1)\tau+1-n-\varepsilon})$$

as $z \to 0, 0 \leqq \arg z \leqq \pi$.

9. Proof of Lemma 5. Suppose that $q(z)$ exists and $\varphi(z) = g(z) - q(z) \in W^N_{\mu+1}((0, \pi])$. Then it follows that $\varphi(z) \in W^{N-1+\alpha}_{\mu+1}((0, \pi])$. Hence we need only show that there exists a polynomial $q(z)$ such that

$$\varphi^{(N)}(z) \equiv g^{(N)}(z) - q^{(N)}(z) \in W^\alpha_{\mu-N+1}((0, \pi]) .$$

We break the proof into three parts, numbered I, II and III.

I. First we assume $0 < \arg z \leqq \delta$. We have

$$g^{(N)}(z) = -\int_0^A (N - 1)! \frac{\beta(t)}{(t - z)^N} dt .$$

We write, with $r = |z|$,

$$\int_0^A = \int_0^{x-r/2} + \int_{x-r/2}^{x+r/2} + \int_{x+r/2}^A = I_1 + I_2 + I_3 .$$

Throughout the proof we shall use constants $C_1, C_2, C_3, \cdots$, which are independent of z; to simplify notation we shall use one symbol C to denote all such constants. I_1 is bounded in absolute value by

$$C\int_0^{x-r/2} t^\mu (\delta x)^{-N} dt \leqq C r^{\mu+1-N} ,$$

where we have used $|\beta(t)| \leqq Ct^\mu$

For I_2 we expand $\beta(t)$ in a Taylor series about the point x and get

$$\begin{aligned} I_2 = \sum_{k=0}^{N-2} \frac{(N-1)!}{k!} \beta^{(k)}(x) \int_{x-r/2}^{x+r/2} \frac{(t-x)^k}{(t-z)^N} dt \\ + \int_{x-r/2}^{x+r/2} \frac{\beta^{(N-1)}(\tau)(t-x)^{N-1}}{(t-z)^N} dt \end{aligned} \tag{9.1}$$

where τ lies between x and t.

The integral term J_k arising from the kth term of (9.1) can be written in the form

$$\begin{aligned} J_k &= C\beta^{(k)}(x) \int_{-r/2}^{r/2} \frac{t^k}{(t-iy)^N} dt \\ &= C \sum_{j=0}^{k} \binom{k}{j} \beta^{(k)}(x) \int_{-r/2}^{r/2} \frac{(t-iy)^j (iy)^{k-j}}{(t-iy)^N} dt \\ &= \sum_{j=0}^{k} C_j \beta^{(k)}(x)(iy)^{k-j} \left[\left(\frac{r}{2} - iy\right)^{j-N+1} - \left(-\frac{r}{2} - iy\right)^{j-N+1} \right], \end{aligned}$$

and thus, since $|\beta^{(k)}(x)| \leqq Cx^{\mu-k}$,

$$|J_k| \leqq C \sum_{j=0}^{k} x^{\mu-k} y^{k-j} 2\left(\frac{r}{2}\right)^{j-N+1} \leqq C r^{\mu-N+1} .$$

The last integral on the right side of (9.1) we write in the form

$$\begin{aligned} &\int_{-r/2}^{r/2} \frac{\beta^{(N-1)}(\tau_1 + x) t^{N-1}}{(t-iy)^N} dt \\ &\qquad = \int_0^{r/2} t^{N-1} \left\{ \frac{\beta^{(N-1)}(\tau_1 + x)}{(t-iy)^N} - \frac{\beta^{(N-1)}(\tau_2 + x)}{(t+iy)^N} \right\} dt \end{aligned}$$

where $0 < \tau_1, -\tau_2 < t$. We write the term in brackets in two parts, and get

$$\int_0^{r/2} t^{N-1} \left\{ \frac{\beta^{(N-1)}(\tau_1 + x)}{(t-iy)^N} - \frac{\beta^{(N-1)}(\tau_1 + x)}{(t+iy)^N} \right\} dt \tag{9.2}$$

$$+ \int_0^{r/2} t^{N-1}\left\{\frac{\beta^{(N-1)}(\tau_1 + x)}{(t + iy)^N} - \frac{\beta^{(N-1)}(\tau_2 + x)}{(t + iy)^N}\right\}dt .$$

The first integral is equal to

$$\begin{aligned} 2i\, Im\int_0^{r/2} t^{N-1}\frac{\beta^{(N-1)}(\tau_1 + x)}{(t - iy)^N}dt \\ = 2i\, Im\int_0^{r/2}\frac{t^{N-1}(t + iy)^N\beta^{(N-1)}(\tau_1 + x)dt}{(t^2 + y^2)^N} . \end{aligned}$$

We make the change of variables $t = sy$ to obtain

$$\begin{aligned} 2i\, Im\int_0^{r/2y}\frac{s^{N-1}(s + i)^N\beta^{(N-1)}(\sigma + x)ds}{(s^2 + 1)^N} \\ = 2i\, Im\sum_{k=1}^{N}\binom{N}{k}\int_0^{r/2y}\frac{s^{2N-1-k}i^k\beta^{(N-1)}(\sigma + x)ds}{(s^2 + 1)^N} \end{aligned}$$

where $\sigma < r/2$; this is bounded in absolute value by

$$C\sum_{k=1}^{N}\int_0^{\infty}\frac{s^{2N-1-k}}{(s^2 + 1)^N}\,| r/2 \pm x |^{\mu-N+1}\, ds \leqq Cr^{\mu-N+1} .$$

The second integral on the right side of (9.2) is bounded absolutely by

$$C\int_0^{r/2} | 2t |^{\alpha}\, | x \pm r/2 |^{\mu-N+1-\alpha}\, t^{-1}dt \leqq Cr^{\mu+1-N} .$$

To handle I_3 we observe that

$$\frac{1}{(t - z)^N} = \sum_{k=0}^{\infty}\binom{k + N - 1}{k}z^k t^{-N-k} .$$

Let m be the integer such that $\mu < m < \mu + 1$ and assume $m \geqq N$. We have

$$\begin{aligned} I_3 = \sum_{k=0}^{m-N}\frac{(k + N - 1)!}{k!}z^k\int_0^A \beta(t)t^{-N-k}\, dt \\ - \sum_{k=0}^{m-N}\frac{(k + N - 1)!}{k!}z^k\int_0^{x+r/2}\beta(t)t^{-N-k}\, dt \\ + \int_{x+r/2}^{A}\sum_{k=m-N+1}^{\infty}\frac{(k + N - 1)!}{k!}z^k\beta(t)t^{-N-k}\, dt . \end{aligned} \tag{9.3}$$

We set

$$q_1(z) \equiv - \sum_{k=0}^{m-N}\frac{(k + N - 1)!}{k!}z^k\int_0^A \beta(t)t^{-N-k}dt .$$

If $m < N$ we set $q_1(z) \equiv 0$, and the last sum of (9.3) begins with

$k = 0$. In any event, $q(z)$ will be taken such that $q^{(N)}(z) \equiv q_1(z)$; its exact form is given in [12].

