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THE DISTRIBUTION OF CUBIC AND QUINTIC NON-RESIDUES

JAMES HENRY JORDAN

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For a prime $p \equiv 1 \pmod{3}$, the reduced residue system S_3 , modulo p , has a proper multiplicative subgroup, C^0 , called the cubic residues modulo p . The other two cosets formed with respect to C^0 , say C^1 and C^2 , are called classes of cubic non-residues. Similarly for a prime $p \equiv 1 \pmod{5}$ the reduced residue system S_5 , modulo p , has a proper multiplicative subgroup, Q^0 , called the quintic residues modulo p . The other four cosets formed with respect to Q^0 , say Q^1 , Q^2 , Q^3 and Q^4 are called classes of quintic non-residues. Two functions, $f_3(p)$ and $f_5(p)$, are sought so that (i) if $p \equiv 1 \pmod{3}$ then there are positive integers $a_i \in C^i$, $i = 1, 2$, such that $a_i < f_3(p)$, and (ii) if $p \equiv 1 \pmod{5}$ then there are positive integers $a_i \in Q^i$, $i = 1, 2, 3, 4$ such that $a_i < f_5(p)$. The results established in this paper are that for p sufficiently large, (i) $f_3(p) = p^{\alpha+\varepsilon}$, where α is approximately .191, and (ii) $f_5(p) = p^{\beta+\varepsilon}$, where $.27 < \beta < .2725$.

Davenport and Erdős [3] raised the general question about the size of the smallest element in any given class of k th power non-residues. The special cases $k = 3$ and $k = 5$ are of primary concern in this paper. They proved a quite general theorem of which two special cases are:

THEOREM A. *For sufficiently large primes $p \equiv 1 \pmod{3}$ and $\varepsilon > 0$ each class of cubic non-residues possess a positive integer smaller than $p^{55/112+\varepsilon}$.*

THEOREM B. *For sufficiently large primes $p \equiv 1 \pmod{5}$ and $\varepsilon > 0$ each class of quintic non-residues possess a positive integer smaller than $p^{197/396+\varepsilon}$.*

In the same paper Davenport and Erdős used a result of de Bruijn [2] to improve the constant of Theorem A to approximately .383.

Recently D. A. Burgess [1] succeeded in improving Polya's inequality concerning character sums. Burgess' result is

THEOREM C. *If p is a prime and if χ is a nonprincipal character, modulo p , and if H and r are arbitrary positive integers then*

$$I: \sum_{m=n+1}^{n+H} \chi(m) \ll H^{1-1/r+1} p^{1/4r} \ln p,$$

for any integer n , where $A \ll B$ is Vinogradov's notation for $|A| < cB$ for some constant c , and in this Theorem c is absolute.

The application of Theorem C to the arguments of Davenport and Erdős cuts each of the exponents of p in half.

The achievement of this paper is to obtain the same result about cubic non-residues by an argument which is independent of the de Bruijn result, reduce the exponent on the result for quintic non-residues by a similar argument, and indicate a method of obtaining results for any primeth power non-residues.

2. Cubic Non-Residues.

A well known result about sums of inverses of primes is:

$$II: \sum_{q \leq x} 1/q = \ln \ln x + K + O(1/\ln x),$$

where K is a positive constant.¹

LEMMA 1. If $0 \leq v < 1/2$ then

$$\sum_{x^{1-v}}^x 1/q = \int_{x^{1-v}}^1 1/y \, dy + O(1/\ln x),$$

where the error term is independent of v .

The proof of Lemma 1 follows directly from II.

LEMMA 2. If $0 \leq v < 1/2$ then

$$\sum_{x^{(1-v)/2}}^{\sqrt{x}} \sum_{q_1}^{x/q_1} 1/q_1 q_2 = \int_{(1-v)/2}^{1/2} \int_y^{1-y} (yz)^{-1} dz dy + O(1/\ln x),$$

where q_1 and q_2 run only over primes and the error term is independent of v .

Proof.

$$\begin{aligned} \sum_{x^{(1-v)/2}}^{\sqrt{x}} \sum_{q_1}^{x/q_1} 1/q_1 q_2 &= \sum_{x^{(1-v)/2}}^{\sqrt{x}} 1/q_1 (\ln \ln (x/q_1) - \ln \ln q_1 + O(1/\ln x)) \\ &= \sum_{x^{(1-v)/2}}^{\sqrt{x}} 1/q_1 (\ln \ln (x/q_1) - \ln \ln q_1) + O(1/\ln x). \end{aligned}$$

Now by a well known summation Theorem,²

¹ See for example LeVeque, *Topics in Number Theory*, Vol. 1, Th. 6-20, p. 108.

