ON THE STABILITY OF THE SET OF EXPONENTS OF A CAUCHY EXPONENTIAL SERIES

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If \( f \in L(-D, D) \) and \( Q(z) \) is a meromorphic function whose poles, all simple, form a sub-set of the set \( \{\lambda_v\} (v = 0, \pm 1, \pm 2, \cdots) \), then the C.E.S. (Cauchy exponential series) of \( f \) with respect to \( Q(z) \) is \( \sum c_v e^{\lambda_v z} \), where

\[
c_v e^{\lambda_v z} = \text{res} \ Q(z) \int_{-D}^D f(t)e^{i\pi(z-v)} dt .
\]

Suppose we are given a class \( A \) of functions \( f \) each of which can be 'represented' in \((-D, D)\) by its C.E.S. with respect to \( Q(z) \). We define a set of neighbourhoods \( U \) of \( \{\lambda_v\} \). Then \( \{\lambda_v\} \) is stable if there is a \( U \) such that to each \( \{\kappa_v\} \in U \) there corresponds a meromorphic function \( q(z) \) whose poles, all simple, form a sub-set of \( \{\kappa_v\} \) and which is such that each \( f \in A \) can be represented in \((-D, D)\) by its C.E.S. with respect to \( q(z) \); and \( \{\lambda_v\} \) is unstable if there is no such neighbourhood.

The case in which \( \lambda_v = iv, A = BV[-D, D], \) 'representation of \( f \) in \((-D, D)\)' means \( \sum_{|\nu| \leq \pi} c_{iv} e^{ivz} \rightarrow 1/2 (f(x+) + f(x-)) \) boundedly within \((D, D)\)' is considered. It is shown, in particular, that with reasonable conditions on the set of neighbourhoods \( U \), \( \{iv\} \) is unstable if \( D > 1/2 \pi \), and stable if \( D = 1/2 \pi \).

Let \( D > 0 \) and \( f \in L(-D, D) \). Let \( Q(z) \) be a meromorphic function whose poles, all simple, form a sub-set of the set \( \{\lambda_v\} (v = 0, \pm 1, \cdots) \). Here, and in what follows, the use of the symbol \( \{\lambda_v\} \) implies that \( \lambda_v \neq \lambda_v' \) if \( v \neq v' \). The C.E.S. (Cauchy exponential series) of \( f \) with respect to \( Q \) is \( \sum c_v e^{\lambda_v z} \) where

\[
c_v e^{\lambda_v z} = \text{res} \ Q(z) \int_{-D}^D f(t)e^{i\pi(z-v)} dt .
\]

Suppose that the set \( \{\lambda_v\} \) is such that, for a class \( A \) of functions \( f \), the C.E.S. of \( f \) 'represents' \( f \) in \((-D, D)\). Then we may consider the question of the stability of the set \( \{\lambda_v\} \). We define, in some way, a set of neighbourhoods \( U \) of \( \{\lambda_v\} \). Then \( \{\lambda_v\} \) is stable if there is a neighbourhood \( U \) such that to each \( \{\kappa_v\} \in U \), there corresponds a meromorphic function \( q(z) \) whose poles, all simple, form a sub-set of \( \{\kappa_v\} \), and which is such that each \( f \in A \) can be represented in \((-D, D)\) by its C.E.S. with respect to \( q(z) \); and \( \{\lambda_v\} \) is unstable if there is no such neighbourhood. The stability of \( \{\lambda_v\} \) depends on the value of \( D \), the class \( A \), the, particular meaning we give to the 'representation' of \( f \),

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and finally on the definition of the set of neighbourhoods $U$. In this note, we confine our attention to the simplest case: $\lambda_v = iv$, $A$ is the class of functions $f$ which are $BV[-D, D]$ and satisfy $2f(x) = f(x+) + f(x-)$ in $(-D, D)$, ‘representation’ of $f$ means ‘bounded convergence’ to $f(x)$ within $(-D, D)$, i.e., for each $\delta$ satisfying $0 < \delta < D$, $\sum_{|v|\leq \varepsilon} c_v e^{ivx} \rightarrow f(x)$ boundedly in the segment $|x| \leq D - \delta$.