Thus to prove that $\varphi^{(N)}(z) \equiv g^{(N)}(z) - q_1(z) \in W_{\mu-N+1}((0, \delta))$, we need only estimate the last two terms on the right side of (9.3). We have

$$\left|\sum_{k=0}^{m-N} \frac{(k+N-1)!}{k!} z^k \int_0^{x+r/2} \beta(t) t^{-N-k} dt\right| \leqq C \sum_{k=0}^{m-N} r^k \int_0^{x+r/2} t^{\mu-N-k} dt = Cr^{\mu+1-N} ,$$

and

$$\begin{aligned} &\left|\int_{x+r/2}^{A} \sum_{k=m-N+1}^{\infty} \frac{(k+N-1)!}{k!} z^k \beta(t) t^{-N-k} dt\right| \\ &\leqq C \sum_{k=m-N+1}^{\infty} \frac{(k+N-1)!}{k!} r^k \int_{x+r/2}^{\infty} t^{\mu-N-k} dt \\ &\leqq C \sum_{k=m-N+1}^{\infty} \frac{(k+N-1)!}{k!} \frac{r^{\mu+1-N}}{|\mu-N-k+1|} \left(\cos\delta + \frac{1}{2}\right)^{\mu-N-k+1} \\ &= Cr^{\mu+1-N} \end{aligned}$$

where we assume $\cos\delta > 1/2$. Thus $\varphi^{(N)}(z) \in W_{\mu-N+1}((0, \delta))$.

II. For $\delta \leqq \arg z \leqq \pi$, observe that $\varphi(z)$ is analytic for $|z| > 0$. That $\varphi(z) \in W_{\mu+1}$ follows from [12]. By Cauchy's theorem

$$\varphi^{(N)}(z) = \frac{N!}{2\pi i} \int \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^{N+1}} \tag{9.4}$$

where the integral is taken around a circle with z as center and radius $\delta_1 |z|$, where δ_1 is a small positive number. Then

$$|\varphi^{(N)}(z)| \leqq C(2\pi\, \delta_1 |z|)(|z| (1 \pm \delta_1))^{\mu+1} (\delta_1 |z|)^{-N-1} = Cr^{\mu+1-N} .$$

III. We will now show that

$$|\varphi^{(N)}(z) - \varphi^{(N)}(\zeta)| \leqq C |z|^{\mu+1-N-\alpha} |z - \zeta|^{\alpha} ,$$

for $|z - \zeta| \leqq \delta |z|, \delta |\zeta|$. First, this inequality follows immediately from (9.4) for $\delta \leqq \arg z, \arg \zeta \leqq \pi$. Thus we will restrict ourselves to the range $0 < \arg z, \arg \zeta \leqq \delta$. We have

$$\begin{aligned} \varphi^{(N)}(z) - \varphi^{(N)}(\zeta) = &-\int_0^{x-r/2} (N-1)!\, \beta(t) \left\{\frac{1}{(t-z)^N} - \frac{1}{(t-\zeta)^N}\right\} dt \\ &- \int_{x-r/2}^{x+r/2} (N-1)!\, \beta(t) \left\{\frac{1}{(t-z)^N} - \frac{1}{(t-\zeta)^N}\right\} dt \\ &+ \sum_{k=0}^{m-N} \frac{(k+N-1)!}{k!} (z^k - \zeta^k) \int_0^{x+r/2} \beta(t) t^{-N-k} dt \end{aligned}$$

$$- \int_{x+r/2}^{A} \sum_{\substack{k=m-N+1 \\ k \geqq 0}}^{\infty} \frac{(k+N-1)!}{k!} (z^k - \zeta^k)\beta(t)t^{-N-k}\, dt$$
$$= I^1 + I^2 + I^3 + I^4 .$$

Then

$$I^1 = -(N-1)! \int_0^{x-r/2} \beta(t) \sum_{k=1}^{N} \frac{t^{N-1}(-1)^k[\zeta^k - z^k]dt}{(t-z)^N(t-\zeta)^N} .$$

Since $|\zeta - z| \leqq \delta |z|$ and $|\zeta| \leqq (1+\delta)|z|$, we have

$$|\zeta^k - z^k| \leqq |\zeta - z|^\alpha |\zeta - z|^{1-\alpha} \sum_{j=0}^{k-1} |\zeta|^{k-j-1} |z|^j$$
$$\leqq C |z|^{k-\alpha} |\zeta - z|^\alpha .$$

Hence

$$|I^1| \leqq C \sum_{k=1}^{N} |z - \zeta|^\alpha |z|^{k-\alpha} \int_0^{x-r/2} t^{\mu+N-k} \left(\frac{r}{2}\right)^{-N} \left(r\left(\frac{1}{2} - \delta\right)\right)^{-N} dt$$
$$\leqq C |z - \zeta|^\alpha |z|^{\mu-N+1-\alpha} .$$

Similarly, we can assume the sum in I_3 begins with $k = 1$, and we obtain

$$|I^3| \leqq C |z - \zeta|^\alpha \sum_{k=1}^{m-N} |z|^{k-\alpha} \int_0^{x+r/2} t^{\mu-N-k} dt$$
$$\leqq C |z - \zeta|^\alpha |z|^{\mu-N+1-\alpha} .$$

Likewise

$$|I^4| \leqq C |z - \zeta|^\alpha \sum_{\substack{k=m-N+1 \\ k>0}}^{\infty} \frac{(k+N-1)!}{k!} |z|^{k-\alpha} \int_{x+r/2}^{\infty} t^{\mu-N-k} dt$$
$$\leqq C |z - \zeta|^\alpha \sum_{\substack{k=m-N+1 \\ k>0}}^{\infty} \frac{(k+N-1)!}{k!} |z|^{\mu-N+1-\alpha} \frac{\left(\cos\delta + \frac{1}{2}\right)^{\mu-N-k+1}}{|\mu - N - k + 1|}$$
$$= C |z|^{\mu-N+1-\alpha} |z - \zeta|^\alpha .$$

Finally we must evaluate I^2. We write

$$\beta(t) = \sum_{k=0}^{N-2} \frac{\beta^{(k)}(x)}{k!} (t-x)^k + \psi(x, t)$$

where

$$\psi(x, t) = \int_x^t \frac{(t-\sigma)^{N-2}}{(N-2)!} \beta^{(N-1)}(\sigma) d\sigma ,$$

and

$$\beta(t) = \sum_{k=0}^{N-2} \frac{\beta^{(k)}(\xi)}{k!}(t-\xi)^k + \psi(\xi, t) .$$

Then we have

$$\begin{aligned} I^2 &= -\sum_{k=0}^{N-2} \frac{(N-1)!}{k!}\int_{x-r/2}^{x+r/2}\left\{\frac{\beta^{(k)}(x)(t-x)^k}{(t-z)^N} - \frac{\beta^{(k)}(\xi)(t-\xi)^k}{(t-\zeta)^N}\right\}dt \\ &\quad -(N-1)\int_{x-r/2}^{x+r/2}\left\{\frac{\psi(x,t)}{(t-z)^N} - \frac{\psi(\xi,t)}{(t-\zeta)^N}\right\}dt \\ &= \sum_{k=0}^{N-2} J^k + J^{N-1} . \end{aligned}$$

