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ON THE DETERMINATION OF CONFORMAL IMBEDDING

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ON THE DETERMINATION OF CONFORMAL IMBEDDING

TILLA KLOTZ

Two imbedding fundamental forms determine (up to motions) the smooth imbedding of an oriented surface in E^3 . The situation is, however, substantially different for the sufficiently smooth conformal imbedding of a Riemann surface R in E^3 . Conventionally such an imbedding is achieved by a conformal correspondence between R and the Riemann surface R_1 determined on a smoothly imbedded oriented surface S in E^3 by its first fundamental form I . We show that except where $H \cdot K = 0$ on S , such an R_1 conformal imbedding of R in E^3 is determined (up to motions) by the second fundamental form II on S , expressed as a form on R . In particular, I is determined by II on R_1 , where $H \cdot K \neq 0$ on S .

Similar remarks are valid for two less standard methods of conformal imbedding. If an oriented surface S is smoothly imbedded in E^3 so that $H > 0$ and $K > 0$, then II defines a Riemann surface R_2 on S . And, if S is imbedded so that $K < 0$, then II' given by

$$H'II' = KI - HII$$

with

$$H' = -\sqrt{H^2 - K}$$

defines a Riemann surface R'_2 on S . Thus a conformal correspondence between R and R_2 (or R'_2) is called an R_2 (or R'_2) conformal imbedding of R in E^3 . We show that I on S , expressed as a form on R , determines the R_2 or (wherever $H \neq 0$ and sign H is known) the R'_2 imbedding of R in E^3 (up to motions). In particular, I determines II on R_2 , and (where $H \neq 0$, and sign H is known) on R'_2 as well. Finally, we give restatements of the fundamental theorem of surface theory in forms appropriate to R_1 , R_2 and R'_2 conformal imbeddings in E^3 .

The two fundamental forms which determine (up to motions) the smooth imbedding of an oriented surface in E^3 are, of course, related by various equations. But neither form determines the other, except in very special cases. Thus, for instance, isometric imbeddings of a surface in E^3 may differ essentially unless (to cite a famous example) the surface is compact, and the common metric imposed by imbedding has positive Gaussian curvature.

2. Consider an oriented surface S which is C^3 imbedded in E^3 .

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We may introduce C^3 isothermal coordinates x, y locally on S , so that

$$I = \lambda(dx^2 + dy^2),$$

with $x + iy$ a conformal parameter on R_1 , and $\lambda > 0$ a C^2 function of x and y . The Codazzi-Mainardi equations involving λ and the C^1 coefficients L, M and N of II become

$$(1) \quad \begin{aligned} L_y - M_x &= \frac{\lambda_y}{2\lambda} (L + N), \\ N_x - M_y &= \frac{\lambda_x}{2\lambda} (L + N). \end{aligned}$$

The theorem egregium formula for

$$K = \frac{LN - M^2}{\lambda^2}$$

may be written in the form

$$(2) \quad LN - M^2 = \frac{-\lambda}{2} \left\{ \left(\frac{\lambda_x}{\lambda} \right)_x + \left(\frac{\lambda_y}{\lambda} \right)_y \right\}.$$

Moreover, since $\lambda > 0$ while

$$H = \frac{L + N}{2\lambda},$$

the equations (1) may be solved for λ_x/λ and λ_y/λ , provided that $H \neq 0$. Substitution in (2) of the expressions so obtained yields

$$(LN - M^2) = \lambda \left\{ \left(\frac{M_y - N_x}{L + N} \right)_x + \left(\frac{M_x - L_y}{L + N} \right)_y \right\}.$$

If we add the assumption that $K \neq 0$, making $LN - M^2 \neq 0$, then

$$(3) \quad \lambda = \frac{LN - M^2}{\left\{ \left(\frac{M_y - N_x}{L + N} \right)_x + \left(\frac{M_x - L_y}{L + N} \right)_y \right\}}.$$

Thus, we have established our original claim that II on R_1 determines I wherever $H \cdot K \neq 0$. It will be convenient to refer to the expression on the right side of (3) as $\lambda(L, M, N)$. Of course, when L_{yy}, N_{xx} and M_{xy} exist,

$$\begin{aligned} \lambda &= \lambda(L, M, N) = \\ &= \frac{(LN - M^2)(L + N)^2}{(L + N)(2M_{xy} - L_{yy} - N_{xx}) + (L_y + N_y)(L_y - M_x) + (L_x + N_x)(N_x - M_y)}. \end{aligned}$$

