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DERIVATIONS AND INTEGRAL CLOSURE

A. SEIDENBERG

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Let \mathcal{O} be an integral domain containing the rational numbers, Σ its quotient field, D a derivation of Σ , and \mathcal{O}' the ring of elements in Σ quasi-integral over \mathcal{O} . It is shown that if $D\mathcal{O} \subset \mathcal{O}$, then $D\mathcal{O}' \subset \mathcal{O}'$.

According to a lemma of Posner [4], which is also used by him in a subsequent paper [5], if \mathcal{O} is a finite integral domain over a ground field F of characteristic 0 and D is a derivation over F sending \mathcal{O} into itself, then D also sends the integral closure of \mathcal{O} into itself. The proof of this in [4] is wrong, but the statement itself is correct and a proof is here supplied. More generally it is proved that if \mathcal{O} is any integral domain containing the rational numbers and D is a derivation such that $D\mathcal{O} \subset \mathcal{O}$, then $D\mathcal{O}' \subset \mathcal{O}'$, where \mathcal{O}' is the ring of elements in the quotient field Σ of \mathcal{O} that are quasi-integral over \mathcal{O} . The theorem is not true for characteristic $p \neq 0$, but if one uses the Hasse-Schmidt differentiations instead of derivations, one gets the corresponding theorem for a completely arbitrary integral domain \mathcal{O} .

Let \mathcal{O} be an arbitrary integral domain containing the rational numbers, and let $\bar{\mathcal{O}}$ be the integral closure of \mathcal{O} . The question whether $D\mathcal{O} \subset \mathcal{O}$ implies $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$ is related to the question whether the ring of formal power series $\bar{\mathcal{O}}[[t]]$ is integrally closed. Thus consider the statements:

A. For every \mathcal{O} , $D\mathcal{O} \subset \mathcal{O}$ implies $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$, and

B. For every \mathcal{O} , $\bar{\mathcal{O}}[[t]]$ is integrally closed. We show that A and B are equivalent statements. (We also show: C. If $\bar{\mathcal{O}}[[t]]$ is integrally closed, then $D\mathcal{O} \subset \mathcal{O}$ implies $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$.) Now according to the last exercise in Nagata's book *Local Rings*, [3; p. 202, Ex. 5], B is a true statement, but we give a counter-example, which also leads to a counter-example for A.

2. Criticism of Posner's proof. Posner purports to prove that if P is a place of the quotient field Σ of \mathcal{O} that has F as residue field and is finite on \mathcal{O} and if $g \in \Sigma$ is finite at P , then Dg is finite at P . This is not so, as the following example shows. Let $\mathcal{O} = F[X, Y]$ be polynomial ring in two indeterminates over F . Let $D = \partial/\partial X$. Let P_1 be the place of $F(X, Y)$ over $F(Y/X)$ obtained by mapping X into 0, let P_2 be the place of $F(Y/X)/F$ obtained by

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mapping Y/X into any element of F , and let P be the composite place. Then $X, Y, Y/X$ are finite at P , but $\partial(Y/X)/\partial X = -Y/X^2$ is not.¹

One reason that Posner's proof fails is that there are no parameters such as those of which he speaks, except in the case that the degree of transcendency of \mathcal{O}/F is 1. In that case, Posner's argument yields a proof.

3. **A generalization.** Let \mathcal{O} be an arbitrary domain, with quotient field Σ . An element $\alpha \in \Sigma$ is said to be quasi-integral over \mathcal{O} if all powers of α are contained in a finite \mathcal{O} -module contained in Σ , or, what comes to the same, if there is a $d \in \mathcal{O}, d \neq 0$, such that $d\alpha^\rho \in \mathcal{O}, \rho = 0, 1, \dots$; (see [2]). If \mathcal{O} is a Noetherian domain, then the concepts of integral dependence and quasi-integral dependence (for elements in Σ) become the same; but it is the concept of quasi-integral dependence, rather than that of integral dependence, which is adapted to our considerations. The elements in Σ that are quasi-integral over \mathcal{O} form a ring \mathcal{O}' , which in the case \mathcal{O} is Noetherian is the integral closure $\bar{\mathcal{O}}$ of \mathcal{O} . The base field F plays little role, and it will be sufficient to assume that \mathcal{O} contains the rational numbers.