With

$$(t-x)^k = \sum_{j=0}^{k}\binom{k}{j}(t-z)^j(z-x)^{k-j}$$

we have, for $k \leqq N-2$,

$$\begin{aligned} J^k &= -\frac{(N-1)!}{k!}\sum_{j=0}^{k}\binom{k}{j} \\ &\quad \times \int_{x-r/2}^{x+r/2}[\beta^{(k)}(x)(t-z)^{j-N}(z-x)^{k-j} - \beta^{(k)}(\xi)(t-\zeta)^{j-N}(\zeta-x)^{k-j}]dt \\ &= -\frac{(N-1)!}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{j-N+1} \\ &\quad \times \left\{\beta^{(k)}(x)(z-x)^{k-j}\left(\left(x+\frac{r}{2}-z\right)^{j-N+1} - \left(x-\frac{r}{2}-z\right)^{j-N+1}\right)\right. \\ &\qquad \left. -\beta^{(k)}(\xi)(\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right)\right\} \end{aligned}$$

To the term in brackets we add and subtract

$$\beta^{(k)}(x)(\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right) .$$

Then the jth bracketed term becomes

$$\begin{aligned} &\beta^{(k)}(x)\left[(z-x)^{k-j}\left(\left(x+\frac{r}{2}-z\right)^{j-N+1} - \left(x-\frac{r}{2}-z\right)^{j-N+1}\right)\right. \\ &\qquad \left. -(\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right)\right] \\ &+(\beta^{(k)}(x)-\beta^{(k)}(\xi)) \\ &\qquad \times\left[(\zeta-\xi)^{k-j}\left(\left(x+\frac{r}{2}-\zeta\right)^{j-N+1} - \left(x-\frac{r}{2}-\zeta\right)^{j-N+1}\right)\right] . \end{aligned} \tag{9.5}$$

Thus to evaluate J^k, $k \leqq N-2$, it suffices to evaluate each term of (9.5).

Since $|\zeta - \xi| = |\eta| \leqq |\zeta| \leqq |z|(1+\delta)$, and

$$\left|x + \frac{r}{2} - \zeta\right| \geqq r\left(\cos\delta - \delta - \frac{1}{2}\right)$$

$$\left|x - \frac{r}{2} - \zeta\right| \geqq r\left(\frac{1}{2} - \delta\right),$$

the second term of (9.5) is bounded absolutely by

$$Cx^{\mu-k-\alpha}|x-\xi|^{\alpha} r^{k-N+1} \leqq Cr^{\mu-N+1-\alpha}|z-\zeta|^{\alpha}.$$

To evaluate the first term of (9.5), consider the function

$$f(p,q) = (iq)^{k-j}\left(\left(x + \frac{r}{2} - p - iq\right)^{j-N+1} - \left(x - \frac{r}{2} - p - iq\right)^{j-N+1}\right)$$

and its first partial derivatives, $f_1(p,q) = f_p(p,q)$, and $f_2(p,q) = f_q(p,q)$. Then we must evaluate

$$\beta^{(k)}(x)(f(x,y) - f(\xi,\eta)).$$

By the mean value theorem we have, for some λ with $0 < \lambda < 1$,

$$\begin{aligned} f(x,y) - f(\xi,\eta) = (x-\xi)&f_1(x + \lambda(\xi - x), y + \lambda(\eta - y)) \\ &+ (y-\eta)f_2(x + \lambda(\xi - x), y + \lambda(\eta - y)). \end{aligned}$$

Then using previously mentioned inequalities for $|\zeta - \xi|$, $|x \pm r/2 - \zeta|$, etc., it follows easily that for $k \leqq N-2$

$$|J^k| \leqq Cr^{\mu-N+1-\alpha}|z-\zeta|^{\alpha}.$$

Thus to finish the proof we must evaluate

$$\begin{aligned} J^{N-1} &= -(N-1)\int_{x-r/2}^{x+r/2}\left\{\frac{\psi(x,t)}{(t-z)^N} - \frac{\psi(\xi,t)}{(t-\zeta)^N}\right\}dt \\ &= -(N-1)\int_{-r/2}^{r/2}\frac{\psi(x,x+t)}{(t-iy)^N} \\ &\qquad + (N-1)\int_{x-\xi-r/2}^{x-\xi+r/2}\frac{\psi(\xi,\xi+t)}{(t-i\eta)^N}dt. \end{aligned}$$

We will assume $x \geqq \xi$; were $x < \xi$ another similar argument would prevail. Then

$$\begin{aligned} J^{N-1} &= (N-1)\int_{-r/2}^{r/2}\left\{\frac{\psi(\xi,\xi+t)}{(t-i\eta)^N} - \frac{\psi(x,x+t)}{(t-iy)^N}\right\}dt \\ &\qquad - (N-1)\int_{-r/2}^{x-\xi-r/2}\frac{\psi(\xi,\xi+t)}{(t-i\eta)^N}dt \qquad (9.6) \\ &\qquad + (N-1)\int_{r/2}^{x-\xi+r/2}\frac{\psi(\xi,\xi+t)}{(t-i\eta)^N}dt. \end{aligned}$$

By definition, for $t \geqq 0$,

$$|\psi(\xi, \xi + t)| \leqq \int_{\xi}^{\xi+t} \frac{(\xi + t - \sigma)^{N-2}}{(N-2)!} |\beta^{(N-1)}(\sigma)| \, d\sigma \leqq C\xi^{\mu-N+1}t^{N-1} ,$$

and thus the third integral on the right side of (9.6) is bounded absolutely by

$$C |z|^{\mu-N+1} \int_{r/2}^{x-\xi+r/2} t^{-1} dt \leqq C |z|^{\mu-N+1} |x + \xi| \left(\frac{r}{2}\right)^{-1}$$
$$\leqq C |z - \zeta|^{\alpha} |z|^{\mu-N+1-\alpha} .$$

We handle the second integral of (9.6) in the same fashion.

Thus we have left to evaluate

$$(9.7) \quad \begin{aligned} &\int_{-r/2}^{r/2} \left\{ \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} - \frac{\psi(x, x + t)}{(t - iy)^N} \right\} dt \\ &= \int_0^{r/2} \left\{ \frac{\psi(\xi, \xi + t)}{(t - i\eta)^N} + (-1)^N \frac{\psi(\xi, \xi - t)}{(t + i\eta)^N} \right. \\ &\qquad \left. - \frac{\psi(x, x + t)}{(t - iy)^N} - (-1)^N \frac{\psi(x, x - t)}{(t + iy)^N} \right\} dt \\ &= \int_0^{r/2} dt \int_0^t \frac{(t - s)^{N-2}}{(N-2)!} \left\{ \frac{\beta^{(N-1)}(\xi + s)}{(t - i\eta)^N} - \frac{\beta^{(N-1)}(\xi - s)}{(t + i\eta)^N} \right. \\ &\qquad \left. - \frac{\beta^{(N-1)}(x + s)}{(t - iy)^N} + \frac{\beta^{(N-1)}(x - s)}{(t + iy)^N} \right\} ds \end{aligned}$$

where we have recalled the definition of ψ.

