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**ON THE CONSTRUCTION OF CERTAIN BOUNDED
CONTINUOUS FUNCTIONS**

JEAN-PIERRE KAHANE

ON THE CONSTRUCTION OF CERTAIN BOUNDED CONTINUOUS FUNCTIONS

J.-P. KAHANE

We give an elementary method for constructing continuous functions fulfilling the hypothesis of Theorem 1 of the preceding paper. Such functions thus constitute counterexamples to the proposition and theorem discussed therein.

THEOREM. *Let $\varphi(x)$ be continuously differentiable on $[0, \infty)$, and suppose*

- (i) $\varphi(0) = 0$
- (ii) $\varphi'(x)$ is nonnegative, and strictly increasing to ∞ on $[0, \infty)$
- (iii) $\varphi'(x)/\varphi(x) \rightarrow \infty, x \rightarrow \infty$.

Put

$$(1) \quad f(x) = \sum_1^{\infty} 2^{-m} \exp\left(\frac{2\pi i}{2^m} x\right), \quad x < 0$$

$$(2) \quad f(x) = e^{i\varphi(x)}, \quad x \geq 0.$$

Then the bounded continuous function $f(x)$ has properties 1, 2, and 3 of Theorem 1 in the previous paper.

Proof. That $0 \in sp f$ follows from (1) as in §2 of the previous paper.

To establish property 3, let us show that

$$\frac{1}{T} \int_0^T f(x+a) dx \rightarrow 0$$

uniformly in a as $T \rightarrow \infty$. If I is any interval of length T , denote by A the part of I lying to the left of 0, and by B that part lying to the right. We have, by (1),

$$\frac{1}{T} \int_A f(x) dx = \frac{|A|}{T} \left\{ \frac{1}{|A|} \int_A \sum_1^{\infty} 2^{-m} \exp\left(\frac{2\pi i}{2^m} x\right) dx \right\}.$$

The quantity in brackets is always in absolute value ≤ 1 , and tends to zero independently of the position of A as $|A| \rightarrow \infty$ (this fact belongs to the rudiments of the theory of almost periodic functions, and can here be verified by direct calculation). Since $|A| \leq T$, we have

$$(3) \quad \frac{1}{T} \int_A f(x) dx \rightarrow 0 \text{ independently of the position of } I \text{ as } T \rightarrow \infty.$$

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The integral $\int_B f(x)dx$ is bounded for all intervals B of the form $[0, b]$. Indeed, if $b > 1$,

$$\int_0^b f(x)dx = \int_0^1 f(x)dx + \int_1^b f(x)dx .$$

Since $\varphi'(x) \geq 0$ we can, by (2), make the substitution $\varphi(x) = \xi$ in the second integral on the right, getting for it the value

$$\int_1^b e^{i\varphi(x)} dx = \int_{\varphi(1)}^{\varphi(b)} e^{i\xi} \frac{d\xi}{\varphi'(x)} .$$

In view of (ii), this last is in absolute value $\leq 4/\varphi'(1)$ by the second mean value theorem. It follows that $\int_B f(x)dx$ is bounded for *all* intervals B lying to the right of the origin, whence

$$(4) \quad \frac{1}{T} \int_B f(x)dx \rightarrow 0 \text{ independently of the position of } I \text{ as } T \rightarrow \infty .$$

From (3) and (4) we see that $1/T \int_I f(x)dx$ is small in absolute value for all intervals I of length T , if only T is sufficiently large, which is property 3.

It remains to verify property 2. We show that if $0 < X_1 < \dots < X_M$ and the A_k are complex numbers

$$(5) \quad \sup_{x > 0} \left| \sum_{k=1}^M A_k e^{i\varphi(x+X_k)} \right| = \sum_1^M |A_k| .$$

So as not to lose the reader in details, we do this for the case $M = 2$; it will be clear how to extend the reasoning to any value of M .

