COMMUTATIVE $F$-ALGEBRAS

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We extend several theorems for commutative Banach algebras to topological algebras with a sequence of semi-norms (F-algebras). The question of what functions "operate" on an F-algebra is considered. It is proven that analytic functions in several complex variables operate by applying a theorem due to Waelbroeck. If all continuous functions operate on an F-algebra, then it is an algebra of continuous functions. However, unlike the situation for Banach algebras [6], it is not true that if $\sqrt{\cdot}$ operates the algebra is $C(\mathcal{D})$. This will be shown by an example. A theorem due to Curtis [4], concerning continuity of derivations when the algebra is regular is extended to F-algebras. The result is applied to an algebra of Lipschitz functions to show that it has only a trivial derivation.

Preliminaries. Throughout this paper the letter $A$ will stand for a commutative F-algebra. An F-algebra is a topological algebra with topology determined by a sequence of algebraic semi-norms. The $n$th semi-norm of an element $x$ in $A$ will be written $\|x\|_n$. We may and shall always assume that for all $x$ in $A$, $\|x\|_n \leq \|x\|_{n+1}$. $\mathcal{D}^+$ will denote the topological space of all continuous multiplicative linear functionals on $A$ with the weak* topology. $\mathcal{D}$ will denote $\mathcal{D}^+$ minus the zero functional with the relativized topology. For $x$ in $A$, $\hat{x}$ will be the function in $C(\mathcal{D}^+)$ (the continuous functions on $\mathcal{D}^+$ with the compact-open topology) defined by $\hat{x}(\varphi) = \varphi(x)$. $A$ will be called regular if given $\varphi_0$ in $\mathcal{D}$ and $V$ a neighborhood of $\varphi_0$, there is an element $x$ in $A$ such that $\varphi_0(x) = 1$ and $\varphi(x) = 0$ for $\varphi \not\in V$. $A$ will be called semi-simple if $\hat{x} = 0$ implies $x = 0$.

A basic device in the study of F-algebras is to represent $A$ as the inverse limit of a sequence of Banach algebras $\{A_n\}$ where $A_n$ is the completion of $A/I_n$ with norm $\|x + I_n\| = \|x\|_n$ and $I_n$ is the ideal of all $x$ in $A$ such that $\|x\|_n = 0$. The homomorphism $\pi_{m,n} : A_n \to A_m$ for $m \leq n$ is defined as the completion of the mapping $x + I_n \to x + I_m$. This representation enables one to construct an element in $A$ by constructing a sequence $\{x_n\}$ such that for each $n,$
Let $x_n \in A_n$ and $\pi_{m,n} x_n = x_m$. The homomorphism $\pi_n: A \rightarrow A_n$ is defined as $x \mapsto x + I_n$. Then $\pi_n^\ast$ (multiplicative linear functionals in $A_n$) $\rightarrow A^+$ is continuous and one-to-one and so its range, which we shall denote by $A^+_n$ is a compact subset of $A^+$. If $K$ is an arbitrary compact subset of $A^+$, there is an integer $n$ such that $K \subseteq A^+_n$ [9].

The following theorem, due to Silov, is also valid for $F$-algebras. If $C$ is a closed and open subset of $A^+$ and the zero homomorphism is not in $C$, then there is an idempotent $e$ in $A$ such that $C = \{ \phi \in A^+: \phi(e) = 1 \}$. The extension to $F$-algebras is proven via the device of the previous paragraph. With the aid of Silov’s theorem the proof that if $A$ is regular, then $A$ is normal is essentially the same as for Banach algebras.

Since so many of the theorems true for Banach algebras are also true for $F$-algebras with almost the same proofs, it is perhaps appropriate to remark that the difficulties introduced by the sequence of semi-norms are sometimes quite subtle. For example such a seemingly innocuous question as whether a multiplicative linear functional is necessarily continuous is still unanswered.

Functions that operate on a commutative semi-simple $F$-algebra. A function $f: D \subseteq C \rightarrow C$ is said to "operate" on an $F$-algebra $A$ if $f \circ \hat{x} \in \hat{A}$ whenever $x \in A$ and the range $\hat{x} \subseteq D$. It is not difficult to adapt Katznelson’s proof in [5] to show that if every continuous function operates on $A$, then $A = C(\Delta)$. However another theorem due to Katznelson which states: If $A$ is a self-adjoint Banach algebra and $\sqrt{--}$ operates on the positive functions in $\hat{A}$, then $A = C(\Delta)$ is no longer true for $F$-algebras; as the following example shows.