We recall that if $D = \pi$, then each $f \in A$ can be represented by its C.E.S. with respect to $Q_v(z) = 1/2 \coth \pi z$, since, in this case, the C.E.S. is the Fourier series of $f$. Let us suppose that to each neighbourhood $U$ there corresponds an $\varepsilon > 0$ such that $\{\mu_i\} \in U$ if $\sum |\mu_i - \lambda_v| < \varepsilon$; and to each $\delta > 0$ there corresponds a neighbourhood $U_{\delta}$ such that if $\{\mu_i\} \in U_{\delta}$ then $\sup |\mu_i - \lambda_v| < \delta$. What we prove, implies that $\{iv\}$ is unstable if $D > \pi/2$, and stable if $D = \pi/2$. We shall, however, prove more than this, viz.

**Theorem 1.** Let $\{l_i\}$ be a real set not containing every integer, such that $l_i$ is an integer for $|v| \geq N$. If $D > \pi/2$, then there is no meromorphic function $q(z)$ whose poles, all simple, form a sub-set of $\{il_i\}$ and which is such that each $f \in A$ can be represented by its C.E.S. with respect to $q$.

**Theorem 2.** Let $l_i = v + \alpha_i + i\beta_i$, where $\alpha_i, \beta_i$ are real numbers which satisfy

$$\lim_{|v| \to \infty} |\alpha_v| < \frac{1}{8}, \quad \lim_{|v| \to \infty} |\beta_v| < \infty.$$  

If $D = \pi/2$, there exists a meromorphic function $q(z)$ whose poles, all simple, form a sub-set of $\{il_i\}$ and which is such that each $f \in A$ can be represented by its C.E.S. with respect to $q$.

**Theorem 3.** The conclusion of Theorem 2 holds if the condition on $\alpha_v$ is replaced by $\sup |\alpha_v| < 1/4$.


2. Let $0 < D \leq \pi$, and let $A$ have the meaning specified in § 1.

**Lemma 1.** If $H_n(t) \in L(-2D, 2D)$ for $n \geq n_0$, then, in order that for each $f \in A$,

$$\int_{-D}^{D} f(t)H_n(t-x)dt \rightarrow f(x)$$
boundedly within \((-D, D)\), it is necessary and sufficient that

\[ \int_0^t H_n(u)du \to \frac{1}{2} \text{sgn } t \]

boundedly within \((-2D, 2D)\).

Proof. Let

\[ J_n(u) = \frac{1}{2\pi} \frac{\sin \left(n + \frac{1}{2}\right)u}{\sin \frac{1}{2}u} \]

Then for each \(f \in A\),

\[ \int_{-D}^D f(t)J_n(t - x)dt \to f(x) \]

boundedly within \((-D, D)\), and

\[ \int_0^t J_n(u)du \to \frac{1}{2} \text{sgn } t \]

boundedly within \((-2D, 2D)\). Let \(K_n(u) = H_n(u) - J_n(u)\). It suffices to prove: in order that for each \(f \in A\),

\[ \int_{-D}^D f(t)K_n(t - x)dt \to 0 \]

boundedly within \((-D, D)\), it is necessary and sufficient that

\[ k_n(t) = \int_0^t K_n(u)du \to 0 \]

boundedly within \((-2D, 2D)\).

Sufficiency. We have

\[ (1) \quad \int_{-D}^D f(t)K_n(t - x)dt = f(D)k_n(D - x) - f(-D)k_n(-D - x) \]

and the second member tends to zero boundedly within \((-D, D)\).

Necessity. In the first place, it is necessary that for each \(\tau \in (-2D, 2D), k_n(\tau) \to 0\) as \(n \to \infty\). For let \(\alpha, \beta \in (-D, D)\) and let \(x = \alpha\). Let \(f(t) = 1\) in the open interval, and let \(f(t) = 0\) outside the closed interval, whose end points are \(\alpha, \beta\). Then
\[ k_n(\beta - \alpha) = \int_{\alpha}^{\beta} K_n(t - \alpha)dt \to 0. \]

Since \( \alpha, \beta \) can be chosen so that \( \beta - \alpha \) has any assigned value in \((-2D, 2D)\), this proves our assertion.

By (1), for each \( x \in (-D, D) \), the functions \( k_n(t - x) \) of \( t \), for \( n \geq n_0 \), form a sequence of elements of \( C[-D, D] \) such that

\[ \int_{-D}^{D} k_n(t - x)df(t) \]

is convergent for each \( f \in A \). By the principle of uniform boundedness, it follows that

\[ \sup_{t \in [-D, D]} |k_n(t - x)| < \infty. \]

Choose \( x = D - \delta \). Then \( k_n(t) \) is uniformly bounded in \([-2D + \delta, \delta]\). Choose \( x = -D + \delta \). Then \( k_n(t) \) is uniformly bounded in \([-\delta, 2D - \delta]\). Hence \( k_n(t) \) is uniformly bounded within \((-2D, 2D)\) as required.