Let ε be given, $0 < \varepsilon < \pi/2$, and, choosing a positive determination of the argument, put, for $k = 1, 2, 3, \dots$

$$(6) \quad a_k = \varphi^{-1}\left(2\pi k + \arg \frac{1}{A_1} - \varepsilon\right) - X_1$$

$$(7) \quad b_k = \varphi^{-1}\left(2\pi k + \arg \frac{1}{A_1} + \varepsilon\right) - X_1 .$$

Clearly $a_k < b_k < a_{k+1}$, $a_k \rightarrow \infty$ as $k \rightarrow \infty$, and by (ii),

$$(8) \quad b_k - a_k \rightarrow 0, k \rightarrow \infty .$$

Also,

$$(9) \quad \Re(A_1 e^{i\varphi(x+X_1)}) \geq (1 - \varepsilon^2) |A_1| \quad \text{for } a_k \leq x \leq b_k .$$

I claim that $\varphi(b_k + X_2) - \varphi(a_k + X_2) \rightarrow \infty$ as $k \rightarrow \infty$. If $c > 0$, by (ii):

$$\frac{\varphi'(x+c)}{\varphi'(x)} \geq c \frac{\varphi'(x+c)}{\varphi(x+c) - \varphi(x)} \geq c \frac{\varphi'(x+c)}{\varphi(x+c)},$$

whence

$$(10) \quad \frac{\varphi'(x+c)}{\varphi'(x)} \rightarrow \infty, x \rightarrow \infty,$$

in view of (iii). Since $X_2 > X_1$, there is, by (8), a $c > 0$ such that, for all sufficiently large k , $c + b_k + X_1 \leq a_k + X_2$. We thus have, from (6), (7), (ii), and (10):

$$\begin{aligned} \varphi(b_k + X_2) - \varphi(a_k + X_2) &= 2\varepsilon \frac{\varphi(b_k + X_2) - \varphi(a_k + X_2)}{\varphi(b_k + X_1) - \varphi(a_k + X_1)} \\ &\geq 2\varepsilon \frac{\varphi'(b_k + X_1 + c)}{\varphi'(b_k + X_1)} \rightarrow \infty \end{aligned}$$

as $k \rightarrow \infty$, since $b_k \rightarrow \infty, k \rightarrow \infty$. This is the desired result which implies, in particular, the existence, for all sufficiently large k , of an $x_k \in [a_k, b_k]$ such that

$$\varphi(x_k + X_2) \equiv \arg \frac{1}{A_2} \pmod{2\pi}.$$

For such x_k we have $A_2 e^{i\varphi(x_k + X_2)} = |A_2|$ which, together with (9), yields (5) for the case $M = 2$, since $\varepsilon > 0$ is arbitrary.

REMARK. Suppose $\varphi(x)$ is *even*, and fulfills condition (i), (ii), and (iii) of the theorem. Besides this, let it be twice continuously differentiable, and be such that $\varphi''(x) \geq C > 0$ (example: $\varphi(x) = e^{x^2}$). Then, if $f(x) = e^{i\varphi(x)}$, $e^{i\lambda x}$ is not, for *any* $\lambda \in sp f$, in the weak closure of any bounded subset of V_f (notation as in the preceding paper). (This observation is due to P. Koosis.)

Indeed, the function $f(x)$ clearly has property 2, according to the above work. A glance at the proof of Theorem 1 in the preceding paper now shows that the desired result will certainly follow if we establish, for all real λ , that

$$\frac{1}{T} \int_0^T f(x+X) e^{-i\lambda x} dx \rightarrow 0 \text{ uniformly in } X \text{ as } T \rightarrow \infty. \text{ But by a}$$

lemma of Van der Corput ([1], vol I, p. 197),

$$\left| \int_0^T e^{i[\varphi(x+X) - \lambda x]} dx \right| \leq 12 \cdot \left\{ \inf_{0 \leq x \leq T} \varphi''(x+X) \right\}^{-1/2} \leq 12C^{-1/2}$$

for all T , which implies the desired statement.

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