We write

$$\beta^{(N-1)}(\xi + s) = \beta^{(N-1)}(\xi) + K_1(\xi, s)s^{\alpha}$$
$$\beta^{(N-1)}(\xi - s) = \beta^{(N-1)}(\xi) + K_2(\xi, s)s^{\alpha}$$
$$\beta^{(N-1)}(x + s) = \beta^{(N-1)}(x) + K_3(x, s)s^{\alpha}$$
$$\beta^{(N-1)}(x - s) = \beta^{(N-1)}(x) + K_4(x, s)s^{\alpha}$$

where $|K_i(s)| \leqq C |z|^{\mu-N+1-\alpha}$. Also

$$(9.8) \quad s^{\alpha} |K_1(\xi, s) - K_3(x, s)| \leqq C |z|^{\mu-N+1-\alpha} |z - \zeta|^{\alpha}$$

and

$$s^{\alpha} |K_1(\xi, s) - K_3(x, s)| \leqq C |z|^{\mu-N+1-\alpha} s^{\alpha} .$$

Similar inequalities hold for $s^{\alpha} |K_2 - K_4|$.

Next, (9.7) becomes

$$
\begin{aligned}
(9.9)\qquad & \int_0^{r/2} dt \int_0^t \frac{(t-s)^{N-2}}{(N-2)!}\Big\{\beta^{(N-1)}(\xi)\Big(\frac{1}{(t-i\eta)^N}-\frac{1}{(t+i\eta)^N}\Big) \\
& \qquad - \beta^{(N-1)}(x)\Big(\frac{1}{(t-iy)^N}-\frac{1}{(t+iy)^N}\Big)\Big\}ds \\
& + \int_0^{r/2} dt \int_0^t \frac{(t-s)^{N-2}}{(N-1)!}\Big\{\frac{K_1(\xi, s)s^\alpha}{(t-i\eta)^N} \\
& \qquad - \frac{K_2(\xi, s)s^\alpha}{(t+i\eta)^N} - \frac{K_3(x, s)s^\alpha}{(t-iy)^N} + \frac{K_4(x, s)s^\alpha}{(t+iy)^N}\Big\}ds\,.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
& \int_0^{r/2} dt \int_0^t \frac{(t-s)^{N-2}}{(N-2)!}\Big(\frac{1}{(t-i\eta)^N}-\frac{1}{(t+i\eta)^N}\Big)ds \\
& = \int_0^{r/2} \frac{t^{N-1}}{(N-1)!}\Big(\frac{1}{(t-i\eta)^N}-\frac{1}{(t+i\eta)^N}\Big)dt \\
& = \sum_{k=0}^{N-1}\binom{N-1}{k}\frac{1}{(N-1)!} \\
& \qquad \times \int_0^{r/2}\{(t-i\eta)^{-1-k}(i\eta)^k-(t+i\eta)^{-1-k}(-i\eta)^k\}dt \\
& = \sum_{k=0}^{N-1} C_k\Big\{\Big(\frac{r}{2}-i\eta\Big)^{-k}(i\eta)^k-\Big(\frac{r}{2}+i\eta\Big)^{-k}(-i\eta)^k\Big\}\,.
\end{aligned}
$$

Thus the first term of (9.9) is bounded absolutely by terms of the form

$$
\begin{aligned}
& \beta^{(N-1)}(\xi)\Big\{\Big(\frac{i\eta}{r/2-i\eta}\Big)^k-\Big(\frac{-i\eta}{r/2+i\eta}\Big)^k\Big\} \\
& \qquad - \beta^{(N-1)}(x)\Big\{\Big(\frac{iy}{r/2-iy}\Big)^k-\Big(\frac{-iy}{r/2+iy}\Big)^k\Big\} \\
& = 2i\beta^{(N-1)}(\xi)\, Im\Big[\Big(\frac{i\eta}{r/2-i\eta}\Big)^k-\Big(\frac{iy}{r/2-iy}\Big)^k\Big] \\
& \qquad + 2i\, Im\Big[\Big(\frac{iy}{r/2-iy}\Big)^k\Big](\beta^{(N-1)}(\xi)-\beta^{(N-1)}(x))\,.
\end{aligned}
$$

The second term is bounded by $C\,|z|^{\mu-N+1-\alpha}|z-\zeta|^\alpha$. The first, using the mean value theorem, is bounded by

$$
(9.10)\qquad C\,|z|^{\mu-N+1}|y-\eta|\,\Big|Im\frac{\sigma^{k-1}ri^k}{(r/2-i\sigma)^{k+1}}\Big|
$$

where σ lies between y and η. But then $\sigma \leqq Cr$, $|r/2-i\sigma| \leqq r/2$, and hence (9.10) is $\leqq$

$$
C\,|z|^{\mu-N+1}|y-\eta|^\alpha r^{1-\alpha}r^{-1} \leqq C\,|z|^{\mu-N+1-\alpha}|z-\zeta|^\alpha\,.
$$

Thus we are left with the second term of (9.9). By symmetry we need only consider

$$\int_0^{r/2} dt \int_0^t (t-s)^{N-2} s^\alpha \left\{ \frac{K_1(\xi, s)}{(t-i\eta)^N} - \frac{K_3(x, s)s^\alpha}{(t-iy)^N} \right\} ds$$

$$(9.11) \quad = \int_0^{r/2} dt \int_0^t (t-s)^{N-2} s^\alpha (K_1(\xi, s) - K_3(x, s))(t-i\eta)^{-N} ds$$

$$+ \int_0^{r/2} dt \int_0^t (t-s)^{N-2} s^\alpha K_3(x, s) \left(\frac{1}{(t-i\eta)^N} - \frac{1}{(t-iy)^N} \right) ds .$$

By (9.8), for any $\varepsilon > 0$ we have

$$s^\alpha \,|\, K_1(\xi, s) - K_3(x, s) \,| \leqq C_1^\varepsilon C_1^{1-\varepsilon} \,|\, z \,|^{\mu - N + 1 - \alpha}\, s^{\alpha\varepsilon} \,|\, x - \xi \,|^{\alpha(1-\varepsilon)} .$$

Thus the first term on the right side of (9.11) is bounded in absolute value by

$$CC_1^\varepsilon C_2^{1-\varepsilon} \,|\, z - \zeta \,|^{\alpha(1-\varepsilon)} \,|\, z \,|^{\mu - N + 1 - \alpha} \int_0^{r/2} dt \int_0^t t^{N-2} s^{\alpha\varepsilon} \,|\, t - i\eta \,|^{-N} ds$$

$$\leqq CC_1^\varepsilon C_2^{1-\varepsilon} \,|\, z - \zeta \,|^{\alpha(1-\varepsilon)} \,|\, z \,|^{\mu - N + 1 - \alpha} \int_0^{r/2} t^{\alpha\varepsilon - 1} dt$$

$$= CC_1^\varepsilon C_2^{1-\varepsilon} \,|\, z - \zeta \,|^{\alpha(1-\varepsilon)} \,|\, z \,|^{\mu - N + 1 - \alpha + \alpha\varepsilon} .$$

Now let $\varepsilon \to 0$.