THEOREM. *Let \mathcal{O} be an arbitrary integral domain containing the rational numbers, let \mathcal{O}' be the ring of elements in the quotient field Σ of \mathcal{O} quasi-integral over \mathcal{O} , and let D be a derivation of Σ . Then: if $D\mathcal{O} \subset \mathcal{O}$, then $D\mathcal{O}' \subset \mathcal{O}'$.*

Proof. Let $\Sigma[[t]]$ be the ring of formal power series in a letter t over Σ and let $\Sigma((t))$ be its quotient field. The mapping $\Sigma c_i t^i \rightarrow \Sigma(Dc_i)t^i, i \geq 0, c_i \in \Sigma$, is a derivation of $\Sigma[[t]]$ into itself and extends D ; it has a unique extension to $\Sigma((t))$, which will also be denoted D . Let E be the expression $1 + tD + (t^2/2!)D^2 + \dots (=e^{tD})$. Then $\alpha + tD\alpha + (t^2/2!)D^2\alpha + \dots$, to be denoted $E\alpha$, has a meaning for every $\alpha \in \Sigma[[t]]$, i.e., the partial sums converge in the topology defined by powers of (t) ; and the mapping $\alpha \rightarrow E\alpha$ is an isomorphism of $\Sigma[[t]]$ into itself, as one easily verifies.² Its unique extension to $\Sigma((t))$ will

¹ Far from all, or even infinitely many, valuation rings \mathfrak{B} centered at (X, Y) being sent into themselves by $D = \partial/\partial X$, there is one and only one. In fact, restricting oneself to valuation rings \mathfrak{B} centered at (X, Y) , if $D\mathfrak{B} \subset \mathfrak{B}$, then $X/Y \notin \mathfrak{B}$, since $D(X/Y) = 1/Y \notin \mathfrak{B}$. Hence $Y/X \in \mathfrak{B}$, and therefore $D(Y/X), D^2(Y/X)$, etc. are also in \mathfrak{B} . Since $D^{n-1}(Y/X) = c_n Y/X^n (c_n \in K), v(Y) \geq n v(X)$ for $n = 1, 2, \dots$, where v is the valuation corresponding to \mathfrak{B} . Thus \mathfrak{B} could not be other than the ring of the valuation in which $v(X)$ is infinitely small with respect to $v(Y)$; and for that ring one checks that $D\mathfrak{B} \subset \mathfrak{B}$.

² We only use that $\alpha \rightarrow E\alpha$ is a monomorphism, but it is actually onto $\Sigma[[t]]$ as one sees from the identity $e^{tD}(e^{-tD}\alpha) = \alpha$.

also be denoted E . Since $D\mathcal{O} \subset \mathcal{O}$, one has $D\mathcal{O}[[t]] \subset \mathcal{O}[[t]]$, and since \mathcal{O} contains the rationals, $E\mathcal{O}[[t]] \subset \mathcal{O}[[t]]$.

Let α be quasi-integral over \mathcal{O} , and let $d \in \mathcal{O}$ be such that $d\alpha^\rho \in \mathcal{O}$, $\rho = 0, 1, \dots$. Then $E(d\alpha^\rho) = Ed(E\alpha)^\rho \in \mathcal{O}[[t]]$, $\rho = 0, 1, \dots$. Hence $dEd(E\alpha - \alpha)^\rho \in \mathcal{O}[[t]]$, $\rho = 0, 1, \dots$; here we use that d and Ed are in $\mathcal{O}[[t]]$. The coefficient of t^ρ in $dEd(E\alpha - \alpha)^\rho$, i.e., the leading coefficient, is $d^2(D\alpha)^\rho$; and this coefficient, as well as all the others, are in \mathcal{O} . Hence $D\alpha$ is quasi-integral over \mathcal{O} .