² See for example LeVeque, *Ibid*, Th. 6-15, p. 103, with λ_n the n th prime, $c_n = 1/\lambda_n$, and $f(y) = \ln \ln (x/y) - \ln \ln y + O(1/\ln x)$.

$$\begin{aligned} \sum_{x^{(1-v)/2}}^{\sqrt{x}} \sum_{q_1}^{x/q_1} 1/q_1 q_2 &= \int_{x^{(1-v)/2}}^{\sqrt{x}} \frac{(\ln \ln t + K + O(1/\ln x)) \ln x dt}{(t \ln t \ln(x/t))} \\ &\quad - (\ln((1-v)/2) + \ln \ln x + K \\ &\quad + O(1/\ln x)) (\ln((1+v)/(1-v)) + O(1/\ln x)). \end{aligned}$$

If the change of variable $t = x^s$ is made then

$$\begin{aligned} \sum_{x^{(1-v)/2}}^{\sqrt{x}} \sum_{q_1}^{x/q_1} 1/q_1 q_2 &= \int_{(1-v)/2}^{1/2} \frac{\ln s}{s(1-s)} ds + \int_{(1-v)/2}^{1/2} \frac{\ln \ln x + K + O(1/\ln x)}{s(1-s)} ds \\ &\quad - (\ln((1-v)/2) + \ln \ln x + K \\ &\quad + O(1/\ln x)) (\ln(1+v)/(1-v)) + O(1/\ln x) \\ &= \int_{(1-v)/2}^{1/2} \frac{\ln s}{s(1-s)} ds - \ln((1-v)/2) \ln((1+v)/(1-v)) \\ &\quad + O(1/\ln x) \\ &= \int_{(1-v)/2}^{1/2} \frac{\ln(1-s) - \ln s}{s} ds + O(1/\ln x) \\ &= \int_{(1-v)/2}^{1/2} \int_y^{1-y} (yz)^{-1} dz dy + O(1/\ln x). \end{aligned}$$

For any positive integer r and primes $p \equiv 1 \pmod{3}$ let $x = 3[p^{(1+1/r)/4}(\ln^2 p)^{r+1}/3]$. Let $C^j(x) = C^j \cap \{m \mid 0 < m \leq x\}$, $j = 0, 1, 2$, and let $N(C^j(x))$ be the cardinality of $C^j(x)$.

The following is a special case of a general theorem of Vinogradov [4], [5].

LEMMA 3. $N(C^j(x)) = x/3 + O(x/\ln x)$, $j = 0, 1, 2$.

Proof. Formula I with $H = x$, and $n = 0$ reads as

$$\begin{aligned} \sum_{m=1}^x \chi(m) &\ll x^{1-1/r+1} p^{1/4r} \ln p \\ &\leq (p^{(1+1/r)/4} (\ln^2 p)^{r+1})^{r/r+1} p^{1/4r} \ln p \\ &= p^{1/4+1/4r} (\ln^2 p)^r \cdot \ln p \\ &= p^{(1+1/r)/4} (\ln^2 p)^{r+1} / \ln p. \end{aligned}$$

In other notation

$$\sum_{m=1}^x \chi(m) = O(x/\ln x).$$

Let $\chi_{3,p}$ be the cubic residue character for primes $p \equiv 1 \pmod{3}$. By the above there is an absolute constant, K_1 , such that

$$\text{III: } \left| \sum_{m=1}^x \chi_{3,p}(m) \right| < K_1 x / \ln x.$$

Set $N(C^j(x)) = x/3 + T_j$, $j = 0, 1, 2$. Notice that $T_0 = -T_1 - T_2$.
Let $w = \cos 2\pi/3 + i\sin 2\pi/3$.

It now follows that

$$\begin{aligned}\sum_{m=1}^{\infty} \chi_{3,p}(m) &= \sum_{j=0}^2 (x/3 + T_j)w^j \\ &= \sum_{j=0}^2 T_j w^j \\ &= -(T_1 + T_2)3/2 + i(T_1 - T_2)\sqrt{3}/2.\end{aligned}$$

Now by III: $|T_1 + T_2| < 2K_1x/3 \ln x$ and $|T_1 - T_2| < 2K_1x/\sqrt{3} \ln x$. These inequalities imply that $|T_1|$ and $|T_2|$ are less than $2K_1x/\sqrt{3} \ln x$. Hence $|T_0| < 4K_1x/\sqrt{3} \ln x$ completing the proof of the lemma.