Let $H$ be the subalgebra of $l^\infty$ consisting of those sequences $\{a_n\}$ for which there is a number, $a$ such that $|a_n - a|^{1/n} \rightarrow 0$. Let $H'$ be the subalgebra of $H$ consisting of those sequences for which $a = 0$. Let $\tau$ be the linear transformation from $H'$ to the entire functions defined by $\tau(\{a_n\})(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$. For each integer $N$ and for $\{a_n\} \in H'$ defined $\| \{a_n\} \|_F = \sup \{ | \tau(\{a_n\})(\lambda) | : |\lambda| \leq N \}$. $\| \cdot \|_F$ is evidently a vector space norm. It is also algebraic; for suppose $\{a_n\}$ and $\{b_n\} \in H'$, $f = \tau(\{a_n\})$, $g = \tau(\{b_n\})$ and $F = \tau(\{a_n b_n\})$. Then

$$F(\lambda) = (1/2\pi i) \int_{|w|=M} f(w) g(\lambda/w) dw / w.$$  

$H'$ is a complete $F$-algebra under the sequence of norms defined above and $H$ is the $F$-algebra obtained by adjoining a unit to $H'$.

For $n = 0, 1, 2, \ldots$, define $z_n$ as the sequence which is 1 in the $n$th coordinate and 0 in all the other coordinates. These elements generate $H'$ (since the polynomials are dense in the entire functions)
and together with the unit of $H$ generate $H$. $\Delta(H)$ is homeomorphic to the one-point compactification of the integers, the point corresponding to the integer $n$ being the functional sending $z_n$ into 1.

It is evident that $\hat{H}$ is a self-adjoint subalgebra of $C(\Delta(H))$, and that $H$ is semi-simple and regular. Yet, although $\sqrt{-1}$ operates on the nonnegative elements of $\hat{H}$, $H \neq C(\Delta(H))$.

For $U$ an open subset of $C^n$ let $H(U)$ be the $F$-algebra of all holomorphic functions on $U$ with the compact-open topology. For $\sigma$ an arbitrary subset of $C^n$, let $H(\sigma)$ be the direct limit of the $F$-algebras $H(U)$ for $U$ ranging over open sets containing $\sigma$ directed as follows: $H(U)^{H(V)}$ if $U \subseteq V$.

Let $a_1, \ldots, a_n$ be elements of a commutative $F$-algebra, say $A$, with unit. For $\varphi \in \Delta = \Delta(A)$, let $\sigma(\varphi)$ be the point in $C^n (\varphi(a_1), \ldots, \varphi(a_n))$ and let $\sigma = \{\sigma(\varphi) : \varphi \in \Delta\}$.

**THEOREM.** There is a continuous homomorphism $\tau$ from $H(\sigma)$ to $A$ such that $\varphi(\tau f) = f(\sigma(\varphi))$ for every $\varphi$ in $\Delta$ and every $f$ in $H(\sigma)$ and $\tau(z_i) = a_i, i = 1, \ldots, n$. (Evidently $f \in H(\sigma)$ defines a function on $\sigma$.)

**Proof.** Waelbroeck, in [11], proved that such a continuous homomorphism exists for even more general topological algebras providing the elements, $a_1, \ldots, a_n$ are regular, i.e. have compact spectrum. An element of an $F$-algebra needn’t be regular, but an element of a Banach algebra is of course regular. We will apply Waelbroeck’s theorem to each of the Banach algebras $A_s$ where $A$ is the inverse limit of $\{A_s\}$.

For every integer $k$ let $\sigma_k$ be defined as above for $\pi_k a_1, \ldots, \pi_k a_n$, let $\tau_k$ be the continuous homomorphism from $H(\sigma_k)$ to $A_s$ \( \forall k : \sigma_k \subseteq \sigma \) and there is a continuous homomorphism $\nu_k : H(\sigma) \to H(\sigma_k)$. The essence of the proof is that the sequence $\{f_k\}$ where $f_k \in A_k$ is defined as $\tau_k \circ \nu_k(f)$ satisfies $\pi_{s+1} f_s = f_s$ for $s = 1$. For then the sequence $\{f_k\}$ defines an element $\hat{f}$ in $A$.