Finally, to evaluate the second term on the right side of (9.11), we observe that

$$\left| \frac{1}{(t-i\eta)^N} - \frac{1}{(t-iy)^N} \right|$$

$$\leqq \sum_{k=1}^N \binom{N}{k} t^{N-k} \,|\, y^k - \eta^k \,|\, |\, t - i\eta \,|^{-N} \,|\, t - iy \,|^{-N}$$

$$\leqq |\, z - \zeta \,|^\alpha \sum_{k=1}^N t^{N-k} y^{k-\alpha} \,|\, t - i\eta \,|^{-N} \,|\, t - iy \,|^{-N}$$

where we have assumed that $y \geqq \eta$. Then

$$\left| \int_0^{r/2} dt \int_0^t (t-s)^{N-2} s^\alpha K_3(x, s)((t-i\eta)^{-N} - (t-iy)^{-N}) ds \right|$$

$$\leqq \sum_{k=1}^N C_k \int_0^{r/2} t^{2N-k-1+\alpha} \,|\, z \,|^{\mu - N + 1 - \alpha} \,|\, z - \zeta \,|^\alpha\, y^{k-\alpha} \,|\, t - i\eta \,|^{-N} \,|\, t - iy \,|^{-N} dt$$

$$\leqq C \,|\, z \,|^{\mu - N + 1 - \alpha} \,|\, z - \zeta \,|^\alpha \sum_{k=1}^N \int_0^{r/2} t^{N-k-1+\alpha} y^{k-\alpha} \,|\, t - iy \,|^{-N} dt .$$

With the change of variables $t = \tau y$, the kth integral becomes

$$\int_0^{r/2y} \tau^{N-k-1+\alpha} \,|\, \tau - i \,|^{-N} dt \leqq \int_0^\infty \tau^{N-k-1+\alpha} \,|\, \tau - i \,|^{-N} dt = C$$

since $1 \leqq k \leqq N$. This completes the evaluation of J^{N-1} and the theorem is proved.

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EMBEDDING A CIRCLE OF TREES IN THE PLANE

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Concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold, it is shown here that there is an embedding if the continuum is decomposable and the manifold is orientable. The embedding is obtained by constructing an annulus on the manifold containing the continuum; in the nonorientable case an annulus or a Möbius strip containing the continuum may be found. Similar results are obtained for continua on a 2-manifold which have a decomposition into continua with zero 1-dimensional Betti numbers such that the decomposition space is a finite planar graph.

This extends the results of [2] concerning the embedding in the plane of homogeneous proper subcontinua of a 2-manifold. Definitions and a summary of other results may be found in [2].

THEOREM 1. *A decomposable homogeneous proper subcontinuum of an orientable 2-manifold can be embedded in the plane.*

Proof. Let X be a decomposable proper subcontinuum of an orientable 2-manifold M. By Theorem 11 of [2] there is a continuous collection G of disjoint continua filling X such that the decomposition space X' is a simple closed curve and the elements of G are mutually homeomorphic, homogeneous, and treelike. Consider the upper semi-continuous decomposition of M whose nondegenerate elements are the elements of G. By Theorem 1 of [1], the decomposition space M' is homeomorphic to M. Let A' be a closed annular neighborhood of the simple closed curve X' on M'. Let f be the projection map from M to M' and A be $f^{-1}(A')$. Consider A' to be filled by a continuous collection of simple closed curves $\{J_\alpha\}$, $\alpha \in [01]$, where $J_{1/2} = X'$. Then f is one-to-one on A-X and $f^{-1}(J_\alpha)$ must be compact since J_α is compact; thus $f/f^{-1}(J_\alpha)$ is a continuous one-to-one map of a compact set for $\alpha \neq 1/2$ and therefore a homeomorphism. For a closed subinterval I of $[01]$ not containing $1/2, f^{-1}(\Sigma_{\alpha \in I} J_\alpha)$ is a closed annulus. If $p \in A\text{-}f^{-1}(J_0 + J_1)$ then $f(p)$ is in the interior of A' and thus p is interior to A. The continuum A must then be a 2-manifold with boundary consisting of $f^{-1}(J_0) + f^{-1}(J_1)$. Fit 2-cells C_0 and C_1 to the boundary curves of A to make a 2-manifold M_1 without boundary. Considering an upper semi-continuous decomposition of M_1 with the elements of G as the nondegenerate elements, we have M_1' as merely

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A' with 2-cells C_0' and C_1' fitted to J_0 and J_1; i.e., M_1' is a 2-sphere. Using Theorem 1 of [1] again, M_1 must be homeomorphic to M_1' and A must be a closed annulus; X is thus planar.

In the same manner we have the following results:

THEOREM 2. *A decomposable homogeneous proper subcontinuum of a nonorientable 2-manifold is contained in an open annulus or open Möbius strip on the manifold.*

THEOREM 3. *If a subcontinuum of an orientable 2-manifold has an upper semi-continuous decomposition into continua with zero mod 2 1-dimensional Betti numbers*[1] *such that the decomposition space is a finite planar graph then the continuum can be embedded in the plane.*

In view of Theorem 1, a nonplanar homogeneous subcontinuum of an orientable 2-manifold would have to be in the class of nontreelike indecomposable continua. No planar homogeneous continuum in this class is known, although the pseudo-circle is a candidate. It would be nice to eliminate the condition of orientability in Theorem 1.

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[1] As in [1], the statement "the continuum X on the 2-manifold M has a zero mod 2, 1-dimensional Betti number" means that for any complex K of M containing X there is a smaller complex L of M containing X such that each of its 1-cycles bounds in K.

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RING-LOGICS AND RESIDUE CLASS RINGS

ADIL YAQUB

Let $(R, \times, +)$ be a commutative ring with unit 1, and let $K = \{\rho_1, \rho_2, \cdots\}$ be a transformation group in R. $(R, \times, +)$ is called a ring-logic, mod K essentially if the "$+$" of R is equationally definable in terms of the "K-logic" $(R, \times, \rho_1, \rho_2, \cdots)$. The Boolean theory results by choosing K to be the group generated by $x^* = 1 - x$ (order 2, $x^{**} = x$). The following result is proved: Let $n = p_1 \cdots p_t$ be square-free, and let R_n be the residue class ring, mod n. Let, $\frown$, be any transitive $0 \to 1$ permutation of $R_{p_i}(i = 1, \cdots, t)$. Let, $\frown$, be the induced permutation of R_n defined by $(x_1, \cdots, x_t)^\frown = (x_1^\frown, \cdots, x_t^\frown)$, $x_i \in R_{p_i}(i = 1, \cdots, t)$, and let K be the transformation group in R_n generated by, $\frown$. Then $(R_n, \times, +)$ is a ring-logic, mod K. An extension of this theorem to the case where n is arbitrary is also considered. The present proofs use the Fermat-Euler Theorem as well as a generalized form of the Chinese Residue Theorem.

The motivation for the study of ring-logics stems from the familiar equational interdefinability of Boolean rings $(R, \times, +)$ and Boolean logics (=Boolean algebras) $(R, \cap, ^*)$ [5]. In a series of recent publications ([1]-[4]), Foster raised this equational interdefinability, as well as the entire Boolean theory, to a more general level. In particular, Foster showed [2; 3] that any p-ring with unit (and more generally, any p^k-ring with unit) is a ring-logic, modulo certain suitably chosen groups. Furthermore, the author proved [6] that R_n, the residue class ring, mod n, is a ring-logic, modulo the "natural group" (generated by $x^\wedge = 1 + x$). Our present object is to further extend these results by considering certain transformation groups in R_n of rather general nature, and with respect to which $(R_n, \times, +)$ is a ring-logic (see Theorem 5).

