ADJOINT QUASI-DIFFERENTIAL OPERATORS OF EULER TYPE

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This paper treats linear quasi-differential operators of the form

$$L[y] = \sum_{j=0}^{n} p_{0j} y^{(j)} - \left( \sum_{j=0}^{n} p_{1j} y^{(j)} - \left( \cdots - \left( \sum_{j=0}^{n} p_{mj} y^{(j)} \right) \cdots \right) \right)',$$

based on an integrable \((m+1) \times (n+1)\) matrix function \([p_{ij}]\), \((i=0, \ldots, m; j=0, \ldots, n)\), about which suitable regularity assumptions are made. Results obtained by Reid (Trans. Amer. Math. Soc. Vol. 85 (1957), pp. 446–461) are extended to operators of the type considered here.

A generalized Green's function for the system \(L[y] = 0, y \in \mathcal{D}\) is defined, where \(\mathcal{D}\) is a linear subspace of the domain of \(L\). Resolvent and deterministic properties of this function are presented, together with the relationship of such a generalized Green's function to the generalized Green's function for the associated adjoint system.

For a large class of two-point boundary problems in which the boundary conditions involve the characteristic parameter linearly it is shown that there exists a simultaneous canonical representation of the boundary conditions for a given problem and those of its adjoint; in particular, in the self-adjoint case this canonical representation has the form of boundary conditions and transversality conditions for a variational problem. Finally, these results are applied to a two-point boundary problem involving a differential operator of the type considered in the paper of Reid above.

Since an important example of an operator of the form of \(L[y]\) is the Euler operator in the calculus of variations, we shall refer to such operators as \textit{quasi-differential operators of Euler type}.

Section 2 gives a more precise description of the operator, and Section 3 is concerned with a discussion of its adjoint. In particular it is shown that if \(\mathcal{D}_0\) is the class of functions \(y\) in the domain of \(L\) with the property that the functions \(y, y', \ldots, y^{(n-1)}, \tilde{y}_m = \sum_{j=0}^{n} p_{mj} y^{(j)}\), \(\tilde{y}_i = \sum_{j=0}^{n} p_{ij} y^{(j)} - \tilde{y}_{i+1}\), \((i = m - 1, \ldots, 1)\), vanish at \(a\) and at \(b\), and if \(T_0\) is the restriction of \(L\) to \(\mathcal{D}_0\), then the adjoint operator \(T_0^*\) is given by

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Section 4 is a study of extensions of the operator $T_0$, and their adjoints. Section 5 is devoted to generalized Green's functions for Euler type quasi-differential systems and their adjoints, and extends the results of Elliott [3] and Reid [5] to the case where the number of linearly independent boundary conditions may differ from the order of the differential equation.

Section 6 is concerned with a certain class of two-point boundary problems in which the boundary conditions involve the characteristic parameter linearly. It is shown that there exists a simultaneous canonical representation of the boundary conditions for a given problem and those of its adjoint; in particular, in the self-adjoint case this canonical representation has the form of boundary conditions and transversality conditions for a variational problem.

Finally, § 7 is devoted to an application of the results of § 6 to a two-point boundary problem involving a differential operator of the type considered by Reid in [7].

The symbol $\mathbb{C}_n$, $(n = 0, 1, 2, \cdots)$, will signify the class of complex-valued functions defined on the compact interval $[\alpha, \beta]$ which have $n$ continuous derivatives. The set of functions $y$ in $\mathbb{C}_{n-1}$ for which $y^{(n-1)}$ is a.c. (absolutely continuous) is denoted by $\mathbb{U}_n$, $(n = 0, 1, 2, \cdots)$. In particular, $\mathbb{C}_0$ and $\mathbb{U}_0$ will signify respectively the classes of continuous and Lebesgue integrable complex-valued functions defined on $[\alpha, \beta]$.