COROLLARY. *If $d \in \mathcal{O}$ and $\alpha \in \Sigma$ are such that $d\alpha^i \in \mathcal{O}$, $i = 0, 1, \dots, \rho$, then $d^2(D\alpha)^i \in \mathcal{O}$, $i = 0, 1, \dots, \rho$.*

Let $\mathfrak{C} = \{c \mid c \in \mathcal{O}, c\mathcal{O}' \subset \mathcal{O}'\}$; then \mathfrak{C} is an ideal, which in the case \mathcal{O}' is the integral closure $\bar{\mathcal{O}}$ of \mathcal{O} is called the conductor of \mathcal{O} .

COROLLARY. *If $D\mathcal{O} \subset \mathcal{O}$, then $D\mathfrak{C} \subset \mathfrak{C}$. In other words, \mathfrak{C} is a differential ideal for any derivation (or any family of derivations) sending \mathcal{O} into itself.*

Proof. If $c \in \mathfrak{C}$ and $\alpha \in \mathcal{O}'$, then $(Dc)\alpha = D(c\alpha) - cD\alpha \in \mathcal{O}$, so that also $(Dc)\mathcal{O}' \subset \mathcal{O}'$.

The last corollary can sometimes be used to prove that a given integral domain \mathcal{O} is integrally closed (see [4]). We first restrict ourselves to a class of integral domains \mathcal{O} such that $\bar{\mathcal{O}} = \mathcal{O}'$, for example, the class of Noetherian domains. Then we restrict ourselves further to a class \mathcal{C} of domains \mathcal{O} such that \mathcal{O} has a conductor $\mathfrak{C}: \bar{\mathcal{O}} \neq (0)$, or equivalently, such that $\bar{\mathcal{O}}$ is contained in a finite \mathcal{O} -module (contained in Σ), for example, the class of finite integral domains (see [7; p. 267]), or quotient rings thereof, or the class of complete local domains (see [3; p. 114]). (For examples of Noetherian domains not having this property, see [3; p. 205 ff]; for an example in characteristic 0, see [1]). Then we can state:

COROLLARY. *Let \mathcal{O} be an integral domain belonging to a class \mathcal{C} defined just above, let \mathcal{O} contain the rational numbers, and let $\{D\}$ be a (finite or infinite) family of derivations of \mathcal{O} into itself. Then, if \mathcal{O} is differentially simple under $\{D\}$ (i.e., has no differential ideal other than (0) or (1)), then \mathcal{O} is integrally closed.*

4. Extension of D to $\bar{\mathcal{O}}$. The above is a simplification of our original proof for a finite integral domain. The idea was that since E sends $\mathcal{O}[[t]]$ into itself, it also sends the integral closure of $\mathcal{O}[[t]]$

into itself. It was then sufficient to prove that $\bar{\mathcal{O}}[[t]]$ is integrally closed; in fact, we have the following theorem for any integral domain \mathcal{O} containing the rational numbers.

THEOREM C. *If $\bar{\mathcal{O}}[[t]]$ is integrally closed and $D\mathcal{O} \subset \mathcal{O}$, then $D\bar{\mathcal{O}} \subset \bar{\mathcal{O}}$. (Here $\bar{\mathcal{O}}$ is the integral closure of \mathcal{O} .)*

Proof. If $\alpha \in \Sigma$, $\alpha = c/d$, $c, d \in \mathcal{O}$, then $E\alpha = Ec/Ed$, so $E\alpha$ is in the quotient field of $\mathcal{O}[[t]]$. If α is integral over \mathcal{O} , then $E\alpha = \alpha + tD\alpha + \dots$ is integral over $\mathcal{O}[[t]]$, hence in $\bar{\mathcal{O}}[[t]]$, whence $D\alpha \in \bar{\mathcal{O}}$.

Our proof that $\bar{\mathcal{O}}[[t]]$ was integrally closed for \mathcal{O} a finite integral domain depended on the following observation, which holds for an arbitrary domain \mathcal{O} .