3. Proof of Theorem 1. We may suppose that \( D \leq \pi \). Let \( \omega \) be chosen to satisfy \( \pi < \omega < 2D \). We choose the notation so that if \( 0 \in \{l_0\} \) then \( 0 = l_0 \). If a meromorphic function \( g(z) \), with the properties mentioned in the enunciation, exists, let \( C_n \) denote a contour which contains in its interior precisely those \( il_v \) for which \( |v| \leq n \), and which does not pass through any of the \( il_v \). Let

\[ H_n(u) = \frac{1}{2\pi i} \int_{C_n} q(z)e^{-iz}dz. \]

If \( \sum c_v e^{it_vx} \) is the C.E.S. of \( f \) with respect to \( q(z) \), then

\[ \sum_{|v| \leq n} c_v e^{it_vx} = \sum_{|v| \leq n} \text{res } q(z) \int_{-D}^{D} f(t)e^{it(x-t)}dt = \int_{-D}^{D} f(t)H_n(t - x)dt. \]

We have

\[ \int_{0}^{\pi} H_n(u)du = \frac{1}{2\pi i} \int_{C_n} q(z) \frac{1 - e^{-iz}}{z}dz = \sum_{|v| \leq n} \frac{r_v}{il_v}(1 - e^{-it_vx}) \]

where \( r_v \) is the residue of \( q(z) \) at \( il_v \) and where, if \( l_0 = 0 \), we use the convention

\[ \frac{1 - e^{-it_0}}{il_0} = \lim_{l \to 0} \frac{1 - e^{-it}}{il} = t. \]
By Lemma 1, it is necessary that

$$\sum_{|\nu| \leq n} \frac{r_{\nu}}{l_{\nu}} (1 - e^{-i\nu x}) \rightarrow \frac{1}{2} \text{sgn } x$$

boundedly within $(-2D, 2D)$, and hence in $[-\omega, \omega]$. Let $x \in (-\omega, \omega - 2\pi)$. Then for $|\nu| \geq N$, the terms on the left are unaltered on replacing $x$ by $x + 2\pi$. By subtraction, it follows that

$$\sum_{|\nu| < N} \frac{r_{\nu}}{l_{\nu}} e^{-i\nu x} (e^{-i\nu 2\pi} - 1) = -1$$

for such $x$, and hence for all $x$. We note that if $l_0 = 0$, the term with $\nu = 0$ is $-r_0 2\pi$. At this point, we distinguish to cases, (a) $l_0 \neq 0$, (b) $l_0 = 0$.

In case (a), we integrate (7) over $(-X, X)$, divide by $2X$, and let $X \rightarrow \infty$. We obtain a contradiction. In case (b), we take mean values as in case (a), and deduce that the term with $\nu = 0$ is $-1$. Then (7) implies that

$$\sum_{0 < |\nu| < N} \frac{r_{\nu}}{l_{\nu}} e^{-i\nu x} (e^{-i\nu 2\pi} - 1) = 0$$

for all $x$. If we multiply this by its conjugate, and take mean values, we deduce that

$$\sum_{0 < |\nu| < N} \left| \frac{r_{\nu}}{l_{\nu}} \right|^2 \sin^2 \pi l_{\nu} = 0.$$

By (6),

$$\sum_{0 < |\nu| < N} \frac{r_{\nu}}{l_{\nu}} (1 - e^{-i\nu x}) \rightarrow \frac{1}{2} \sin x - \frac{x}{2\pi}$$

boundedly within $(-2D, 2D)$. Considering odd parts, its follows that

$$\sum_{0 < |\nu| < N} \frac{r_{\nu}}{l_{\nu}} \sin l_{\nu} x \rightarrow \frac{1}{2} \text{sgn } x - \frac{x}{2\pi}$$

boundedly within $(-2D, 2D)$. By hypothesis, there is an integer $\mu$ say, which is not one of the $l_{\nu}$; and $\mu \neq 0$ since $l_0 = 0$. By (8), $r_{\mu} = 0$ if $l_{\nu}$ is not an integer. Hence, on multiplying both sides of (9) by $\mu \sin \mu x$ and integrating over $(-\pi, \pi)$, we obtain $0 = 1$, a contradiction.