In any case, substitution of $\lambda = \lambda(L, M, N)$ in (1) yields conditions

$$(4) \quad \begin{aligned} L_y - M_x &= \frac{\{\lambda(L, M, N)\}_y}{2\lambda(L, M, N)} (L + N) \\ N_x - M_y &= \frac{\{\lambda(L, M, N)\}_x}{2\lambda(L, M, N)} (L + N) \end{aligned}$$

on L, M and N wherever $H \cdot K \neq 0$, or, equivalently, wherever

$$(L + N)(LN - M^2) \neq 0.$$

Suppose now that a C^1 quadratic form

$$\Omega = Ldx^2 + 2Mdx dy + Ndy^2$$

is given on a Riemann surface R . (Here $x + iy$ is a conformal parameter on R .) Suppose also that $(L + N)(LN - M^2) \neq 0$. Then the previous discussion establishes

$$\lambda(L, M, N)(dx^2 + dy^2)$$

as the only possible I for a $C^3 R_1$ conformal imbedding of R in E^3 with $\Omega = II$. Thus, if such an imbedding exists, $\lambda(L, M, N)$ must be positive and C^2 , while (4) must be valid. On the other hand, if $\lambda(L, M, N)$ is a positive C^2 function, and if (4) does hold, then both (1) and (2) are valid with $\lambda = \lambda(L, M, N)$. Thus the fundamental theorem of surface theory (see p. 124 of [3]) immediately implies the following result.

THEOREM 1. *If $\Omega = Ldx^2 + 2Mdx dy + Ndy^2$ is a C^1 quadratic form on R with $(L + N)(LN - M^2) \neq 0$, then necessary and sufficient conditions for the existence (locally) and uniqueness (up to motions) of a $C^3 R_1$ conformal imbedding of R in E^3 with $\Omega = II$ are that $\lambda(L, M, N)$ be positive and C^2 , and that (4) be valid.*

3. Consider an oriented surface S which is C^4 imbedded in E^3 so that $H > 0$ and $K > 0$. We may introduce C^3 bisothermal coordinates x, y locally on S , so that

$$II = \mu(dx^2 + dy^2)$$

with $x + iy$ a conformal parameter on R_2 , and $\mu > 0$ a C^1 function of x and y . Here the Codazzi-Mainardi equations involving μ , and the Christoffel symbols for the coefficients E, F and G of I become

$$(5) \quad \begin{aligned} \mu_x &= \mu(\Gamma_{12}^2 - \Gamma_{22}^1), \\ \mu_y &= \mu(\Gamma_{12}^1 - \Gamma_{11}^2). \end{aligned}$$

And the theorem egregium yields a complicated expression for

$$K = \frac{\mu^2}{EG - F^2} > 0$$

as a function of E, F, G and their first and second partial derivatives, which we refer to for convenience as $K(E, F, G)$. Thus

$$(6) \quad \mu = \sqrt{K(E, F, G)(EG - F^2)},$$

and we have established our original claim that I on R_2 determines II . We will refer to the expression on the right side of (6) as $\mu(E, F, G)$. Here, substitution of $\mu = \mu(E, F, G)$ in (5) yields conditions

$$(7) \quad \begin{aligned} \{\mu(E, F, G)\}_x &= \mu(E, F, G)(\Gamma_{12}^2 - \Gamma_{22}^1), \\ \{\mu(E, F, G)\}_y &= \mu(E, F, G)(\Gamma_{12}^1 - \Gamma_{11}^2), \end{aligned}$$

on E, F and G .

Suppose now that a C^2 quadratic form

$$\Omega = Edx^2 + 2Fdxdy + Gdy^2$$

is given on a Riemann surface R . Suppose also that $K(E, F, G) > 0$. Then the previous discussion establishes

$$\mu(E, F, G)(dx^2 + dy^2)$$

as the only possible II for a $C^3 R_2$ conformal imbedding of R in E^3 with $\Omega = I$. Thus, if such an imbedding exists, $\mu(E, F, G)$ must be positive and C^1 , while (7) must be valid. On the other hand, if $\mu(E, F, G)$ is a positive C^1 function, and if (7) does hold, then both (5) and $K = K(E, F, G)$ are valid, with $\mu = \mu(E, F, G)$. Thus the fundamental theorem of surface theory immediately implies the following result.

THEOREM 2. *If $\Omega = Edx^2 + 2Fdxdy + Gdy^2$ is a C^2 quadratic form on R , then necessary and sufficient conditions for the existence (locally) and uniqueness (up to motions) of a $C^3 R_2$ conformal imbedding of R in E^3 with $\Omega = I$ are that $K(E, F, G)$ be positive, that $\mu(E, F, G)$ be positive and C^1 , and that (7) be valid.*

4. Finally, consider an oriented surface S which is C^4 imbedded in E^3 so that $K < 0$. We may introduce C^3 isothermal coordinates x, y locally on S , so that

$$II' = \mu'(dx^2 + dy^2)$$

with $x + iy$ a conformal parameter on R'_2 , and $\mu' > 0$ a C^1 function of x and y . Since $H'II' = KI - HII$,

$$(8) \quad \begin{aligned} HL + H'\mu' &= KE, \\ HN + H'\mu' &= KG, \\ HM &= KF. \end{aligned}$$