If $f$ and $g$ belong to $\mathbb{U}_n$ and $f(x) = g(x)$ almost everywhere, we will simply write $f = g$. If $f$ is a complex-valued function on $[\alpha, \beta]$, then $\overline{f}$ denotes the function with domain $[\alpha, \beta]$ whose value at $x$ is the complex conjugate of $f(x)$. If $u$ and $v$ are functions on $[\alpha, \beta]$ and $v \in \mathbb{U}_n$, then we define $(u, v)$ as

$$ (u, v) = \int_{\alpha}^{\beta} \overline{v} u. $$

Matrix notation will be used except where it is impracticable. If $M$ is a matrix, then the conjugate transpose of $M$ is denoted by $M^*$. Vectors are treated as matrices with one column. The symbols $E_n$ and $0_{mn}$ are used to represent the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively; the subscripts will be omitted when there is no danger of confusion.

A matrix function is said to be continuous, integrable, etc. whenever each of its elements possesses the specified property. If $A$ is an a.c. matrix function, then $A'(x)$ signifies the matrix of derivatives at values for which these derivatives exist and the zero matrix elsewhere.

2. Description of the operator. Suppose that $[p_{ij}]$, $(i = 0, \cdots,$
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For suitable $y$ in $\mathfrak{U}_n$ define functions $\bar{y}_1, \cdots, \bar{y}_m$ as follows:

\begin{equation}
\bar{y}_m(x) = \sum_{j=0}^{n} p_{m,j}(x)y^{(j)}(x) ;
\end{equation}

if $\bar{y}_{j+1} \in \mathfrak{U}_i$, then

\begin{equation}
\bar{y}_i(x) = \sum_{j=0}^{n} p_{i,j}(x)y^{(j)}(x) - \bar{y}_{i+1}(x) ,
\end{equation}

($i = m-1, \cdots, 1$).

The class of functions $y$ in $\mathfrak{U}_n$ for which $\bar{y}_1, \cdots, \bar{y}_m$ are a.c. will be denoted by $\mathfrak{A}_n$. For convenience the vector functions $(y^{(j-1)})$, $(j = 1, \cdots, n)$, and $(\bar{y}_i)$, ($i = 1, \cdots, m$), will be denoted by $\hat{y}$ and $\bar{y}$, respectively; the $(n + m)$-vector function $(y, \cdots, y^{(n-1)}, \bar{y}_1, \cdots, \bar{y}_m)$ will be represented by $\bar{y}$.

Denote by $L$ the operator with domain $\mathfrak{A}_n$ which is defined by

\begin{equation}
L[y] = \sum_{j=0}^{n} p_{o,j}y^{(j)} - \bar{y}_1 .
\end{equation}

The operator $L$ is a quasi-differential operator in the sense of Böcher [1]; in particular, it is a generalization of the Euler operator in the calculus of variations and, as was stated in the introduction, it will be called a quasi-differential operator of the Euler type.

Let $\mathfrak{A}_m^n$ be the collection of functions $y$ in $\mathfrak{A}_n$ for which $\bar{y}(a) = 0 = \bar{y}(b)$, and denote by $T_0$ the restriction of $L$ to $\mathfrak{A}_m^n$. Suppose that $\mathcal{B}_m^n$ is the class of functions $z$ in $\mathfrak{A}_m$ which are essentially bounded and have the property that there exists a function $f_i$ in $\mathfrak{A}_m$ such that $(L[y], z) = (y, f_i)$ for all $y$ in $\mathfrak{A}_m^n$.

A second operator $L^\times$ will now be defined. For suitable functions $z$ in $\mathfrak{A}_m$ define functions $\bar{z}_1, \cdots, \bar{z}_m$ as follows:

\begin{equation}
\bar{z}_m(x) = \sum_{i=0}^{m} \bar{p}_{i,m}(x)z^{(i)}(x) ;
\end{equation}

if $\bar{z}_{j+1} \in \mathfrak{U}_i$, then

\begin{equation}
\bar{z}_i(x) = \sum_{i=0}^{m} \bar{p}_{i,j}(x)z^{(i)}(x) - \bar{z}_{i+1}(x) ,
\end{equation}

($j = n - 1, \cdots, 1$).

