ON THE CONSTRUCTION OF CERTAIN BOUNDED CONTINUOUS FUNCTIONS

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We give an elementary method for constructing continuous functions fulfilling the hypothesis of Theorem 1 of the preceding paper. Such functions thus constitute counterexamples to the proposition and theorem discussed therein.

**Theorem.** Let \( \varphi(x) \) be continuously differentiable on \([0, \infty)\), and suppose

(i) \( \varphi(0) = 0 \)
(ii) \( \varphi'(x) \) is nonnegative, and strictly increasing to \( \infty \) on \([0, \infty)\)
(iii) \( \varphi'(x)/\varphi(x) \to \infty, x \to \infty \).

Put

\[ f(x) = \sum_{m=1}^{\infty} 2^{-m} \exp\left(\frac{2\pi i}{2^m} x\right), \quad x < 0 \]
\[ f(x) = e^{i\varphi(x)}, \quad x \geq 0. \]

Then the bounded continuous function \( f(x) \) has properties 1, 2, and 3 of Theorem 1 in the previous paper.

**Proof.** That \( 0 \in sp f \) follows from (1) as in § 2 of the previous paper.

To establish property 3, let us show that

\[ \frac{1}{T} \int_{-T}^{0} f(x + a)dx \to 0 \]

uniformly in \( a \) as \( T \to \infty \). If \( I \) is any interval of length \( T \), denote by \( A \) the part of \( I \) lying to the left of 0, and by \( B \) that part lying to the right. We have, by (1),

\[ \frac{1}{T} \int_{A} f(x)dx = \frac{|A|}{T} \left\{ \frac{1}{|A|} \int_{A} \sum_{m=1}^{\infty} 2^{-m} \exp\left(\frac{2\pi i}{2^m} x\right)dx \right\}. \]

The quantity in brackets is always in absolute value \( \leq 1 \), and tends to zero independently of the position of \( A \) as \( |A| \to \infty \) (this fact belongs to the rudiments of the theory of almost periodic functions, and can here be verified by direct calculation). Since \( |A| \leq T \), we have

\[ \frac{1}{T} \int_{A} f(x)dx \to 0 \] independently of the position of \( I \) as \( T \to \infty \).
The integral \( \int_0^b f(x) \, dx \) is bounded for all intervals \( B \) of the form \([0, b]\). Indeed, if \( b > 1 \),
\[
\int_0^b f(x) \, dx = \int_0^1 f(x) \, dx + \int_1^b f(x) \, dx.
\]
Since \( \varphi'(x) \geq 0 \) we can, by (2), make the substitution \( \varphi(x) = \xi \) in the second integral on the right, getting for it the value
\[
\int_1^b e^{i\varphi(x)} \, dx = \int_{\varphi(1)}^{\varphi(b)} e^{it} \frac{d\xi}{\varphi'(x)}.
\]
In view of (ii), this last is in absolute value \( \leq 4/\varphi'(1) \) by the second mean value theorem. It follows that \( \int_b^\infty f(x) \, dx \) is bounded for all intervals \( B \) lying to the right of the origin, whence
\[
(4) \quad \frac{1}{T} \int_B f(x) \, dx \rightarrow 0 \text{ independently of the position of } I \text{ as } T \rightarrow \infty.
\]
From (3) and (4) we see that \( 1/T \int_I f(x) \, dx \) is small in absolute value for all intervals \( I \) of length \( T \), if only \( T \) is sufficiently large, which is property 3.

It remains to verify property 2. We show that if \( 0 < X_1 < \cdots < X_M \) and the \( A_k \) are complex numbers
\[
(5) \quad \sup_{x>0} \left| \sum_{k=1}^M A_k e^{i\varphi(x+x_k)} \right| = \sum_{k=1}^M |A_k|.
\]
So as not to lose the reader in details, we do this for the case \( M = 2 \); it will be clear how to extend the reasoning to any value of \( M \).