But we show in [2] that on R'_2 ,

$$(9) \quad L = -N.$$

Thus

$$K = \frac{-(L^2 + M^2)}{EG - F^2} < 0$$

must be given by the theorem egregium expression $K(E, F, G)$. Using (8), we obtain

$$(10) \quad \begin{aligned} HL &= K(E - G), \\ HM &= KF, \end{aligned}$$

so that

$$H^2(L^2 + M^2) = K^2\langle(E - G)^2 + F^2\rangle.$$

Division of this last equation by $(L^2 + M^2) = -K(E, F, G)(EG - F^2) \neq 0$ yields

$$(11) \quad H = \pm \sqrt{\frac{-K(E, F, G)}{EG - F^2} \{E - G\}^2 + F^2}.$$

Thus H vanishes if and only if $(E - G) + iF = 0$. Of course, where $H \neq 0$, the orientation of S determines the sign of H . On the other hand, where $H \neq 0$, we may set

$$\begin{aligned} L(E, F, G) &= (G - E) \sqrt{\frac{-K(E, F, G)(EG - F^2)}{\{(E - G)^2 + F^2\}}}, \\ M(E, F, G) &= -F \sqrt{\frac{-K(E, F, G)(EG - F^2)}{\{(E - G)^2 + F^2\}}}, \end{aligned}$$

and

$$N(E, F, G) = -L(E, F, G).$$

Using (10), we conclude that so long as $H \neq 0$,

$$(12) \quad \begin{aligned} L &= \pm L(E, F, G), \\ M &= \pm M(E, F, G), \\ N &= \pm N(E, F, G), \end{aligned}$$

with plus or minus signs consistently chosen in accordance with the

sign of H . Thus we have established our original claim that I on R'_2 determines II (if $H \neq 0$, and sign H is known).

Note, however, that the Codazzi-Mainardi equations on R'_2 which read

$$(13) \quad \begin{aligned} L_y - M_x &= L(\Gamma_{12}^1 + \Gamma_{11}^2) + M(\Gamma_{12}^2 - \Gamma_{11}^1) \\ L_x + M_y &= L(\Gamma_{22}^1 + \Gamma_{12}^2) + M(\Gamma_{22}^2 - \Gamma_{12}^1) \end{aligned}$$

are not affected by the sign of H . (In particular, if L, M and N solve (13), so will $-L, -M$ and $-N$.) Thus, whichever the choice of signs in (12), the Codazzi-Mainardi equations yield the following conditions

$$(14) \quad \begin{aligned} \{L(E, F, G)\}_y - \{M(E, F, G)\}_x &= L(E, F, G)(\Gamma_{12}^1 + \Gamma_{11}^2) + M(E, F, G)(\Gamma_{12}^2 - \Gamma_{11}^1), \\ \{L(E, F, G)\}_x - \{M(E, F, G)\}_y &= L(E, F, G)(\Gamma_{22}^1 + \Gamma_{12}^2) + M(E, F, G)(\Gamma_{22}^2 - \Gamma_{12}^1), \end{aligned}$$

on E, F and G , wherever $(E - G) + iF \neq 0$.

Suppose now that a C^2 quadratic form

$$\Omega = Edx^2 + 2Fdxdy + Gdy^2$$

is given on a Riemann surface R . Suppose also that $K(E, F, G) < 0$ while $(E - G) + iF \neq 0$. Then the previous discussion establishes

$$(15) \quad L(E, F, G)(dx^2 - dy^2) + 2M(E, F, G)dxdy$$

and

$$(16) \quad -L(E, F, G)(dx^2 - dy^2) - 2M(E, F, G)dxdy$$

as the only possible forms which could serve as II for a $C^3 R'_2$ conformal imbedding of R in E^3 with $\Omega = I$. Thus, if such an imbedding exists, $L(E, F, G)$ and $M(E, F, G)$ must be C^1 functions, while (14) must be valid. Finally, should such an imbedding exist with one choice (15) or (16) for II , composition with a reflection of S in a plane will leave I invariant while yielding the remaining choice for II . On the other hand, if $L(E, F, G)$ and $M(E, F, G)$ are C^1 functions, and if (14) does hold, then, given either choice (15) or (16) for II , both (13) and $K = K(E, F, G)$ are valid. Thus the fundamental theorem of surface theory immediately implies the following result.

Theorem 3. *If $\Omega = Edx^2 + 2Fdxdy + Gdy^2$ is a C^2 quadratic form on R with $(E - G) + iF \neq 0$, then necessary and sufficient conditions for the existence (locally) and uniqueness (up to motions and reflections in planes) of a $C^3 R'_2$ conformal imbedding of R in*

E^3 with $\Omega = I$ are that $K(E, F, G)$ be negative, that $L(E, F, G)$ and $M(E, F, G)$ be C^1 functions, and that (14) be valid.

5. We close by noting a pair of statements of the type one gets by slight rewording of the results described above. Isometric oriented surfaces imbedded C^4 in E^3 so that $H > 0$ and $K > 0$ are congruent if and only if the isometry between them is conformal between their R_2 structures. Similarly, such surfaces on which $H \neq 0$ and $K < 0$ are congruent if and only if the isometry between them is conformal between their R'_2 structures and preserves the sign of H .

The weakness of these results amply illustrates the sense in which II , while inessential on R_2 or R'_2 , is of fundamental importance in determining the imbedding of a surface, as distinct from the R_2 or R'_2 conformal imbedding of a Riemann surface. None-the-less, more significant applications of Theorems 1, 2 and 3 should be possible.

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