If each $A_k$ were semi-simple, then it would follow that $\pi_s f_s = f_s$ for $s = 1$. For Waelbroeck’s theorem implies that $(\pi_{s+1} f_{s+1})^\wedge = \hat{f}_s$. However, even if $A$ is semi-simple, it does not follow that each $A_k$ is semi-simple.

Let $s$ and $t$ be two fixed integers with $s \leq t$. We shall examine the construction of $f_s$. Let $b_i = \pi_i a_i$ for $i = 1, \ldots, n$. $f \in H(\sigma)$ may be considered as a function holomorphic in a neighborhood, say $W$, of $\sigma$ and, therefore, of $\sigma_s$. The following assertions are proven in [11].

1. $\sigma_s$ is convex in the following sense. There is a finite set of polynomials in $n$ variables, say $p_1, \ldots, p_r$ and neighborhoods $D_1, \ldots, D_n$ of the spectrum of $b_1, \ldots, b_n$ respectively and neighborhoods $D_{n+1}, \ldots, D_{n+r}$.
of the spectrum of \( b_{n+1} = p_i(b_1, \ldots, b_n), \ldots, b_{n+r} = p_r(b_1, \ldots, b_n) \) respectively such that the following two facts are true:

(a) \( \sigma_s \subseteq D \subseteq W \) where \( D = \{ \lambda \in D_1 \times \cdots \times D_n : p_i(\lambda) \in D_{n+i} \text{ for } i = 1, \ldots, r \} \).

(b) If \( E = D_1 \times \cdots \times D_n \times \cdots \times D_{n+r} \) and \( X = \{ (\lambda, p_1(\lambda), \ldots, p_r(\lambda)) : \lambda \in D \} \), then the restriction mapping, \( \rho \), from \( E \) to \( X \) is a continuous open homemorphism of \( H(E) \) onto \( H(X) \) with kernel the ideal generated by \( \{ z^i_{n+k} - p_k(z, \ldots, z_n) : k = 1, \ldots, r \} \). By (a), \( f \) is a holomorphic function on \( D \) and determines a function \( F \in H(X) \) where \( F(\lambda, p(\lambda)) \) is defined to be \( f(\lambda) \) (i.e. \( F \) depends only on the first \( n \) coordinates). By (b), \( F = \rho(G) \) where \( G \in H(E) \).

(2) Define \( \alpha : H(E) \rightarrow A_s \) by

\[
\alpha(H) = (1/2\pi i)^{n+r} \int_{\Gamma_1} \cdots \int_{\Gamma_{n+r}} H(\lambda_1, \ldots, \lambda_{n+r})(\lambda_1^i - b_1)^{-1} \cdots (\lambda_{n+r}^i - b_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r}
\]

where \( \Gamma_i \) is a rectifiable curve in \( D_i \) including in its interior the spectrum of \( b_i \) for \( i = 1, \ldots, n + r \). \( \alpha \) is a continuous homomorphism and \( \alpha(z_i) = b_i \) for \( i = 1, \ldots, n \). Thus, by (b), if \( \rho(G_i) = \rho(G) = F \), then \( \alpha(G_i) = \alpha(G) \). \( f_s \) is defined as \( \alpha(G) \).

(3) If the system of polynomials \( p_1, \ldots, p_r \) and the neighborhoods \( D_1, \ldots, D_{n+r} \) are replaced by another system which meets the condition \( \sigma_s \subseteq D \subseteq W \), then the same element \( f_s \in A_s \) arises.