1. The ring of residues mod p^k. Let $(R_{p^k}, \times, +)$ be the residue class ring, mod p^k, where p is prime and $k \geqq 1$. Let G denote the group of units in R_{p^k}. Then, as is well known, the order of G is $\varphi(p^k) = p^k - p^{k-1}$, where $\varphi(n)$ is the familiar Euler φ-function (=number of positive integers which do not exceed n and which are relatively prime to n). Let, $\frown$, be a permutation of R_{p^k}. We call, $\frown$, a *transitive* $0 \to 1$ permutation if (i) $0^\frown = 1$, and (ii) for any elements α, β in R_{p^k}, there exists an integer r such that $\alpha^{\frown r} = \beta$, where $\alpha^{\frown r} = (\cdots((\alpha^\frown)^\frown)^\frown \cdots)^\frown$ (r-iterations).

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We recall from [4] the *characteristic function* $\delta_\mu(x)$, defined as follows: for any given $\mu \in R_{pk}$, $\delta_\mu(x) = 1$ if $x = \mu$ and $\delta_\mu(x) = 0$ if $x \neq \mu$. Following [4], we also define: $a \times_\frown b = {}_\frown(a^\frown \times b^\frown)^\smile$, where, $^\smile$, is the inverse of the $0 \to 1$ permutation, $^\frown$. One readily verifies that $a \times_\frown 0 = 0 \times_\frown a = a$. Hence, we have the following "normal expansion formula" [4]:

$$(1.1) \qquad f(x, y, \cdots) = \sum_{\alpha, \beta, \cdots \in R_{pk}}^{\times_\frown} f(\alpha, \beta, \cdots)(\delta_\alpha(x)\delta_\beta(y)\cdots) .$$

In (1.1), $\alpha, \beta, \cdots$ range independently over all the elements of R_{pk} while $x, y, \cdots$ are indeterminates over R_{pk}. Also, $\sum_{\alpha_i \in R}^{\times_\frown} \alpha_i$ denotes $\alpha_1 \times_\frown \alpha_2 \times_\frown \cdots$, where $\alpha_1, \alpha_2, \cdots$ are all the elements of R.

We now have the following

LEMMA 1. *Let, $^\frown$, be any transitive permutation of R_{pk}, and let K be the transformation group in R_{pk} generated by, $^\frown$. Then all the elements of R_{pk} are equationally definable in terms of the K-logic $(R_{pk}, \times, ^\frown)$.*

Proof. Since, $^\frown$, is a *transitive* permutation of R_{pk}, therefore, $R_{pk} = \{0, 0^\frown, 0^{\frown 2}, \cdots, 0^{\frown pk-1}\}$. Similarly, we have, $xx^\frown x^{\frown 2} \cdots x^{\frown pk-1} = 0$, for all x in R_{pk}. The last equation shows that 0 (and with it $0^\frown, 0^{\frown 2}, \cdots, 0^{\frown pk-1}$) is expressible in terms of the K-logic, and the lemma is proved.

LEMMA 2. *Let $G = \{1, \zeta_2, \zeta_3, \cdots, \zeta_\varphi\}$ be the group of units in the residue class ring $(R_{pk}, \times, +)$. Let, $^\frown$, be a transitive $0 \to 1$ permutation of R_{pk} satisfying $1^\frown = \zeta_2, \zeta_2^\frown = \zeta_3, \cdots, \zeta_{\varphi-1}^\frown = \zeta_\varphi$, but otherwise, $^\frown$, is entirely arbitrary. Let K be the transformation group in R_{pk} generated by, $^\frown$. Then each characteristic function $\delta_\mu(x)$, $\mu \in R_{pk}$, is equationally definable in terms of the K-logic $(R_{pk}, \times, ^\frown)$.*

Proof. Since, $^\frown$, is *transitive*, therefore, there exists an integer r such that $\mu^{\frown r} = 0$. Now, one readily verifies that

$$\delta_\mu(x) = (x^{\frown r+1}x^{\frown r+2}x^{\frown r+3} \cdots x^{\frown r+\varphi})^{pk-pk-1} ,$$

since, by the Fermat-Euler Theorem, $a^{pk-pk-1} = 1$ for all a in G. This proves the lemma.

THEOREM 3. *Let K, $^\frown$, be as in Lemma 2. Then the residue class ring $(R_{pk}, \times, +)$ is a ring-logic, mod K.*

Proof. By (1.1), $x + y = \sum_{\alpha, \beta \in R_{pk}}^{\times_\frown} (\alpha + \beta)(\delta_\alpha(x)\delta_\beta(y))$. By Lemma 1 and Lemma 2, each of $\alpha + \beta$, $\delta_\alpha(x)$, and $\delta_\beta(y)$, is expressible in terms of the K-logic. Hence, the "$+$" of R_{p^k} is equationally definable in terms of the K-logic. Next, we show that $(R_{p^k}, \times, +)$ is *fixed* by its K-logic. Suppose that $(R_{p^k}, \times, +')$ is another ring with the same class of elements R_{p^k} and the same "$\times$" as $(R_{p^k}, \times, +)$ and which has the *same logic* as $(R_{p^k}, \times, +)$. To prove that $+' = +$. But this follows, since, up to isomorphism, there is only one cyclic group of order p^k.

2. The general case. In attempting to generalize Theorem 3 to the residue class ring $(R_n, \times, +)$, n arbitrary, we need the following concept of independence, introduced by Foster [4].

DEFINITION. Let $\{U_1, \cdots, U_t\}$ be a finite set of algebras of the same species S. We say that the algebras $U_1, \cdots, U_t$ are *independent* or satisfy the *Chinese Residue Theorem*, if, corresponding to each set $\{\Psi_i\}$ of expressions of species S, there exists a single expression X such that $\Psi_i = X \pmod{U_i}$ $(i = 1, \cdots, t)$. By an *expression* we mean some composition of one or more indeterminate-symbols $x, \cdots$ is terms of the primitive operations of $U_1, \cdots, U_t$; $\Psi_i = X (\operatorname{mod} U_i)$ means that this is an identity of the algebra U_i.

As usual, we shall use the *same* symbols to denote the operation symbols of the algebras $U_1, \cdots, U_t$ when these algebras are of the same species. We now have the following

LEMMA 4. *Let $p_1, \cdots, p_t$ be distinct primes. Let, $\frown$, be any transitive $0 \to 1$ permutation of $R_{p_i^{k_i}}$, and let K_i be the transformation group in $R_{p_i^{k_i}}$ generated by, $\frown$, $(i = 1, \cdots, t)$. Then the K_i-logics $(R_{p_i^{k_i}}, \times, \frown)(i = 1, \cdots, t)$ are independent.*

Proof. Let $n = p_1^{k_1} \cdots p_t^{k_t}$ and let $E = xx^\frown x^{\frown 2} \cdots x^{\frown n-1}$. Let $p_i^{k_i} n_i = n$. Since $(p_i^{k_i}, n_i) = 1$, therefore, there exist integers r_i, s_i such that $r_i n_i - s_i p_i^{k_i} = 1$. Now, one readily verifies that

$$\omega_i = \text{def} = E^{\frown r_i n_i} = \begin{cases} 1 (\operatorname{mod} R_{p_i^{k_i}}) , \\ 0 (\operatorname{mod} R_{p_j^{k_j}}) \end{cases} \qquad (j \neq i) .$$

To prove the independence of the logics $(R_{p_i^{k_i}}, \times, \frown)$, let $\{\Psi_i\}$ be a set of t expressions of species $\times, \frown$; i.e., primitive composition of indeterminate-symbols in terms of the operations $\times, \frown$. Define

$$X = \Psi_1 \omega_1 \times_\frown \cdots \times_\frown \Psi_t \omega_t .$$

It is readily verified that $\Psi_i = X (\operatorname{mod} R_{p_i^{k_i}})(i = 1, \cdots, t)$, since $a \times_\frown 0 = 0 \times_\frown a = a$. This proves the lemma.