THEOREM. *If \mathcal{O} is completely integrally closed (i.e., if $\mathcal{O}' = \mathcal{O}$), then so is $\mathcal{O}[[t]]$. More generally, for any \mathcal{O} , $(\mathcal{O}[[t]])' \subset \mathcal{O}'[[t]]$.*

Proof. Let $\alpha(t)$ be quasi-integral over $\mathcal{O}[[t]]$. Then there is a $d \in \mathcal{O}[[t]]$, $d = d(t) \neq 0$, such that $d\alpha^\rho \in \mathcal{O}[[t]]$, $\rho = 0, 1, \dots$. Since $\text{ord } d + \rho \text{ ord } \alpha \geq 0$, $\rho = 0, 1, \dots$, one first observes that $\alpha \in \Sigma[[t]]$. Let $d = d_s t^s + d_{s+1} t^{s+1} + \dots$, $d_s \neq 0$, and let $\alpha = \alpha_r t^r + \alpha_{r+1} t^{r+1} + \dots$. Since the leading coefficient of $d\alpha^\rho$ is in \mathcal{O} , we have $d_s \alpha_r^\rho \in \mathcal{O}$, whence α_r is quasi-integral over \mathcal{O} . Now $\alpha - \alpha_r t^r$ is quasi-integral over $\mathcal{O}[[t]]$, whence α_{r+1} is quasi-integral over \mathcal{O} ; and in this way one sees that all the coefficients of α are quasi-integral over \mathcal{O} .

If \mathcal{O} is Noetherian, then so is $\mathcal{O}[[t]]$. Hence:

COROLLARY. *If \mathcal{O} is an integrally closed Noetherian domain, then so is $\mathcal{O}[[t]]$.*

This is Nagata's (47.6) in [3; p. 200].

Finally, if \mathcal{O} is a finite integral domain, then so is $\bar{\mathcal{O}}$, whence in this case $\bar{\mathcal{O}}[[t]]$ is integrally closed. Recalling that $\bar{\mathcal{O}}$ is a finite \mathcal{O} -module (see [7; p. 267]), one sees that $\bar{\mathcal{O}}[[t]]$ is even the integral closure of $\mathcal{O}[[t]]$ in accordance with the following:

THEOREM. *Let \mathcal{O} be an integral domain whose integral closure is Noetherian and is a finite \mathcal{O} -module. Then the integral closure of $\mathcal{O}[[t]]$ is $\bar{\mathcal{O}}[[t]]$.*

Proof. Let $\bar{\mathcal{O}} = \mathcal{O}w_1 + \dots + \mathcal{O}w_s$. Then

$$\mathcal{O}[[t]] = \mathcal{O}[[t]]w_1 + \cdots + \mathcal{O}[[t]]w_s,$$

whence $\bar{\mathcal{O}}[[t]]$ is a finite $\mathcal{O}[[t]]$ -module and thus integral over $\mathcal{O}[[t]]$. Let d be a common denominator of the w_i when written as quotients of elements in \mathcal{O} . Then $d\bar{\mathcal{O}}[[t]] \subset \bar{\mathcal{O}}[[t]]$, whence $\mathcal{O}[[t]]$ and $\bar{\mathcal{O}}[[t]]$ have the same quotient field. As we have already seen that $\bar{\mathcal{O}}[[t]]$ is integrally closed, the proof is complete.

Although not necessary for our considerations, we mention the following:

THEOREM. *If \mathcal{O} is a Noetherian domain, then $\bar{\mathcal{O}}[[t]]$ is integrally closed, where t abbreviates a set t_1, \dots, t_n of n distinct letters.*

Proof. $\bar{\mathcal{O}}$ is a Krull ring (see [3; p. 118]), hence from the definition [3; p. 115], $\bar{\mathcal{O}}_p$ is a Noetherian valuation ring for every minimal prime ideal p of $\bar{\mathcal{O}}$. Moreover $\bar{\mathcal{O}} = \bigcap \bar{\mathcal{O}}_p$, where the intersection is taken over the minimal prime ideals of $\bar{\mathcal{O}}$ (see [3; p. 116]). Since $\bar{\mathcal{O}}_p[[t]]$ is integrally closed, also $\bar{\mathcal{O}}[[t]] = \bigcap \bar{\mathcal{O}}_p[[t]]$ is integrally closed.