THEOREM 1. *Let d be the solution of*

$$1/3 = \int_{1-y}^1 1/y \, dy + \int_{(1-y)/2}^{1/2} \int_y^{1-y} (yz)^{-1} dz dy.$$

For all sufficiently large primes $p \equiv 1 \pmod{3}$ there is in each class of cubic non-residues modulo p , a positive integer smaller than $p^{(1-a)/4+\varepsilon}$. (d satisfies the inequality $.234 < d < .235$).

Proof. Given $\varepsilon > 0$ let $r = [1/\varepsilon] + 1$. Define x in terms of p as above and notice that as long as $\varepsilon < d$ then

$$\begin{aligned}x^{1-a+\varepsilon} &= [p^{(1+1/r)/4}(ln^2 p)^{r+1}/3]^{1-a+\varepsilon} 3^{1-a+\varepsilon} \\ &\leq p^{(1-a+\varepsilon)/4+(1-a+\varepsilon)/4r} \cdot (ln^2 p)^{(1-a+\varepsilon)(r+1)} \\ &< p^{(1-a+\varepsilon)/4+1/4r} \cdot (ln^2 p)^{r+1} \\ &< p^{(1-a)/4+\varepsilon}, \quad \text{for sufficiently large primes } p.\end{aligned}$$

It therefore suffices to prove that for sufficiently large primes, each class of cubic non-residues contains a positive integer smaller than $x^{1-a+\varepsilon}$.

Assume that Theorem 1 is false. Then, for some fixed $\varepsilon > 0$, there are infinitely many primes $p \equiv 1 \pmod{3}$ such that one of their classes of non-residues fails to contain a positive integer less than $x^{1-a+\varepsilon}$. Let p_1 be one such prime. Notice that x, C^0, C^1 and C^2 are defined in terms of p_1 and will therefore be fixed in this argument.

Without loss of generality C^2 can represent that class of non-residues modulo p_1 that has no positive integers less than $x^{1-a+\varepsilon}$. Since C^2 has this property it follows that C^1 has no positive integer less than $x^{(1-a+\varepsilon)/2}$ because a in C^1 implies a^2 in C^2 .

Since C^0 is closed under multiplication, modulo p_1 , an integer w in $C^2(x)$ must have prime factors not in C^0 . If w has exactly one

prime factor, q , not in C^0 then q must be in C^2 . If w has exactly two prime factors, q_1 and q_2 , not in C^0 then both q_1 and q_2 must be in C^0 . Further w cannot have more than two prime factors not in C^0 since the product of any three or more prime factors not in C^0 exceeds x .

A consequence of the above conditions on positive integers in $C^2(x)$ is the following upper bound for $N(C^2(x))$:

$$\text{IV} : \sum_{x^{1-d+\varepsilon}}^x [x/q_1] + \sum_{\sqrt{x^{1-d+\varepsilon}}}^{\sqrt{x}} \sum_{q_1}^{x/q_1} [x/q_1 q_2]$$

where the q_1 and q_2 are taken only over primes. But

$$\begin{aligned} \text{IV} &\leq x \left(\sum_{x^{1-d+\varepsilon}}^x 1/q_1 + \sum_{\sqrt{x^{1-d+\varepsilon}}}^{\sqrt{x}} \sum_{q_1}^{x/q_1} 1/q_1 q_2 \right) \\ &< x \left(-\ln((1-d+\varepsilon)/(1-d)) + \int_{1-d}^1 1/y dy \right. \\ &\quad \left. + \int_{(1-d)/2}^{1/2} \int_y^{1-y} (yx)^{-1} dz dy + K_2/\ln x \right) \\ &= x(-\ln(1+\varepsilon/(1-d)) + 1/3) + K_2 x/\ln x, \end{aligned}$$

where K_2 is a constant independent of x . But this inequality can hold only for finitely many primes to be compatible with Lemma 3.