4. Proof of Theorem 2. For all sufficiently large $n$, the circle $\Gamma_n : |z| = n + 1/2$, contains in its interior the points $il_{\nu}$ for $|\nu| \leq n$, and every point on $\Gamma_n$ is at a distance greater than 3/8 from all the points $il_{\nu}$. Let $q(z)$ be a meromorphic function whose poles, all simple,
form a sub-set of \( \{i\nu_i\} \), and define \( H_n(u) \) by (2) with \( C_n \) replaced by \( \Gamma_n \). Using the notation of §§ 1, 2, we have

\[
J_n(u) = \frac{1}{2\pi i} \int_{\Gamma_n} Q_\delta(z) e^{-zu} dz,
\]

and therefore, as in § 2, it suffices to prove that we can choose \( q(z) \) so that

\[
\int_0^\Gamma K_n(u) du = \frac{1}{2\pi i} \int_{\Gamma_n} (q(z) - Q_\delta(z)) \frac{1 - e^{-z\gamma}}{z} dz \to 0
\]

boundedly within \(( -\pi, \pi ) \).

Write

\[
P(z) = (z - i\nu) \prod_{i} \left( 1 - \frac{z}{i\nu_i} \right) \left( 1 - \frac{z}{i\nu_{-i}} \right).
\]

In § 5, we shall prove

**Lemma 2.** As \(|z| \to \infty\), \( P(z) = o(|z|^{|\nu|/\pi + |\nu|}) \). On \( \Gamma_n \), \(|P(z)|^{-1} = o(n^{|\nu|/\pi + |\nu|}) \) as \( n \to \infty \).

The meromorphic function \( Q_\delta(z)P(z) \) is regular, except possibly at the points \( i\nu \), which are at most simple poles of residue \( P(i\nu)/2\pi \). By Lemma 2, \( P(i\nu) = o(|\nu|^{|\nu|}) \). Hence we can define the meromorphic function

\[
R(z) = \frac{1}{2\pi} \left[ \frac{P(0)}{z} + \sum' P(i\nu) \left( \frac{1}{z - i\nu} + \frac{1}{i\nu} \right) \right]
\]

which has the same principal parts as \( Q_\delta(z)P(z) \). Thus

\[
Q_\delta(z)P(z) = R(z) + S(z)
\]

where \( S(z) \) is an integral function. We can write \( q(z)P(z) = F(z) \), where \( F(z) \) is an integral function. Then

\[
q(z) - Q_\delta(z) = \frac{F(z) - S(z) - R(z)}{P(z)}.
\]

In § 5, we shall prove

**Lemma 3.** On \( \Gamma_n \), \( R(z) = o(n^{|\nu|}) \) as \( n \to \infty \).

We choose \( F(z) \) so that the numerator in (10) will not be of a greater order of magnitude than \( R(z) \). This means, since \( F \) and \( S \) are integral functions, that \( F = S + c \) where \( c \) is a constant. Theorem 2 will follow if we show that
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tends to zero boundedly within (−π, π). Write \( z = (n + 1/2)e^{i\theta} \). By Lemmas 2 and 3, \[
\frac{c - R(z)}{P(z)} = o(ne^{-n\pi|\cos \theta|}).
\]
If then \( |x| \leq \pi - \delta, \delta > 0 \), we have
\[
I_n(x) = o\left(\int_0^{2\pi} e^{-n\pi|\cos \theta|} d\theta\right) = o(1).
\]

5. In order to prove Lemmas 2 and 3, it will be convenient to write
\[
P(iz) = ip(z),
\]
so that
\[
p(z) = (z - l_v) \prod \left(1 - \frac{z}{l_v}\right)\left(1 - \frac{z}{l_{-v}}\right),
\]
and
\[
R(iz) = r(z) = \frac{1}{2\pi} \left[ \frac{p(0)}{z} + \sum' p(\nu) \left(\frac{1}{z - \nu} + \frac{1}{\nu}\right) \right].
\]

We need the following result, which is a special case \((a = 0)\) of [3] Theorem 1 (with a change of notation).

**Lemma 4.** Let \( L, M \) be positive numbers. Let \( s_v = \nu + \sigma_v + i\tau_v \), where \( \sigma_v, \tau_v \) are real numbers which satisfy \( |\sigma_v| \leq L, |\tau_v| \leq M \) for all \( \nu \). Suppose that there is a \( \delta > 0 \) such that \( |s_v| \geq \delta \) for all \( \nu \). Let
\[
\psi(z) = \left(1 - \frac{z}{s_v}\right) \prod \left(1 - \frac{z}{s_v}\right)\left(1 - \frac{z}{s_{-v}}\right).
\]
Then there is a positive constant \( C \) (depending only on \( L, M, \delta \)) such that,
(i) for all \( z, |\psi(z)| < C(1 + |z|)^{\nu e^{\pi (|\nu| + 1)}} \);
(ii) if \( |z - s_v| \geq \delta \) for all \( \nu \), then \( |\psi(z)|^{-1} < C(1 + |z|)^{\nu e^{-\pi (|\nu| + 1)}} \).