Now consider the statements A and B mentioned at the beginning. We say that A and B are equivalent. Recall that we are assuming that \mathcal{O} contains the rational numbers.

$B \Rightarrow A$. This follows at once from C, the first theorem of this section.

$A \Rightarrow B$. Let α be in the quotient field of $\bar{\mathcal{O}}[[t]]$ and integral over $\bar{\mathcal{O}}[[t]]$. Then $\alpha \in \Sigma[[t]]$, $\alpha = \alpha_0 + \alpha_1 t + \cdots$. From an equation of integral dependence for α on $\bar{\mathcal{O}}[[t]]$, by placing $t = 0$, one sees that $\alpha_0 \in \bar{\mathcal{O}}$. Now apply A to the ring $\bar{\mathcal{O}}[[t]]$ and the derivation $D = \partial/\partial t$. Then $\partial\alpha/\partial t, \partial^2\alpha/\partial t^2, \dots$ are integral over $\bar{\mathcal{O}}[[t]]$, whence all the coefficients of α are in $\bar{\mathcal{O}}$.

Now according to the last exercise in Nagata's *Local Rings*, B is a true statement; however, we will show that this is incorrect.

THEOREM. *If \mathcal{O} is an (integrally closed) integral domain containing a field and there is a nonunit $b \in \mathcal{O}$ such that $\bigcap (b^p) \neq (0)$, then $\mathcal{O}[[t]]$ is not integrally closed.*

Proof. Let p be the characteristic and $n > 1$, an integer such that $n \not\equiv 0(p)$. Then $b^n + b^{n-2}t$ has an n th root $\alpha = b[1 + (t/b^2)]^{1/n} = b[1 + c_1(t/b^2) + c_2(t^2/b^4) + \cdots]$ in $\Sigma[[t]]$, where c_1, c_2, \dots are in the prime field of Σ and $c_1 \neq 0$. If $a \in \bigcap (b^p)$ and $a \neq 0$, then $a\alpha \in \mathcal{O}[[t]]$, so

that α is in the quotient field of $\mathcal{O}[[t]]$. Now α is integral over $\mathcal{O}[[t]]$, but is not in $\mathcal{O}[[t]]$. Hence $\mathcal{O}[[t]]$ is not integrally closed.

THEOREM. *Let \mathfrak{B} be a (proper) valuation ring containing a field. Then $\mathfrak{B}[[t]]$ is integrally closed if and only if \mathfrak{B} is of rank 1, i.e., if and only if there is no chain $0 < p_1 < p_0 < \mathfrak{B}$ of prime ideals.*

Proof. If \mathfrak{B} is of rank 1, then it is well-known and can be checked at once, that \mathfrak{B} is completely integrally closed. Hence $\mathfrak{B}[[t]]$ is completely integrally closed, hence integrally closed.

On the other hand, if \mathfrak{B} is of rank > 1 and $0 < p_1 < p_0 < \mathfrak{B}$ is a chain of prime ideals in \mathfrak{B} and $b \in p_0 - p_1$, then $p_1 \subset \cap (b^p)$, whence $\mathfrak{B}[[t]]$ is not integrally closed.