3. Quintic Non-Residues. It is helpful to adopt the following notations: Let

$$\begin{aligned} I_1 &= \int_{(1-v)}^1 1/y dy \\ I_2 &= \int_{(1-v)/2}^{1/2} \int_y^{1-y} (yz)^{-1} dz dy \\ I_3 &= \int_{(1-v)/4}^{(1-v)/2} \int_{1-v-y}^{1-y} (yz)^{-1} dz dy \\ I_4 &= \int_{(1-v)/4}^{(1-v)/3} \int_y^{(1-v-y)/2} \int_{1-v-y-z}^{1-y-z} (yzu)^{-1} du dz dy \\ I_5 &= \int_{(1-v)/4}^{(1-v)/3} \int_{(1-v-y)/2}^{(1-y)/2} \int_z^{1-y-z} (yzu)^{-1} du dz dy \\ I_6 &= \int_{(1-v)/3}^{1/3} \int_y^{(1-y)/2} \int_z^{1-y-z} (yzu)^{-1} du dz dy \\ I_7 &= \int_{(1-v)/4}^{1/4} \int_y^{(1-y)/3} \int_z^{(1-y-z)/2} \int_u^{1-y-z-u} (yzut)^{-1} dt du dz dy. \end{aligned}$$

In the following summation the q_i run only over primes.

$$\begin{aligned} S_1 &= \sum_{x^{1-v}}^x 1/q_1 \\ S_2 &= \sum_{x^{(1-v)/2}}^{\sqrt{x}} \sum_{q_1}^{x/q_1} 1/q_1 q_2 \end{aligned}$$

$$\begin{aligned}
S_3 &= \sum_{x^{(1-v)/4}}^{x^{(1-v)/2}} \sum_{x^{1-v}/q_1}^{x/q_1} 1/q_1 q_2 \\
S_4 &= \sum_{x^{(1-v)/4}}^{x^{(1-v)/3}} \sum_{q_1}^{\sqrt{x^{(1-v)}/q_1}} \sum_{x^{1-v}/q_1 q_2}^{x/q_1 q_2} 1/q_1 q_2 q_3 \\
S_5 &= \sum_{x^{(1-v)/4}}^{x^{(1-v)/3}} \sum_{\sqrt{x^{(1-v)}/q_1}}^{\sqrt{x}/q_1} \sum_{q_2}^{x/q_1 q_2} 1/q_1 q_2 q_3 \\
S_6 &= \sum_{x^{(1-v)/3}}^{\sqrt[3]{x}} \sum_{q_1}^{\sqrt{x}/q_1} \sum_{q_2}^{x/q_1 q_2} 1/q_1 q_2 q_3 \\
S_7 &= \sum_{x^{(1-v)/4}}^{\sqrt[4]{x}} \sum_{q_1}^{\sqrt[3]{x}/q_1} \sum_{q_2}^{\sqrt{x}/q_1 q_2} \sum_{q_3}^{x/q_1 q_2 q_3} 1/q_1 q_2 q_3 q_4 .
\end{aligned}$$

We can now restate

LEMMA 1. *If $0 \leq v < 1/2$ then $S_1 = I_1 + O(1/\ln x)$, and*

LEMMA 2. *If $0 \leq v < 1/2$ then $S_2 = I_2 + O(1/\ln x)$, and similarly*

LEMMA $j + 1$. *If $0 \leq v < 1/2$ then $S_j = I_j + O(1/\ln x)$ for $j = 3, 4, 5, 6, 7$.*

The proofs of Lemmas 4, 5, 6, 7, and 8 are straight forward generalizations of Lemmas 1 and 2 and are much too lengthy to be exhibited here.

For any positive integer r and primes $p \equiv 1 \pmod{5}$ let $x = 5[p^{(1+1/r)/4}(\ln^2 p)^{r+1}/5]$. Let $Q^j(x) = Q^j \cap \{m \mid 0 < m \leq x\}$, $j = 0, 1, 2, 3, 4$, and let $N(Q^j(x))$ be the cardinality of $Q^j(x)$.

The following is another special case of the general Theorem of Vinogradov [4], [5]:

LEMMA 9. $N(Q^j(x)) = x/5 + O(x/\ln x)$, $j = 0, 1, 2, 3, 4$.

Proof. Let $\chi_{5,p}$ be the quintic residue character for primes $p \equiv 1 \pmod{5}$. By the argument in the proof of Lemma 3 there is an absolute constant K_3 such that

$$V : \left| \sum_{m=1}^x \chi_{5,p}(m) \right| < K_3 x / \ln x .$$

Set $N(Q^j(x)) = x/5 + T_j$, $j = 0, 1, 2, 3, 4$. Notice that $T_0 = -\sum_{j=1}^4 I_j$. Let $\rho = \cos 2\pi/5 + i \sin 2\pi/5$.