**Proof of Lemma 2.** We can find a positive number \( L < 1/8 \) such that \( |\alpha_v| \leq L \) for \( |\nu| > N \) say; and a positive number \( M \) such that \( |\beta_v| \leq M \) for all \( \nu \). In Lemma 4, choose \( s_v = l_v \) for \( |\nu| > N; \beta_v = \nu \) for
0 < \mid \nu \mid \leq N; = 3/8 for \nu = 0. Then \( p(z)/\psi(z) \) tends to a nonzero constant as \( |z| \to \infty \). By Lemma 4 (with \( \delta = 3/8 \)), there is a positive constant \( D \) such that

(i) \( |p(z)| < D |z|^{4L} e^{-|z|^{4L}} \) if \( |z| \) is sufficiently large;

(ii) if \( z \) is on \( C_n \) and \( n \) is sufficiently large then \( |p(z)|^{-1} < D n^{4L} e^{-|z|^{4L}} \) (the condition \( |z - s| \geq 3/8 \) for all \( \nu \) being satisfied). Since \( P(z) = \nu p(-iz), \) and \( 4L < 1/2 \), the lemma follows.

**Proof of Lemma 3.** By (i) above, \( p(\nu) = O(|\nu|^{4L}). \) By (11), it will suffice to prove that if \( z \) is on \( C_n \), then

\[
\sum' \frac{z)p(\nu)}{\nu(z - \nu)} = o(n^{4L}).
\]

The left hand side is

\[
O\left[ \sum_{0 < \nu \leq 3n} \frac{n^{4L}}{\nu(n + \frac{1}{2} - \nu)} + \sum_{n < \nu \leq 2n} \frac{n^{4L}}{\nu(\nu - n + \frac{1}{2})} + \sum_{\nu > 2n} n^{4L - 2} \right].
\]

The first and second sums are \( O(n^{4L} \log n) \). The third sum is \( O(n^{4L}). \) This proves the lemma.

In Lemma 4, we could replace \( 4L \) by \( 2L \), if the \( \sigma_\nu \) satisfy the further condition

\[
\sum_{|\nu| \leq n} \frac{\sigma_\nu}{\nu + \frac{1}{2}} = O(1).
\]

This follows from [3] Theorem 2. Hence, as the preceding proof shows, we can replace \( 1/8 \) by \( 1/4 \) in Theorem 2 if we add the condition

\[
\sum_{|\nu| \leq n} \frac{\alpha_\nu}{\nu + \frac{1}{2}} = O(1).
\]

**6.** The function \( q(z) \) of § 4 is given by

\[
q(z) = \frac{1}{2} \coth \pi z + \frac{c - R(z)}{P(z)}.
\]

Let

\[
q_0(z) = iq(iz) = \frac{1}{2} \cot \pi z + \frac{c - r(z)}{p(z)}.
\]

If \( \sum c_\nu e^{i\nu z} \) is the C.E.S. of \( f \) with respect to \( q(z) \), then, for all sufficiently large \( n \),
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(12) \[ \sum_{|\nu| \leq n} c_{\nu} e^{i\nu x} = \frac{1}{2\pi i} \int_{r_n} q(z)dz \int_{-\pi/2}^{\pi/2} f(t)e^{i(x-t)}dt \]

Suppose now that \( \beta_{\nu} = 0 \) for all \( \nu \), and that \( c \) is real. Then \( q_{\nu}(z) \) is real for real \( z \), so that \( q_{\nu}(\bar{z}) = \overline{q_{\nu}(z)} \). If \( r_{\nu} = \text{res} q(z) = \text{res} q_{\nu}(z) \), then \( r_{\nu} \) is real. Let \( f \) be real. Write

\[ a_{\nu} - ib_{\nu} = c_{\nu} = r_{\nu} \int_{-\pi/2}^{\pi/2} f(t)e^{-i\nu t}dt . \]

Equating real parts in (12), we get

(13) \[ \sum_{|\nu| \leq n} a_{\nu} \cos l_{\nu}x + b_{\nu} \sin l_{\nu}x = \frac{1}{2\pi i} \int_{r_n} q(z)dz \int_{-\pi/2}^{\pi/2} f(t) \cos z(x-t)dt \]

We thus obtain the class of trigonometric series investigated by Korous [1]. Theorem 2 shows, in this special case, not only that (13) converges boundedly to \( f(x) \) within \(( -\pi/2, \pi/2) \), but also that

\[ \sum_{|\nu| \leq n} a_{\nu} \sin l_{\nu}x - b_{\nu} \cos l_{\nu}x \]

converges boundedly to zero.

