SOME AVERAGES OF CHARACTER SUMS

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Let \( \chi \) and \( \phi \) be nonprincipal characters mod \( p \). Let \( f \) be a polynomial mod \( p \) and let \( a_1, \ldots, a_p \) be complex constants. We will assume \( a_j = a_k \) for \( j \equiv k(p) \), and thus have \( a_n \) defined for all \( n \). Define

\begin{equation}
S = \sum_r a_r \chi(f(r))
\end{equation}

and

\begin{equation}
J_s(c) = \sum_r \phi(r) \chi(r^n - c),
\end{equation}

where the variables of summation run through a complete system of residues mod \( p \).

The averages in question are

\begin{equation}
A_1 = \sum_{a=1}^{p-1} |J_s(a)|^2,
\end{equation}

and

\begin{equation}
A_2 = \sum |S|^2,
\end{equation}

where the sum in (4) is over the coefficients mod \( p \) of certain fixed powers of the variables in \( f \). Exact formulae for \( A_1 \) will be obtained in all cases, and for \( A_2 \) in an extensive class of cases.

Specifically, the following theorems are true.

THEOREM I. Let \( f(r) = y r^{m_1} + x r^{m_2} + g(r) \) and assume \( (m_2 - m_1, p - 1) = 1 \). Let the sum in (4) be over all \( x \) and \( y \) mod \( p \). If \( g \) has a nonzero constant term and neither \( m_1 \) nor \( m_2 \) is zero, then

\begin{equation}
A_2 = p(p - 1) \sum_{r=1}^{p-1} |a_r|^2 + p^2 |a_0|^2.
\end{equation}

Otherwise,

\begin{equation}
A_2 = p(p - 1) \sum_{r=1}^{p-1} |a_r|^2.
\end{equation}

THEOREM II. Let \( d = (n, p - 1), \psi(t) = e^{2\pi i (r \text{ind} (t) / s)}, \) where, naturally, \( s \mid (p - 1), (r, s) = 1 \) and \( g^{\text{ind} (t)} \equiv t(p) \) for \( g \) a primitive root mod \( p \). If \( ds \mid (p - 1) \), then \( A_1 = 0 \). If \( ds \mid (p - 1) \) and \( \psi \chi^s \) is nonprincipal, then \( A_1 = p(p - 1)d \). If \( ds \mid (p - 1) \) and \( \psi \chi^s \) is principal, then \( A_1 = p(p - 1)(d - 1) - (p - 1) \).

The following is an immediate consequence of the first theorem.

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Theorem III. Let \( f \) be as in Theorem I, and assume \( |a_r| = 1, r = 1, \ldots, p \). Then there exist \( x_0, y_0, x_1 \) and \( y_1 \) depending on \( \chi \), such that the \( S_r \) as in (1), for \( x_0 \) and \( y_0 \), satisfies \( |S| < \sqrt{p} \) and the \( S_r \), for \( x_1 \) and \( y_1 \), satisfies \( \sqrt{(p - 2)} < |S| \).

Proof of Theorem II. Our principal device is the fact that a function which is periodic mod \( p \) has a unique expansion by means of the characters mod \( p \). That is if \( h(r) = h(s) \) for \( r \equiv s(p) \), then for \( \chi \)

\[
(7) \quad h(n) = \sum_{\theta} b_\theta \theta(n),
\]

where \( \theta \) runs through the characters mod \( p \). \( b_\theta \) is given by

\[
(8) \quad (p - 1)b_\theta = \sum_r h(r)\theta(r).
\]

Regarding \( J_\chi(c) \) as a periodic function mod \( p \) of \( c \), and expanding \( J_\chi(c) \) in the form (7), we obtain, by standard methods,

\[
(9) \quad J_\chi(c) = \sum_{\rho^n = \chi} \pi(\bar{\rho}, \chi)\rho(c)
\]

where \( \pi(\alpha, \beta) \) is a Jacobi sum [1]

\[
(10) \quad \pi(\alpha, \beta) = \sum_r \alpha(r)\beta(1 - r).
\]

The sum in (9) is over all characters \( \rho \) which satisfy the indicated condition.

The expansion (7) has a Parseval identity

\[
(11) \quad \sum_{i=1}^{p-1} |h(t)|^2 = (p - 1) \sum_\theta |a_\theta|^2.
\]

Thus we can evaluate \( A_i \) by means of (11) and (9) when we know the value of \( |\pi(\alpha, \beta)|^2 \). Now [1] \( |\pi(\alpha, \beta)|^2 = p \) when \( \alpha \neq \varepsilon, \beta \neq \varepsilon \) and \( \alpha\beta = \varepsilon \), where \( \varepsilon \) is the principal character. If \( \alpha = \varepsilon \) or \( \beta = \varepsilon \), then \( |\pi(\alpha, \beta)|^2 = 1 \). If \( \alpha\beta = \varepsilon \) with \( \alpha \neq \varepsilon \) or \( \beta \neq \varepsilon \), then \( |\pi(\alpha, \beta)|^2 = p \). By hypothesis, \( \chi \) is nonprincipal. Thus \( |\pi(\bar{\rho}, \chi)|^2 \) is \( p \) unless \( \bar{\rho} = \varepsilon \) or \( \bar{\rho}\chi = \varepsilon \). If \( \bar{\rho} = \varepsilon \), then \( \bar{\rho} = \varepsilon \) and \( \psi\chi^* \) is principal. If \( \bar{\rho}\chi = \varepsilon \), then \( \rho = \chi \) and \( \rho^* = \psi\chi^* \) implies \( \psi = \varepsilon \) which is excluded by hypothesis. Let \( N \) be the number of solutions of \( \rho^* = \psi\chi^* \). If \( \psi\chi^* \) is nonprincipal then \( |\pi(\bar{\rho}, \chi)|^2 = p \) for all \( N \) of the \( \rho \) and \( A_i = p(p - 1)N \). If \( \psi\chi^* \) is principal, then \( |\pi(\bar{\rho}, \chi)|^2 = p \) for \( N - 1 \) of the \( \rho \) and \( |\pi(\bar{\rho}, \chi)|^2 = 1 \) for \( \rho = \varepsilon \). Thus, in this case, \( A_i = (p - 1)(p(N - 1) + 1) = Np(p - 1)^2 \).