The class of functions $z$ in $\mathfrak{A}_m$ for which $\bar{z}_1, \cdots, \bar{z}_m$ are a.c. will be denoted by $\mathfrak{A}_m$. Let $L^\times$ be the operator with domain $\mathfrak{A}_m$ defined by

\begin{equation}
L^\times[z] = \sum_{i=0}^{m} \bar{p}_{i,m}z^{(i)} - \bar{z}_1 .
\end{equation}

If $z \in \mathfrak{A}_m$, then $\bar{z}$ and $\bar{z}$ will signify the vector functions $(z^{(i-1)})$, ($i = 1, \cdots, m$), and $(\bar{z}_i)$, ($j = 1, \cdots, n$), respectively. The $(m + n)$-vector function $(z, \cdots, z^{(m-1)}, \bar{z}_1, \cdots, \bar{z}_m)$ will be denoted by $\bar{z}$. 
Except when a statement is made to the contrary, the following hypothesis will be assumed throughout this paper.

**HYPOTHESIS (H).** The matrix \([p_{ij}(x)], (i = 0, \ldots, m; j = 0, \ldots, n)\), is integrable and there exists an \(\epsilon > 0\) such that \(|p_{mn}(x)| \geq \epsilon\) almost everywhere on \([a, b]\). Moreover, \(p_m\) and \(p_n\) are essentially bounded and \(p_ip_{m}^{-1}p_{m}j\) is integrable, \((i = 1, \ldots, m-1; j = 1, \ldots, n-1)\).

It is to be noted that if \(y \in \mathbb{A}_n\) and \(z \in \mathbb{A}_m\), then \(L[y]\) and \(L^*[z]\) are integrable.

Let \(\mathcal{A}(x), \mathcal{A}_1(x), \mathcal{A}_2(x), \) and \(\mathcal{A}_3(x)\) be \(m \times n, m \times m, n \times n,\) and \(n \times m\) matrices, respectively, defined as follows:

\[
\mathcal{A}_1(x) = [p_{ij}(x) - p_{in}(x)p_{nm}^{-1}(x)p_{mj}(x)],
\]
\((i = 0, \ldots, m - 1; j = 0, \ldots, n - 1)\),

\[
\mathcal{A}_2(x) = \begin{bmatrix}
0_{m-1} & p_{in}(x)p_{nm}^{-1}(x) \\
-E_{m-1} & p_{in}(x)p_{nm}^{-1}(x)
\end{bmatrix},
\]
\((i = 1, \ldots, m - 1)\),

\[
\mathcal{A}_3(x) = \begin{bmatrix}
0_{n-1} & -E_{n-1} \\
p_{mn}^{-1}(x)p_{m0}(x) & p_{mn}^{-1}(x)p_{m1}(x)
\end{bmatrix},
\]
\((j = 1, \ldots, n - 1)\),

\[
\mathcal{A}_4(x) = \begin{bmatrix}
0_{m-1} & 0_{n-1} \\
0_{m-1} & -p_{mn}^{-1}(x)
\end{bmatrix}.
\]

If \(f\) and \(g\) belong to \(\mathbb{A}_m\), then the equation \(L[y] = f\) is equivalent to the following system in the vector functions \(\tilde{y} = (\tilde{y}_i), (i = 1, \ldots, n)\), and \(\tilde{y} = (\tilde{y}_j), (j = 1, \ldots, m)\):

\[
\begin{align}
\tilde{y}' + \mathcal{A}_1 \tilde{y} + \mathcal{A}_2 \tilde{y} &= 0, \\
\tilde{y}' - \mathcal{A}_1 \tilde{y} - \mathcal{A}_2 \tilde{y} &= -fe^{(m,1)};
\end{align}
\]