It now follows that

$$\begin{aligned}
\sum_{m=1}^x \chi_{5,p}(m) &= \sum_{j=0}^4 (x/5 + T_j) \rho^j \\
&= \sum_{j=0}^4 T_j \rho^j \\
&= \sum_{j=1}^4 T_j (\cos 2\pi j/5 - 1) + i \sum_{j=1}^4 T_j \sin 2\pi j/5 .
\end{aligned}$$

Now from V it follows that

$$(i) : |(T_1 + T_4)(\cos 2\pi/5 - 1) + (T_2 + T_3)(\cos 4\pi/5 - 1)| < K_3 x / \ln x$$

and

$$(ii) : |(T_1 - T_4) \sin 2\pi/5 + (T_2 - T_3) \sin 4\pi/5| < K_3 x / \ln x .$$

Notice that $\chi_{5,p}^2$ is also a character and by the argument in the proof of Lemma 3 of § 2 there is an absolute constant K_4 such that

$$VI : \left| \sum_{m=1}^x \chi_{5,p}^2(m) \right| < K_4 x / \ln x .$$

But on the other hand

$$\begin{aligned}
\sum_{m=1}^x \chi_{5,p}^2(m) &= \sum_{j=0}^4 (x/5 + T_j) \rho^{2j} \\
&= \sum_{j=0}^4 T_j \rho^{2j} \\
&= \sum_{j=1}^4 T_j (\cos 4\pi j/5 - 1) + i \sum_{j=1}^4 T_j \sin 4\pi j/5 .
\end{aligned}$$

Now by VI it follows that

$$(iii) : |(T_1 + T_4)(\cos 4\pi/5 - 1) + (T_2 + T_3)(\cos 2\pi/5 - 1)| < K_4 x / \ln x$$

$$(iv) : |(T_1 + T_4) \sin 4\pi/5 + (T_3 - T_2) \sin 2\pi/5| < K_4 x / \ln x .$$

With a little manipulation of (i), (ii), (iii), and (iv) one can obtain $|T_j| < K_5 x / \ln x$, $j = 0, 1, 2, 3, 4$, where K_5 is an absolute constant independent of x , proving Lemma 9.

THEOREM 2. *Let d denote the solution of $1/5 = \sum_{i=1}^d I_i$. For all sufficiently large primes $p \equiv 1 \pmod{5}$ there is in each class of quintic non-residues, modulo p , a positive integer smaller than $p^{(1+d)/4+\varepsilon}$ (d satisfies the inequality $.08 < d < .09$).*

Proof. Given $\varepsilon > 0$ let $r = [1/\varepsilon] + 1$. Define x in terms of p as above and notice as long as $\varepsilon < d$ then $x^{1-d+\varepsilon} < p^{(1-d)/4+\varepsilon}$ for sufficiently large values of p . It will suffice to prove that for sufficiently large primes $p \equiv 1 \pmod{5}$ that each class of quintic non-residues modulo p contains a positive integer less than $x^{1-d+\varepsilon}$.

Assume that Theorem 2 is false. Then, for some fixed $\varepsilon > 0$, there are infinitely many primes $p \equiv 1 \pmod{5}$ such that one of their classes of non-residues fails to contain a positive integer less than $x^{1-d+\varepsilon}$.

Let p_1 be one such prime. Notice that x, Q^0, Q^1, Q^2, Q^3 and Q^4 are defined in terms of p_1 and will therefore be fixed in this argument.

Without loss of generality Q^4 can represent that class of non-residues modulo p_1 that has no positive integers less than $x^{1-a+\varepsilon}$. Since Q^4 has this property it follows that Q^1, Q^2 and Q^3 have no positive integers less than $x^{(1-a+\varepsilon)/4}$ because a in Q^1 or Q^3 implies a^4 in Q^4 and a in Q^2 implies a^2 in Q^4 .

Since Q^0 is closed under multiplication, modulo p_1 , an integer w in $Q^4(x)$ must have prime factors not in Q^0 . One of the following conditions holds depending on the exact number of primes, q_i , not in Q^0 that divide w .

(i) There exists a prime q_1 such that $q_1 | w$ and $x^{1-a+\varepsilon} \leq q_1 \leq x$, since q_1 is in $Q^4(x)$.

(ii) There exist primes q_1 and q_2 such that $q_1 q_2 | w$ and $x^{1-a+\varepsilon} < q_1 q_2 < x$, since $q_1 q_2$ is in $Q^4(x)$.