Let \( \{ p_1, \ldots, p_r, D_1, \ldots, D_{n+r} \} \) be a system used to to define \( f_s \). Suppose \( c_i = \pi_a z_i \) for \( i = 1, \ldots, n \) and \( c_{n+k} = p_k(c_1, \ldots, c_n) \) for \( k = 1, \ldots, r \). Then

\[
\pi_s f_s = \pi_{s,t}(1/2\pi i)^{n+r} \int \cdots \int G(\lambda_1 - c_1) \cdots (\lambda_{n+r} - c_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r} = (1/2\pi i)^{n+r} \int \cdots \int G(\lambda_1 - b_1) \cdots (\lambda_{n+r} - b_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r} = f_s.
\]

For the system \( \{ p_1, \ldots, p_n, D_1, \ldots, D_{n+r} \} \) may be used to define \( f_s : sp(b_i) \subseteq sp(c_i) \subseteq D_i \) for \( i = 1, \ldots, n + r \) and \( \sigma_s \subseteq \sigma_t \subseteq D \subseteq W \). Thus \( \tau f \) is well defined.

If \( \varphi \in A \), then \( \varphi \in A_k \) for some integer \( k \), say \( \varphi = \pi^*_k \varphi \) for \( \psi \in A_k \), then \( f(\sigma(\varphi)) = f(\sigma_k(\psi)) = \psi(f_k) = \varphi(\tau f) \). \( \tau z_i = a_i \), since \( (z_i)_s = \pi_s a_i \) for every integer \( s \), for \( i = 1, \ldots, n \). \( \tau \) is continuous, since \( f_a \rightarrow f_0 \) for all \( k \nu_k f_a \rightarrow \nu_k f_0 \) for all \( k \tau_k \nu_k f_a \rightarrow \tau_k \nu_k f_0 \) (i.e. for all \( k \) \( (f_a)_k \rightarrow (f_0)_k \) in \( \tau f_a \rightarrow \tau f_0 \).
This theorem, except for continuity of the operational calculus, is also proven in [1] via the Arens-Calderon theorem [2].

Continuity of derivations. A derivation on an algebra $A$ is a linear operator $D$ satisfying $D(xy) = xDy + (Dx)y$ for every $x$ and $y$ in $A$. If $A$ is a commutative $F$-algebra, a linear transformation $D: A \rightarrow C(\Delta)$ satisfying $D(xy) = xDy + (Dx)y$ will be called a derivation into $C(\Delta)$. It is conjectured that a derivation on a Banach algebra must be continuous. Curtis [4] proved that if a Banach algebra is regular, then any derivation is continuous, in fact any derivation from the algebra to $C(\Delta)$ is continuous. This theorem will be extended to allow the algebra to be an $F$-algebra. It will then be applied to some $F$-algebras to determine all derivations in these algebras.

The following lemma is a modification of one in [3] and its proof is essentially the same.

**Lemma.** Let $t$ be an algebraic homomorphism from a commutative $F$-algebra $A$ to a semi-normed algebra $B$. Let $\{g_n\}$ and $\{h_n\}$ be two sequences of elements in $A$ such that for all $n$: $g_n h_n = g_n$ and if $m \neq n$, then $h_n h_m = 0$. Then it is not possible that for all $n$ \[ n \|tg_n\| > n \|g_n\|_n \|h_n\|_n. \]

**Corollary.** If $D$ is a derivation from a regular commutative semi-simple $F$-algebra $A$ to $C(\Delta)$, then $D$ is continuous.

**Proof.** Let $\{A_k\}$ and $\{\Delta_k\}$ be defined as in the preliminaries. Since every compact subset of $\Delta$ is contained in some $\Delta_N$, it suffices to prove that if $x_n \rightarrow 0$, then $Dx_n \rightarrow 0$ uniformly on each $\Delta_N$. The procedure will be to show:

1. For all $N$ there is an at most finite set $F_N \subseteq \Delta_N$ such that $Dx_n \rightarrow 0$ uniformly on the closure of $[\Delta_N \setminus F_N]$;
2. If $\varphi$ is isolated in $\Delta$, then $Dx(\varphi) = 0$ for every $x$ in $A$; and
3. If $\varphi \in \Delta_N$ is isolated in $\Delta_m$ for every $m \geq N$, then $\varphi$ is isolated in $\Delta$. (1), (2), and (3) imply that $Dx_n(\varphi) \rightarrow 0$ for every $\varphi$ and this together with (1) implies that $Dx_n \rightarrow 0$ uniformly on $\Delta_N$. This is basically the same proof as in [4]. The third step is the only novel point in the proof. It does not follow from the fact that every compact set is contained in some $\Delta_N$. The example of Arens' ([7] problem 2E) shows this. (3) may be proven as follows: Suppose $\varphi \in \Delta_N$ is isolated in $\Delta_m$ for all $m \geq N$. By Silov's theorem, for each $m \geq N$, there is an idempotent $e_m \in A_m$ such that $\varphi(e_m) = 1$ and $\varphi'(e_m) = 0$ if $\varphi' \in \Delta_m$ and $\varphi' \neq \varphi$ (identifying $\Delta_m$ with $\Delta(A_m)$). Then, because each $e_m$ is an idempotent and $(\pi_r e_s) \hat{=} e_r$ for $N \leq r \leq s$, $\pi_r e_s = e_r$ for $N \leq r \leq s$ (two idempotents in $A_r$ equal modulo the radical are iden-
tical). Thus \( \{e_m\} \) defines an idempotent \( e \) in \( A \) such that \( \varphi(e) = 1 \) and \( \varphi'(e) = 0 \) for \( \varphi' \neq \varphi \) and \( \varphi' \in \Delta 
abla \).

Steps (1) and (2) will be sketched. Proof of (1): Let \( B \) be the semi-normed algebra which as an algebra is \( A \), but with semi-norm \( \| x \| = \| x \|_N + \| D x \|_N \). Let \( F = \{ \varphi \in \Delta_N : x \to D x(\varphi) \) is not a continuous linear functional} \}. Since \( A \) is an \( \mathcal{F} \)-space, the principle of uniform boundedness applies. Since for each \( x \) in \( A \) \( \{ D x(\varphi) : \varphi \in \Delta_N \setminus F \} \) is bounded (by \( \| D x \|_N \)), \( D x_n \to 0 \) uniformly on \( \Delta_N \setminus F \). \( F \) is a finite set. If not, then there is an infinite sequence \( \{ \varphi_n \} \subseteq F \) with mutually disjoint neighborhoods. Since the algebra is by hypothesis regular, there are sequences \( \{y_n\}, \{z_m\} \) such that \( \varphi_n(\varphi_m) = 1 \), \( y_n z_n = y_n \) and \( z_n z_m = 0 \) if \( m \neq n \). Then since \( \varphi_n \in F \), there is an \( x_n \) in \( A \) such that \( |D x_n(\varphi_n)| > n \| x_n \|_N \cdot \| y_n \| \cdot \| z_n \|_N \). Thus letting \( g_n = x_n y_n \) and \( h_n = z_n \), we have \( \| g_n \| \geq \| D g_n \|_N > n \| g_n \|_N \cdot \| h_n \|_N \) and this contradicts the previous lemma. Thus we may let \( F \) be \( \Delta_N \). Proof of (2): Let \( \varphi \in \Delta \) be isolated. Choose, by Silov's theorem an idempotent \( e \) such that \( \varphi(e) = 1 \) and \( \varphi'(e) = 0 \) for \( \varphi' \neq \varphi \). Then \( D e(\varphi) = 0 \) and, by semi-simplicity, \( e x = \varphi(x)e \) for any \( x \) in \( A \). Hence

\[
0 = D(e x)(\varphi) = x(\varphi) D e(\varphi) + D x(\varphi) = D x(\varphi)
\]

for any \( x \) in \( A \).

By the closed graph theorem and the previous corollary, if \( D \) is a derivation on a regular commutative semi-simple \( \mathcal{F} \)-algebra, then \( D \) is continuous.

Let \( C^\infty(R) \) be the algebra of infinitely differentiable functions on the real line. For \( f \) in \( C^\infty(R) \), let

\[
\| f \|_a = \sum_{k=0}^n \sup \left[ \frac{1}{k!} |f^{(k)}(t)| : -n \leq t \leq n \right].
\]

\( C^\infty(R) \) is a regular semi-simple \( \mathcal{F} \)-algebra. If \( D \) is a derivation on \( C^\infty(R) \) and \( x \) is the function mapping \( t \) into \( t \), then for any polynomial \( p \) in \( x \), \( D p(x) = p'(x) D x \). Since the polynomials in \( x \) are dense in \( C^\infty(R) \) and since \( D \) is continuous, \( D f = f'D x \) for any \( f \) in \( C^\infty(R) \).