Therefore, the number of solutions of \( \rho^* = \psi\chi^* \), is the number of solutions of \( \sigma^* = \psi \). It is a standard lemma from the theory of cyclic groups of order \( k \) that \( a^* = b \) has \((n, k) = 0 \) solutions according to whether...
or not order $b \mid k/(n, k)$. Also, $N$ is the number of solutions of $x^n = \psi(g)$, for $x$, in $(p - 1) - st$ roots of unity. From either description of $N$, it follows that $N = d$ or $N = 0$ according as $ds \mid (p - 1)$ or $ds \mid (p - 1)$, and the theorem follows.

**Proof of Theorem I.** Referring to the hypotheses of Theorem I,

$$|S|^2 = \sum_{r,s} a_r a_s \chi(yr^{m_1} + xr^{m_2} + g(r)) \bar{\chi}(ys^{m_1} + xs^{m_2} + g(s))$$

and thus,

(12) $A_2 = \sum a_r a_s \sum \chi(yr^{m_1} + xr^{m_2} + g(r)) \bar{\chi}(ys^{m_1} + xs^{m_2} + g(s)) = T_1 + T_2.$

$T_1$ is the sum of the terms in (12) such that $r \neq 0$ and $s \neq 0$. $T_2$ is the sum of the terms in (12) such that $r = 0$ or $s = 0$. $T_1$ can be written

(13) $T_1 = \sum_{r \neq 0, s} a_r a_s \chi(r/s) A(p^{m_2-m_1}, r^{-m_2}g(r); s^{m_2-m_1}, s^{-m_1}g(s))$

where

$$A(a, b; c, d) = \sum_{y + cx \equiv 0} \chi \left( \frac{y + ax + b}{y + cx + d} \right).$$

Now,

$$A(a, b; c, d) = \sum_x \sum_{y \neq 0} \chi \left( \frac{y + x(a - c) + (b - d)}{y} \right).$$

Except when $(a - c)x + (b - d) \equiv 0(p),$

$$\sum_{y \neq 0} \chi \left( \frac{y + (a - c)x + (b - d)}{y} \right) = -1.$$

Also, $(a - c)x + (b - d) \equiv 0(p)$ when $x \equiv ((b - d)/(a - c))(p)$ or when $a \equiv c$ and $b \equiv d$. Thus, if $a \not\equiv c$ or $b \not\equiv d$, then

$$A(a, b; c, d) = -(p - 1) + p - 1 = 0.$$

If $a \equiv c$ and $b \equiv d$, then

$$A(a, b; c, d) = p(p - 1).$$

In view of this (13) becomes the sum over all $r$ and $s$ such that $r \neq 0 \not\equiv s$ and $r^{m_2-m_1} = s^{m_2-m_1}$, $r^{-m_2}g(r) = s^{-m_1}g(s)$. Since $(m_2 - m_1, p - 1) = 1$, we have $r \equiv s$. Thus the sum in (13) is over those $r$ and $s$ such that $r \neq 0 \not\equiv s$ and $r \equiv s$. Thus

$$T_1 = p(p - 1) \sum_{r=1}^{p-1} |a_r|^2.$$
Now

\begin{equation}
T_2 = \sum_{r \geq 0} a_r \sum_{x,y} \chi(yr^{m_1} + xr^{m_2} + g(r))\overline{\chi}(g(0)) \\
+ \sum_{x \geq 0} a_x \sum_{x,y} \chi(g(0))\overline{\chi}(ys^{m_1} + xs^{m_2} + g(s)) \\
+ \lvert a_0 \rvert^2 \sum_{x,y} \chi(g(0))\overline{\chi}(g(0)) = p^2 \lvert a_0 \rvert^2 \lvert \chi(g(0)) \rvert^2,
\end{equation}

except when \( m_1 = 0 \) or \( m_2 = 0 \).

Thus, if \( g(0) = 0 \),

\[ A_2 = p(p - 1) \sum_{r \neq 0} |a_r|^2 \]

and if \( g(0) \neq 0 \), then

\[ A_2 = p(p - 1) \sum_{r \neq 0} |a_r|^2 + p^2 \lvert a_0 \rvert^2, \]

when \( m_1 = 0 \) or \( m_2 = 0 \), then \( \chi(g(0)) \) in (14) must be changed to \( \chi(y + g(0)) \) or \( \chi(x + g(0)) \), and \( A_2 \) is given by (6).

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