(iii) There exist primes q_1, q_2 and q_3 such that $q_1 q_2 q_3 | w$ and $x^{1-a+\varepsilon} < q_1 q_2 q_3 < x$, since $q_1 q_2 q_3$ is in $Q^4(x)$.

(iv) There exist primes q_1, q_2, q_3 , and q_4 such that $q_1 q_2 q_3 q_4 | w$ and $x^{1-a+\varepsilon} < q_1 q_2 q_3 q_4 < x$, since $q_1 q_2 q_3 q_4$ is in $Q^4(x)$.

It should be noticed that w cannot have more than four prime divisors which are not in Q^0 since the product of any five or more primes not in Q^0 would exceed x . The number of w 's that could possibly satisfy (i) is less than or equal to

$$\sum_{x^{1-d+\varepsilon}}^x [x/q_i].$$

The number of w 's that could possibly satisfy (ii) is less than $x(S_2 + S_3)$. The number of w 's that could possibly satisfy (iii) is less than $x(S_4 + S_5 + S_6)$. The number of w 's that could possibly satisfy (iv) is less than xS_7 . Combining the above we have

$$\begin{aligned} N(Q^4(x)) &< \sum_{x^{1-d+\varepsilon}}^x [x/q_i] + x \sum_{i=2}^7 S_i \\ &\leq - \sum_{x^{1-d}}^{x^{1-d+\varepsilon}} [x/q_i] + x \sum_{i=1}^7 S_i \\ &= -x \ln((1-d+\varepsilon)/(1-d)) + x \left(\sum_{i=1}^7 I_i + K_6 / \ln x \right) \\ &= x(-\ln(1+\varepsilon/(1-d)) + 1/5) + K_6 x / \ln x, \end{aligned}$$

where K_6 is a constant independent of x . But this inequality can hold only for finitely many primes to be compatible with Lemma 9.

4. Remarks. The techniques of the previous sections generalize for K th power non-residues when K is a prime. In these cases the definition of d involves $(K^2 - 3K + 4)/2$ integrals ranging from multi-

plicity 1 through $K - 1$. There are $K - 1$ possible divisibility conditions imposed on the elements of $A^{K-1}(x)$. The upper bound for $N(A^{K-1}(x))$ involves $(K^2 - 3K + 4)/2$ summations ranging from multiplicity 1 through $K - 1$. The contradiction is reached in the same manner. The details are lengthy but straightforward. For example for seventh power residues the results of Davenport and Erdős imply an exponent of p equal to $959/3840$. While using methods exhibited in § 3 one obtains an exponent smaller than $25/104$.

When K is composite the job is more difficult since the subgroup of K th power residues and the $K - 1$ cosets form a cyclic group of composite order. These cyclic groups have proper subgroups. The "without loss of generality" comment is no longer valid and some arguments concerning the number of prime factors of K must be called upon. The author intends to present these techniques at a future date.

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November, 1966

Larry Armijo, <i>Minimization of functions having Lipschitz continuous first partial derivatives</i>	1
Edward Martin Bolger and William Leonard Harkness, <i>Some characterizations of exponential-type distributions</i>	5
James Russell Brown, <i>Approximation theorems for Markov operators</i>	13
Doyle Otis Cutler, <i>Quasi-isomorphism for infinite Abelian p-groups</i>	25
Charles M. Glennie, <i>Some identities valid in special Jordan algebras but not valid in all Jordan algebras</i>	47
Thomas William Hungerford, <i>A description of $\text{Mult}_i(A^1, \dots, A^n)$ by generators and relations</i>	61
James Henry Jordan, <i>The distribution of cubic and quintic non-residues</i>	77
Junius Colby Kegley, <i>Convexity with respect to Euler-Lagrange differential operators</i>	87
Tilla Weinstein, <i>On the determination of conformal imbedding</i>	113
Paul Jacob Koosis, <i>On the spectral analysis of bounded functions</i>	121
Jean-Pierre Kahane, <i>On the construction of certain bounded continuous functions</i>	129
V. V. Menon, <i>A theorem on partitions of mass-distribution</i>	133
Ronald C. Mullin, <i>The enumeration of Hamiltonian polygons in triangular maps</i>	139
Eugene Elliot Robkin and F. A. Valentine, <i>Families of parallels associated with sets</i>	147
Melvin Rosenfeld, <i>Commutative F-algebras</i>	159
A. Seidenberg, <i>Derivations and integral closure</i>	167
S. Verblunsky, <i>On the stability of the set of exponents of a Cauchy exponential series</i>	175
Herbert Walum, <i>Some averages of character sums</i>	189