As a second application of the previous corollary, we show that the following algebra of Lipschitz functions has no nontrivial derivations.

Let \( \alpha \leq 1 \). Let \( L_\alpha \) be the subalgebra of \( C(R) \) consisting of functions of period 1 with finite norm \( \| - \|_\alpha \) where \( \| f \|_\alpha \) is defined to be

\[
\sup \left[ \frac{|f(t)|}{|t|} : t \in R \right] + \sup \left[ \frac{|f(s + h) - f(s)|}{|h|^\alpha} : s \in R, h \neq 0 \right].
\]

Let \( 1_\alpha = \{ f \in L_\alpha : \varliminf s \rightarrow 0 f(s + h) - f(s) \| h | \rightarrow 0 : h \rightarrow 0 \} \) for \( s \in R \). For \( \alpha < 1 \), \( L_\alpha \) is a Banach space, \( 1_\alpha \) a closed subspace, and \( L_\alpha \) is isomorphic to \( l_\alpha^* \) \[8\]. Let \( \alpha_n = 1 - 1/n \) and \( L \) be \( \bigcap L_{\alpha_n} \) with the sequence of algebraic norms \( \{ \| - \|_{\alpha_n} \} \). \( L \) may also be defined as the inverse limit of \( \{ L_{\alpha_n} \} \). \( L_{\alpha_{n+1}} \subseteq 1_{\alpha_n} \subseteq L_{\alpha_n} \) and so \( L \) is also the inverse limit
of \( \{1_{\alpha_n}\} \). This implies that \( L = L^{**} \), however even more is true: A bounded subset of \( L \) must have compact closure, i.e., \( L \) is a Montel space. For let \( S \) be a bounded set in \( L \subseteq 1_{\alpha_n} \). \( 1_{\alpha_n} \) is isometrically isomorphic as a Banach space with a subspace of \( C(W^*) \) where \( W^* \) is a compact set obtained as follows: Let \( U = \{t \in \mathbb{R}: 0 \leq t \leq 1\} \), \( V = \{(r, s): 0 \leq r \leq 1, 0 < r - s \leq 1/2\} \) and \( W = U \cup V \), then \( W \) is a locally compact space and \( W^* \) is its one-point compactification. The isomorphism \( f \rightarrow \tilde{f} \) is defined by \( \tilde{f}(\infty) = 0 \), \( \tilde{f}(t) = f(t) \), and

\[
\tilde{f}(r, s) = [f(r) - f(s)]/(r - s)^{\alpha_n}.
\]

To see that \( S \) is precompact in \( L \) it suffices to show that \( S \) is precompact in each \( 1_{\alpha_n} \), or, equivalently, that \( \tilde{S} \) is equicontinuous. This follows from the fact that there is a number \( K \) such that

\[
f \in S \Rightarrow ||f||_{\alpha_n+1} \leq K.
\]

The representation of \( 1_{\alpha_n} \) as \( C(W^*) \) is due to DeLeeuw [8].

A derivation \( D \) on \( L \) must map every element into 0. For \( L \) is a regular, commutative, semi-simple \( F \)-algebra and so it suffices to show that if \( f \in L \), then \( \varphi(Df) = 0 \) for any \( \varphi \in \Delta(L) \). \( D(f - \varphi(f)) = Df \) and \( f - \varphi(f) \) is in the kernel, \( M \), of \( \varphi \). So it suffices to show that \( D[M] \subseteq M \). Since \( M \) is an ideal, \( D[M^2] \subseteq M \), \( M^2 \neq M \), but \( M^2 \) is dense in \( M \) and so, since \( D \) must be continuous, \( D[M] \subseteq M \). (Any maximal ideal \( M \) must be the set of all functions in \( L \) vanishing at some \( t_0 \) where \( 0 \leq t_0 < 1 \). The function \( \sin \left(\left|t - t_0\right|/2\pi\right) \) is in \( M \) but not in \( M^2 \). Sherbert [10] proved that \( M^2 \) is dense in \( M \) for the Banach algebra \( 1_{\alpha} \), in fact for algebras of Lipschitz functions on more general spaces than the unit interval. His proof works as well for \( L \).)

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