To get a counter-example to Nagata's last exercise, one has but to take \mathcal{O} to be a valuation ring of rank > 1 that contains a field.³

To get an example of a ring \mathcal{O} and derivation D such that $D\mathcal{O} \subset \mathcal{O}$ but $D\bar{\mathcal{O}} \not\subset \bar{\mathcal{O}}$, let \mathfrak{B} be a valuation ring of rank 2 containing the rational numbers, let $\mathcal{O} = \mathfrak{B}[[t]]$ and $D = \partial/\partial t$. Let b be a nonunit in \mathfrak{B} such that $\cap (b^p) \neq (0)$, and let

$$\alpha = (b^2 + t)^{1/2} = b \left[1 + c_1 \frac{t}{b^2} + c_2 \frac{t^2}{b^4} + \cdots \right],$$

where c_1, c_2, \dots are rational numbers. Then α is integral over $\mathcal{O} = \mathfrak{B}[[t]]$ but $D\alpha$ is not.

Concerning the proof spoken of at the beginning of this section, the author is obliged to Professor Mumford for the remark in context that if D is a derivation, then e^D , formally at any rate, is an isomorphism. The introduction of the parameter t on the one hand prevents the computations from collapsing into meaninglessness, and on the other allows one to recover D from e^{tD} .

5. The case of characteristic $p \neq 0$. For $p \neq 0$, the theorem of § 3 is not true, even for curves. Thus consider the curve given by $Y^p - X^p - X^{p+1} = 0$. One checks that $Y^p - X^p - X^{p+1}$ is irreducible (over the ground field F). Let (x, y) be a generic point of the curve over F . Let D be a derivation of $F(y)/F$ with $Dy = 1$; since x is separable over $F(y)$, D can be extended uniquely to a derivation, still to be denoted D , of $F(y, x)$. One finds $-(p+1)x^p Dx = 0$, hence $Dx = 0$. Let $\mathcal{O} = F[x, y]$. Then $D\mathcal{O} \subset \mathcal{O}$. Now y/x is integral

³ In reference to the exercise, Nagata [3; p. 221] cites Sugaku, Vol. 9, No. 1 (1957), p. 61, which we have not been able to locate; and while he notes that the proof there is not complete, he remarks that "a supplement is expected to appear soon".

over \mathcal{O} , since $(y/x)^p = 1 + x$, but $D(y/x) = 1/x$ is not, as otherwise it would be integral over $F[x]$.

However, if one uses the Hasse-Schmidt differentiations [6] instead of derivations, one gets the corresponding theorem.⁴ Recall that a differentiation D of a field Σ into itself is a sequence $D = (\delta_0, \delta_1, \delta_2, \dots)$ of mappings of Σ into itself with $\delta_0 = 1$ and satisfying the properties:

$$\begin{aligned} \delta_i(x + y) &= \delta_i x + \delta_i y \\ \delta_i xy &= \sum_{i+j=v} \delta_i x \delta_j y . \end{aligned}$$

By $D\mathcal{O} \subset \mathcal{O}$ we now mean $\delta_i \mathcal{O} \subset \mathcal{O}$ for every i . Then

$$E = \delta_0 + t\delta_1 + t^2\delta_2 + \dots$$

still yields an isomorphism and can be used instead of our previous E to get the conclusion $D\mathcal{O}' \subset \mathcal{O}'$. (After obtaining $\delta_1 \mathcal{O}' \subset \mathcal{O}'$ as before, we argue that $d^3 E d(E\alpha - \alpha - t\delta_1 \alpha)^\rho \in \mathcal{O}[[t]]$, $\rho = 0, 1, \dots$, whence $d^4(\delta_2 \alpha)^\rho \in \mathcal{O}$, $\rho = 0, 1, \dots$, and $\delta_2 \alpha$ is quasi-integral over \mathcal{O} , etc.) In the case of characteristic 0, the same argument shows one can drop the assumption that \mathcal{O} contains the rationals (i.e., if one uses differentiations instead of derivations).

The corollaries of the theorem of § 3 also have easily stated generalizations, with similar proofs.

REMARK. Since $(1 + (1 + 4t)^{1/2})/2 \in Z[[t]]$, the last two theorems of §4 hold without the field condition.

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⁴ For some useful information on differentiations, see K. Okugawa, "Basic properties of differential fields of an arbitrary characteristic and the Picard-Vessiot theory", J. of Math. of Kyoto Univ., Vol. 2 (1963), pp. 295-322.

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