7. We now turn to the proof of Theorem 3. We again suppose that the notation has been chosen so that if \( 0 \in \{l_{\nu}\} \), then \( 0 = l_{\nu} \). It will suffice to prove

**Lemma 5.** Under the conditions of Theorem 3, there are complex numbers \( w_{\nu} \), such that

\[ \sum_{|\nu| \leq n} w_{\nu} e^{i\nu x} \to \frac{1}{2} \text{sgn } x \]

boundedly within \(( -\pi, \pi) \).

For then, by the classical theorem of Mittag-Leffler, there is a meromorphic function \( q(z) \) whose poles form a sub-set of \( \{il_{\nu}\} \), the principal part at \( il_{\nu} \) being \( il_{\nu}w_{\nu}/(z-il_{\nu}) \) if \( l_{\nu} \neq 0 \). If \( l_{\nu} = 0 \), we allow the origin to be a regular point. Defining \( H_n(u) \) by (2), we have

\[ \int_{0}^{\pi} H_n(u)du = \frac{1}{2\pi i} \int_{c_n} q(z) \frac{1-e^{-ux}}{z}dz \]
By Lemma 5,
\[
\sum_{|\nu| \leq n} w_\nu (1 - e^{-i\nu x}) = \sum_{|\nu| \leq n} w_\nu e^{-i\nu x} \to -\frac{1}{2} \text{sgn } x
\]
boundedly within \((-\pi, \pi)\). Thus, Theorem 3 will follow from Lemma 1.

One way of proving Lemma 5 is to generalize the following theorem of Levinson [2, 48]: if the real numbers \(\lambda_\nu\) satisfy \(|\lambda_\nu| \leq P < 1/4\), then there are numbers \(w_\nu\) such that
\[
\sum_{|\nu| \leq n} \left[ w_\nu e^{i\nu x} - \frac{e^{-i\nu x}}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-i\nu t} dt \right]
\]
converges uniformly to zero within \((-\pi, \pi)\) if \(f \in L^1(-\pi, \pi)\). The generalization consists in showing that we can replace the real \(\lambda_\nu\) by \(\nu + \alpha_\nu + i\beta_\nu\), where \(|\alpha_\nu| \leq P\) and \(\lim_{|\nu| \to \infty} |\beta_\nu| < \infty\). However, we only need the result for the function \(f(t) = 1/2 \text{sgn } t\). It seems worthwhile to prove this special case, for which the argument of Levinson can be given a rather simple form. This is done in § 9.

8. We need the following deduction from Lemma 4.

**Lemma 6.** Let \(S_\nu = \nu + \sigma_\nu + i\tau_\nu\), where \(\sigma_\nu, \tau_\nu\) are real numbers which satisfy \(|\sigma_\nu| \leq P, |\tau_\nu| \leq Q\) for all \(\nu\), where \(0 < P < 1/4\) and \(Q > 0\). Let
\[
\mathcal{F}(z) = (z - S_0) \prod_{\nu \neq 0} \left(1 - \frac{z}{S_\nu}\right) \left(1 - \frac{z}{S_{-\nu}}\right).
\]

Then there is a constant \(K\) (depending only on \(P\) and \(Q\)) such that
\[
|\mathcal{F}(z)| < K(1 + |z|)^iP e^{\pi|\text{Im } z|}.
\]
and there is a constant \(K_\varepsilon\) (depending only on \(P, Q\) and \(\varepsilon\)) such that
\[
|\mathcal{F}(z)|^{-1} < K_\varepsilon(1 + |z|)^{P e^{-\pi\varepsilon|\text{Im } z|}}
\]
if \(|z - S_\nu| \geq \varepsilon\) for all \(\nu\).

**Proof.** In the following proof, and in § 9, the symbols \(K, K_\varepsilon\) do not necessarily denote the same constants at each occurrence. In Lemma 4, choose \(s_0 = \frac{1}{2} P, s_\nu = S_\nu\) for \(\nu \neq 0\). For \(|\nu| \geq 1\), we have
\[
|s_\nu| > \frac{3}{4}.\]
By Lemma 4 (with \(\delta = \min (1/2P, 3/4)\)),
\[
|\psi(z)| < K(1 + |z|)^{P e^{\pi|\text{Im } z|}}.
\]
Now
\begin{equation}
\Psi(z) = -\frac{P}{2} \left( \frac{z - S_0}{z - S_0} \right) \Psi(z)
\end{equation}
and \(|z - S_0)/(z - S_0)| < K\) for \(|z - S_0| \geq 1/4\). For such \(z\), (14) follows from (16). Finally, \(|\Psi(z)| \leq K\) inside \(|z - S_0| \leq 1/4\) since this is true on the boundary. This proves (14).