We are now in a position to consider $(R_n, \times, +)$ in regard to the concept of ring-logic. Indeed, let $n = p_1^{k_1} \cdots p_t^{k_t}$, where the p_i are distinct primes $(i = 1, \cdots, t)$, and let $G_i = \{1, \zeta_{i2}, \zeta_{i3}, \cdots, \zeta_{i\varphi_i}\}$ be the group of units in the residue class ring $(R_{p_i^{k_i}}, \times, +)$. For each i, define, $\frown$, to be a transitive $0 \rightarrow 1$ permutation of $R_{p_i^{k_i}}$ satisfying $1^\frown = \zeta_{i2}, \zeta_{i2}^\frown = \zeta_{i3}, \cdots, (\zeta_{i,\varphi_i - 1})^\frown = \zeta_{i\varphi_i}$, but otherwise, $\frown$, is entirely arbitrary, and let K_i be the transformation group in $R_{p_i^{k_i}}$ generated by, $\frown$. Now, it is well known that the residue class ring R_n is isomorphic to the direct product of $R_{p_1^{k_1}}, \cdots, R_{p_t^{k_t}}$:

$$R_n \cong R_{p_1{}^1} \times \cdots \times R_{p_t^{k_t}} \text{ (direct product)}, \ n = p_1^{k_1} \cdots p_t^{k_t} .$$

Furthermore, it is easily seen that by defining $(x_1, \cdots, x_t)^\frown = (x_1^\frown, \cdots, x_t^\frown)$, $(x_1, \cdots, x_t) \in R_n$, we obtain a transitive $0 \rightarrow 1$ permutation of R_n. Let K be the transformation group in R_n generated by the above permutation, $\frown$. We now have the following

THEOREM 5. *The residue class ring $(R_n, \times, +)$, n arbitrary, is a ring-logic, mod K, where K is the transformation group in R_n above.*

Proof. Let $n = p_1^{k_1} \cdots p_t^{k_t}$, where the p_i are distinct primes $(i = 1, \cdots, t)$. By Theorem 3, each $(R_{p_i^{k_i}}, \times, +)$ is a ring-logic, mod K_i, where K_i is as defined above $(i = 1, \cdots, t)$. Hence, for each i, there exists an expression Ψ_i such that

$$x_i + y_i = \Psi_i(x_i, y_i; \times, \frown), \text{ for all } x_i, y_i \text{ in } R_{p_i^{k_i}} .$$

But, by Lemma 4, the K_i-logics $(R_{p_i^{k_i}}, \times, \frown)$ are independent $(i = 1, \cdots, t)$, and hence there exists a single expression X such that $X = \Psi_i \pmod{R_{p_i^{k_i}}}$ $(i = 1, \cdots, t)$. Now, let $x = (x_1, \cdots, x_t)$, $y = (y_1, \cdots, y_t)$ be any elements of $R_n(\cong R_{p_1^{k_1}} \times \cdots \times R_{p_t^{k_t}})$. Since the operations are component-wise in this direct product, therefore,

$$\begin{aligned} X(x, y; \times, \frown) &= X((x_1, \cdots, x_t), (y_1, \cdots, y_t); \times, \frown) \\ &= (X(x_1, y_1; \times, \frown), \cdots, X(x_t, y_t; \times, \frown)) \\ &= (\Psi_1(x_1, y_1; \times, \frown), \cdots, \Psi_t(x_t, y_t; \times, \frown)) \\ &= (x_1 + y_1, \cdots, x_t + y_t) \\ &= x + y . \end{aligned}$$

Hence, the "+" of R_n is *equationally* definable in terms of the K-logic $(R_n, \times, \frown)$. The proof that $(R_n, \times, +)$ is fixed by its K-logic follows as in the "fixed" part of the proof of Theorem 3, since again, up to isomorphism, there is only one cyclic group of order n. This completes the proof of the theorem.

We shall now take a closer look at the case where $n = p_1 \cdots p_t$ is *square-free*. In this case the group G_i of units in R_{p_i}(=field) is precisely the set of all nonzero elements of $R_{p_i}(i = 1, \cdots, t)$, and the, $\frown$, described above (see paragraph preceding Theorem 5) for R_{p_i} is now simply *any* transitive $0 \to 1$ permutation of R_{p_i}. Hence, we have the following

COROLLARY 6. *Let $n = p_1 \cdots p_t$ be square-free, and let, $\frown$, be any transitive $0 \to 1$ permutation of $R_{p_i}(i = 1, \cdots, t)$. Let, $\frown$ be the induced permutation of R_n defined by $(x_1, \cdots, x_t)^\frown = (x_1^\frown, \cdots, x_t^\frown)$, $x_i \in R_{p_i}(i = 1, \cdots, t)$, and let K be the transformation group in R_n generated by, $\frown$. Then $(R_n, \times, +)$ is a ring-logic, mod K.*

Thus, if, in particular, we choose $x^\frown = 1 + x$ in the above Corollary, we obtain the following (compare with [6]).

COROLLARY 7. *Let n be square-free, and let N be the "natural group", generated by $x^\frown = 1 + x$. Then $(R_n, \times, +)$ is a ring-logic, mod N.*

Upon choosing, $\frown$, in Theorem 5 in all of the various available ways, we obtain the corresponding transformation groups K with respect to which $(R_n, \times, +)$ is a ring-logic.

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ERRATA

Correction to

CHAINS OF INFINITE ORDER AND THEIR APPLICATION TO LEARNING THEORY

JOHN LAMPERTI AND PATRICK SUPPES

Volume 9 (1959), 739–754

Professor M. Iosifescu has pointed out to us an error in our paper [1]. The difficulty lies in the positivity condition

$$p_{j_0}^{(n_0)}(x) \geqq \delta \geqq 0 \qquad \text{for every } x \,, \tag{2.3}$$

which is not strong enough when $n_0 > 1$. Iosifescu has in fact given an example of a second order Markov chain satisfying (2.3) with $n_0 = 0$ for which $\lim_{n\to\infty} p_i^{(n)}(x)$ is not independent of x as asserted by Theorem 2.1.