Let \( \varepsilon \) be given, \( 0 < \varepsilon < \pi/2 \), and, choosing a positive determination of the argument, put, for \( k = 1, 2, 3, \cdots \)
\[
(6) \quad a_k = \varphi^{-1}\left(2\pi k + \arg \frac{1}{A_1} + \varepsilon\right) - X_i
\]
\[
(7) \quad b_k = \varphi^{-1}\left(2\pi k + \arg \frac{1}{A_1} - \varepsilon\right) - X_i.
\]
Clearly \( a_k < b_k < a_{k+1}, a_k \rightarrow \infty \text{ as } k \rightarrow \infty \), and by (ii),
\[
(8) \quad b_k - a_k \rightarrow 0, \; k \rightarrow \infty.
\]
Also,
\[
(9) \quad \Re(A_1 e^{i\varphi(x+x_1)}) \geq (1 - \varepsilon^5) |A_1| \text{ for } a_k \leq x \leq b_k.
\]
I claim that \( \varphi(b_k + X_0) - \varphi(a_k + X_2) \rightarrow \infty \text{ as } k \rightarrow \infty \). If \( c > 0 \), by (ii):
\[
\frac{\varphi'(x + c)}{\varphi'(x)} \geq \frac{c \varphi'(x + c)}{\varphi(x + c) - \varphi(x)} \geq \frac{\varphi'(x + c)}{\varphi(x + c)},
\]
whence

\[
(10) \quad \frac{\varphi'(x + c)}{\varphi'(x)} \to \infty, \quad x \to \infty,
\]
in view of (iii). Since \( X_2 > X_1 \), there is, by (8), a \( c > 0 \) such that, for all sufficiently large \( k, c + b_k + X_1 \leq a_k + X_2 \). We thus have, from (6), (7), (ii), and (10):

\[
\varphi(b_k + X_2) - \varphi(a_k + X_k) = 2c \frac{\varphi(b_k + X_k) - \varphi(a_k + X_k)}{\varphi(b_k + X_1) - \varphi(a_k + X_1)} \geq 2c \frac{\varphi'(b_k + X_1 + c)}{\varphi'(b_k + X_1)} \to \infty
\]
as \( k \to \infty \), since \( b_k \to \infty, k \to \infty \). This is the desired result which implies, in particular, the existence, for all sufficiently large \( k \), of an \( x_k \in [a_k, b_k] \) such that

\[
\varphi(x_k + X_2) = \arg \frac{1}{A_2} \pmod{2\pi}.
\]
For such \( x_k \) we have \( A_k e^{i \varphi(a_k + X_2)} = |A_2| \) which, together with (9), yields (5) for the case \( M = 2 \), since \( \varepsilon > 0 \) is arbitrary.

**Remark.** Suppose \( \varphi(x) \) is even, and fulfills condition (i), (ii), and (iii) of the theorem. Besides this, let it be twice continuously differentiable, and be such that \( \varphi''(x) \geq C > 0 \) (example: \( \varphi(x) = e^{x^2} \)). Then, if \( f(x) = e^{i \varphi(x)}, e^{i \lambda x} \) is not, for any \( \lambda \in \text{sp} \ f \), in the weak closure of any bounded subset of \( V_f \) (notation as in the preceding paper). (This observation is due to P. Koosis.)

Indeed, the function \( f(x) \) clearly has property 2, according to the above work. A glance at the proof of Theorem 1 in the preceding paper now shows that the desired result will certainly follow if we establish, for all real \( \lambda \), that

\[
\frac{1}{T} \int_0^T f(x + X) e^{-i \lambda x} \, dx \to 0 \text{ uniformly in } X \text{ as } T \to \infty.
\]
But by a lemma of Van der Corput ([1], vol I, p. 197),

\[
\left| \int_0^T e^{i \varphi(x + X) - \lambda x} \, dx \right| \leq 12 \cdot \left\{ \inf_{0 \leq x \leq T} \varphi''(x + X) \right\}^{-1/2} \leq 12C^{-1/2}
\]
for all \( T \), which implies the desired statement.
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