Let \(|z - S_\nu| \geq \varepsilon\) for all \(\nu\). If \(|z - S_0| \geq \varepsilon\) then
\begin{equation}
|\Psi(z)| \leq K(1 + |z|)^{4\rho e^{-\pi/mz}}
\end{equation}
by Lemma 4, and \(|(z - s_\nu)/(z - S_\nu)| < K\) so that (15) follows from (17) and (18). If, however, \(|z - S_0| < \varepsilon\), then for small \(\varepsilon\) the disc \(\mathcal{A} : |z - S_0| < \varepsilon\) is outside each disc \(|z - S_\nu| < \varepsilon\) (\(\nu = \pm 1, \pm 2, \cdots\)). If it is outside the disc \(\mathcal{A} : |z - S_\nu| < \varepsilon\), then \((\Psi(z))^{-1}\) is regular in \(\mathcal{A}\) and so \(|\Psi(z)|^{-1} \leq K\) in \(\mathcal{A}\) since this is true on the boundary. If \(\mathcal{A}\) meets \(\mathcal{A}'\) we apply this argument to the portion of \(\mathcal{A}\) which is outside \(\mathcal{A}'\).

9. Proof of Lemma 5. By the hypothesis (of Theorem 3), there are positive numbers \(P, Q\) such that \(|\alpha_\nu| \leq P < 1/4, |\beta_\nu| \leq Q\), for all \(\nu\). Let \(C_n\) denote the rectangular contour whose vertices are \(\pm (n + 1/2) \pm ni\). Let
\[G(z) = (z - l_0) \prod_{\nu=1}^{\infty} \left( 1 - \frac{z}{l_\nu}\right) \left(1 - \frac{z}{l_{-\nu}}\right).\]
We define
\[w_\nu = \frac{1}{2\pi i} \int_{-\infty}^{\infty} G(u, \nu) (u - l_\nu) \frac{du}{G'(l_\nu)(u - l_\nu)}\]
where
\[\varphi(u) = \frac{1 - \cos \pi u}{u}.\]
Then
\[
\sum_{|\nu| \leq n} w_\nu e^{i\nu x} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} G(u) \varphi(u) du \int_{\sigma_n} \frac{e^{i\zeta}}{G(\zeta)(u - \zeta)} d\zeta \\
- \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \varphi(u) e^{iux} du \int_{\sigma_n} \frac{d\zeta}{u - \zeta}.
\]
The last term is
\[
\frac{1}{2\pi i} \int_{-(n + 1/2)}^{n + 1/2} \varphi(u) e^{iux} du = \frac{1}{2\pi} \int_{-(n + 1/2)}^{n + 1/2} \frac{1 - \cos \pi u}{u} \sin ux du \\
\rightarrow \frac{1}{2} \text{sgn } x.
\]
boundedly within \((-\pi, \pi)\). Hence it suffices to prove that \(I_n(x) \to 0\) boundedly within \((-\pi, \pi)\), where

\[
I_n(x) = \int_{-\infty}^{\infty} G(u)\varphi(u)du \int_{\zeta} e^{iz} d\zeta.
\]

Since \(G(z)\) is a function \(\Psi(z)\), we have by (15), \(\left|G(\zeta)\right|^{-1} < Kn^{i\pi}e^{-n\pi}\) on the horizontal sides of \(C_n\). Further,

\[
|e^{iz}| \leq e^{n|x|}, \quad |u - \zeta|^{-1} < K(1 + |u|)^{-1}, \quad |\varphi(u)| < K(1 + |u|)^{-1}.
\]