The difficulty can be overcome by making the stronger assumption that for some state j_0, some positive integer n_0, and some sequence of positive numbers δ_m,

$$p_{j_0^* m}^{(n_0)}(x) \geqq \delta_m \qquad \text{for every } x \text{ and } m \,. \tag{2.3$'$}$$

Here $p_{x_m}^{(n)}(x)$ is the joint probability (defined formally by (2.11) and (2.12)) of executing the sequence x_m after n steps, given x, and $j_0^* m$ means a sequence of m repetitions of j_0. Thus we are asserting that the event, consisting of m consecutive visits to j_0 starting after a lapse of time n_0, has positive probability uniformly in x (not in m). If $n_0 = 1$, (2.3′) follows from (2.3) with $\delta_m = \delta^m$, and our error lay in the tacit use of (2.3′), rather than (2.3), in proving Lemma 2.2 in our paper. When (2.3′) is assumed the argument given is valid. Lemma 2.1 does in fact follow from (2.3) and (2.5) as asserted, and so with the new hypothesis the conclusions of § 2 are justified.

Let us consider the effect of this change on the application to linear learning models. Assumption (ii) (b) of Theorem 4.1, which is used to derive (2.3), is now seen to be inadequate for the conclusions of the theorem. However the special case (4.5), when $m_0 = 0$, yields (2.3) with $n_0 = 1$ and so the results are valid in this situation. Although (2.3′) could be adapted to yield greater generality, we take it that essentially all cases of interest are actually covered by (4.5), and

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shall leave the matter so. A similar remark applies to Theorem 4.2.

Reference

1. John Lamperti and Patrick Suppes, '*Chains of infinite order and their application to learning theory*,' Pacific J. Math. **9** (1959), 739-754.

Correction to

NON-LINEAR DIFFERENTIAL EQUATIONS ON CONES IN BANACH SPACES

Charles V. Coffman

Volume 14 (1964), 9–15

In [1] the proof of a main lemma, Lemma 3.1, contains an error. The lemma itself is false without stronger hypotheses. The purpose of this note is to state and prove a lemma which can be used in place of Lemma 3.1 in the proofs of Theorem 4.1 and 5.1 in [1].

Let Y be a Banach space, let Γ be a closed linear manifolds in Y^* which is total for Y.[1] Assume that I is some real interval. The differential equation with which [1] is concerned is

$$dy/dt = f(t, y)\ , \tag{1}$$

where f is a function from $I \times C \to Y$ which is continuous with respect to the weak Γ-topology on Y; C is a subset of Y. The notation and terminology used here will be the same as that employed in [1]; the definition of a weak Γ-derivative, a weak Γ-solution of (1), etc., are to be found in [1].

Let $\mathscr{C}$ be the space of weakly Γ-continuous functions on I with values in C, furnished with the topology of uniform convergence (in the weak Γ-topology) on compact subintervals of I. If C is compact in the weak Γ-topology, then Ascoli's theorem implies that a set of equicontinuous functions in $\mathscr{C}$ is relatively compact in $\mathscr{C}$. However unless the topology on $\mathscr{C}$ satisfies the first axiom of countability one cannot conclude from Ascoli's theorem, as is done in [1], that an equicontinuous sequence of functions in $\mathscr{C}$ has a convergent subsequence. ($\mathscr{C}$ will satisfy the first axiom of countability, for example,

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if C is bounded and Γ is separable in its norm topology.)

Let Y, Γ, C, I and $\mathscr{C}$ be as above, the following Lemma can be used in place of Lemma 3.1 of [1] in the proofs of Theorem 4.1 and 5.1.

LEMMA. *Let $\{y_n(t)\}$ be a sequence of weakly Γ-continuous functions defined on I with values in C. Let C be compact in the weak Γ-topology. For each neighborhood V of 0 in Y, in the weak Γ-topology, and for each compact subinterval I' of I, let there exist an $N = N(V, I')$ such that for all $n \geqq N$, $y_n(t)$ is a V-approximate weak Γ-solution of* (1) *on I'. Then, in $\mathscr{C}$, the sequence $\{y_n(t)\}$ has a cluster point $y_0(t)$ and $y_0(t)$ is a weak Γ-solution of* (1) *on I.*

Proof. As is shown in the proof of Lemma 3.1 in [1], the sequence $\{y_n(t)\}$ is equicontinuous in the weak Γ-topology on Y, thus it follows from Ascoli's theorem that the sequence $\{y_n(t)\}$ has a cluster point $y_0(t)$ in $\mathscr{C}$. To complete the proof it will be shown that given $\gamma \in \Gamma$, there exists a subsequence $\{y_{n_k}(t)\}$ of the original sequence such that

$$(2) \qquad \gamma(y_{n_k}(t)) \longrightarrow \gamma(y_0(t)) \qquad \text{as } k \longrightarrow \infty ,$$

and

$$(3) \qquad \gamma(f(t, y_{n_k}(t))) \longrightarrow \gamma(f(t, y_0(t))) \qquad \text{as } k \longrightarrow \infty ,$$

uniformly on compact subintervals of I. To this end let $\{I_k\}$ be an expanding sequence of compact intervals whose union is I. Since $f(t, y)$ is uniformly continuous on $I_k \times C$ for each k, there is a neighborhood V_k of 0 such that $|\gamma(f(t, y'(t)) - f(t, y_0(t)))| < (1/k)$ on I_k for any function $y'(t)$ with $y'(t) - y(t) \in V_k$ on I_k. Let

$$V\left[\gamma, \frac{1}{k}\right] = \left\{y \in Y : |\gamma(y)| < \frac{1}{k}\right\}, \quad \text{if} \quad V'_k = V_k \cap V\left[\gamma, \frac{1}{k}\right]$$

then for each k it is possible to choose an element $\{y_{n_k}(t)\}$ of the original sequence such that $y_{n_k}(t) - y_0(t) \in V'_k$ on I_k. It easily follows that a subsequence selected in this manner satisfies (2) and (3), and the limits are uniform on compact subintervals of I. Finally since the hypothesis implies that

$$\gamma(D_\Gamma y_{n_k}(t) - f(t, y_{n_k}(t))) \longrightarrow 0 , \qquad \text{as } k \longrightarrow \infty ,$$

uniformly on compact subintervals of I, it follows from (2) and (3) that

$$(4) \qquad \gamma(y_0(t_1) - y_0(t_0)) = \int_{t_0}^{t_1} \gamma(f(t, y_0(t))dt , \quad t_1, t_0 \in I .$$

As γ was arbitrary, (4) holds for each $\gamma \in \Gamma$, consequently $D_\Gamma y_0(t)$

exists on I and $y_0(t)$ is a weak Γ-solution of (1) on I.

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CARNEGIE INSTITUTE OF TECHNOLOGY

Correction to

A SUFFICIENT CONDITION THAT AN ARC IN S^n BE CELLULAR

P. H. DOYLE

Volume 14 (1964), 501-503

In Corollary 1 add to the hypothesis: each subarc of A_2 is p-shrinkable.

Correction to

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PARFENY P. SAWOROTNOW

Volume 14 (1964), 1399-1403

Page 1400, line 6 from the bottom: Should read $\|R_x\|$ instead of $\|R\|$.

Page 1401, line 15: Should read $|\lambda - \lambda_0| \, |y_{\lambda_0} x| < 1$ instead of $|\lambda - \lambda_0| \, \|y_{\lambda_0} x\| < 1$.

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Page 508, line 15: Change §2 to read §3.

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