Since \(G(u)|< K(1 + |u|)^{1p}\) by (14), the contribution to \(I_n\) of a horizontal side of \(C_n\) does not exceed in absolute value

\[
Kn^{1+4p}e^{-n(1+|\zeta|)} \int_{-\infty}^{\infty} \frac{du}{(1 + |u|)^{1p-4p}},
\]

and tends to zero uniformly within \((-\pi, \pi)\). It remains to consider the contribution to \(I_n\) of a vertical side of \(C_n\), say the right side. This contribution is

\[
J_n(x) = \int_{-\infty}^{\infty} G(u)\varphi(u)du \int_{-i\alpha}^{i\alpha} e^{iz(u+1/2+\zeta)} d\zeta
\]

\[
= e^{iz(u+1/2)} \int_{-\infty}^{\infty} G(u + n + \frac{1}{2}) \varphi(u + n + \frac{1}{2}) du
\]

\[
\times \int_{-n}^{n} \frac{e^{iz\zeta}}{G(n + \frac{1}{2} + \zeta)(u - \zeta)} d\zeta.
\]

For all \(v\), we define \(l'_v = -n + l_{v+n}\). Then

\[
\frac{G(z)}{G(w)} = \frac{(z - l_0)}{(w - l_0)} \prod_{v=1}^{n} \frac{(z - l_v)(z - l_{-v})}{(w - l_v)(w - l_{-v})}
\]

\[
= \frac{z - n - l'_0}{w - n - l'_0} \prod_{v=1}^{n} \frac{(z - n - l'_{v+n})(z - n - l'_{-v-n})}{(w - n - l'_{v+n})(w - n - l'_{-v-n})}
\]

\[
= \frac{G_z(z - n)}{G_z(w - n)}
\]

where

\[
G_z(z) = (z - l'_0) \prod_{v=1}^{n} \left(1 - \frac{z}{l'_v}\right)\left(1 - \frac{z}{l'_{-v}}\right)
\]

and \(l'_v = v + \alpha'_v + i\beta'_v\), \(\alpha'_v = \alpha_{v+n}\), \(\beta'_v = \beta_{v+n}\). Then \(|\alpha'_v| \leq P\), \(|\beta'_v| \leq Q\). Hence \(G_z(z)\) is a function \(\Psi(z)\) (of Lemma 6) and satisfies the inequalities (14), (15) with constants \(K, K\epsilon\) independent of \(n\). In (19), we use the equation
It follows that
\[ |J_n(x)| \leq \int_{-\infty}^{\infty} \left| G_n(u + \frac{1}{2}) \right| \varphi(u + n + \frac{1}{2}) |J| \, du \]
where
\[ J = \int_{\gamma} \frac{e^{i\xi \zeta}}{G_n(\zeta + \frac{1}{2})(u - \zeta)} \, d\zeta \]
and \( \gamma \) denotes the path from \(- i \infty \) to \( i \infty \) modified by replacing the segment \((- i/8, i/8)\) by the right half or the left half of the circle \(|\zeta| = 1/8\), according as \( u < 0 \) or \( u > 0 \). On \( \gamma \), \( \text{re}(\zeta + 1/2) \) is between \(3/8\) and \(5/8\), and therefore \( \zeta + 1/2 \) is at a distance greater than \(1/8\) from all the zeros of \( G_n(z) \). By Lemma 6, \( |G_n(\zeta + 1/2)|^{-1} < Ke^{-\pi|\eta|}(1 + |\eta|) \), where \( \eta = im \zeta \). Further \( |u - \zeta|^{-1} < K(1 + |u|)^{-1} \), and so
\[ |J| < \frac{K}{1 + |u|} \int_{-\infty}^{\infty} e^{-\pi|\zeta - x|}(1 + |\eta|) \, d\eta \]
\[ < \frac{K}{(1 + |u|)(\pi - |x|)^2} . \]
Since \( |G_n(u + 1/2)| < K(1 + |u|)^{4p} \), it remains to prove that \( H_n \to 0 \) where
\[ H_n = \int_{-\infty}^{\infty} \frac{du}{(1 + |u|)^{d}(1 + |u + n + \frac{1}{2}|)} \]
and \( d = 1 - 4P > 0 \).

If \( m \) is a positive integer, then
\[ H_n = \int_{|u| \leq m} + \int_{|u| > m} \]
and the first integral tends to zero as \( n \to \infty \). Choose \( p \) so that \( pd > 1 \) and let \( q^{-1} + p^{-1} = 1 \). Then
\[ \int_{|u| > m} \leq \left( \int_{|u| > m} \frac{du}{(1 + |u|)^{pd}} \right)^{1/p} \left( \int_{-\infty}^{\infty} \frac{du}{(1 + |u + n + \frac{1}{2}|)^{q}} \right)^{1/q} \]
\[ < Km^{1/p - d} . \]
so that \( \lim H_n = 0 \), as required.

*Added in proof.* A result similar to Theorem 2 was proved in a Ph. D thesis by J. A. Anderson.

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