and the equation \(L^*[z] = g\) is equivalent to the following system in the vector functions \(\tilde{z} = (\tilde{z}_i), (j = 1, \ldots, m)\), and \(\tilde{z} = (\tilde{z}_i), (i = 1, \ldots, n)\):

\[
\begin{align}
\tilde{z}' + \mathcal{A}_3 * \tilde{z} + \mathcal{A}_4 * \tilde{z} &= 0, \\
\tilde{z}' - \mathcal{A}_3 * \tilde{z} - \mathcal{A}_4 * \tilde{z} &= -ge^{(n,1)},
\end{align}
\]

where \(e^{(k,i)}\), \((k = 1, 2, 3, \ldots)\), is used to denote the \(k\)-dimensional vector whose first coordinate is one, and whose remaining coordinates are zero. If \(\mathcal{J}\) is the \((m + n) \times (m + n)\) matrix

\[
\mathcal{J} = \begin{bmatrix}
0_{mn} & -E_m \\
E_n & 0_{nm}
\end{bmatrix},
\]

and \(\mathcal{A}\) is the \((m + n) \times (m + n)\) matrix function defined by

\[
\mathcal{A} = \begin{bmatrix}
0_m & -E_m \\
E_n & 0_m
\end{bmatrix},
\]

and \(\mathcal{A}\) is the \((m + n) \times (m + n)\) matrix function defined by
then (2.5) and (2.6) may be written as

\begin{equation}
\mathcal{L}[\tilde{y}] \equiv \mathcal{J} \tilde{y}' + \mathcal{A} \tilde{y} = f e^{(m+n,1)},
\end{equation}

and

\begin{equation}
\mathcal{L}^*[\tilde{z}] \equiv -\mathcal{J}^* \tilde{z}' + \mathcal{A}^* \tilde{z} = g e^{(m+n,1)},
\end{equation}

respectively.

Theorems on existence and uniqueness of solutions of $L[y] = f$ and $L^*[z] = g$ follow from corresponding theorems for the respective first order systems (2.8) and (2.9). It also follows that $y \in \mathfrak{A}_n$ if and only if there exists an integrable function $f$ such that $y$ is the first coordinate of a vector function $\tilde{y}$ satisfying (2.8), and $z \in \mathfrak{A}_m$ if and only if there is an integrable function $g$ such that $z$ is the first coordinate of a vector function $\tilde{z}$ satisfying (2.9).

The differential system (2.5) is identically normal in the sense that if $\tilde{y}(x)$ is a solution of $\mathcal{L}[\tilde{y}] = 0$ with $\tilde{y}(x) = 0$ on a subinterval $X$ of $[a, b]$, then $\tilde{y}(x) = 0$ on $X$. Indeed, if $\tilde{y}$ is such a solution of (2.5), then $\tilde{y}$ is a solution of $\tilde{y}' - \mathcal{A} \tilde{y} = 0$ satisfying $\mathcal{A} \tilde{y} = 0$ on $X$. This latter condition implies that $\tilde{y}_m(x) = 0$ on this subinterval, and the differential equation $\tilde{y}' - \mathcal{A} \tilde{y} = 0$ implies in turn that $\tilde{y}_j(x) = 0$ on $X$ for $j = m - 1, \cdots, 1$. Similarly, system (2.6) is also identically normal. It follows from the identical normality of (2.5) that functions $y_\ast$ in $\mathfrak{A}_n$ are linearly independent solutions of $L[y] = 0$ if and only if the corresponding vector functions $\tilde{y}_\ast$ are linearly independent solutions of $\mathcal{L}[\tilde{y}] = 0$. Similarly, it follows from the identical normality of (2.6) that functions $z_\ast$ in $\mathfrak{A}_m$ are linearly independent solutions of $L^*[z] = 0$ if and only if the corresponding vector functions $\tilde{z}_\ast$ are linearly independent solutions of $\mathcal{L}^*[\tilde{z}] = 0$.

3. The adjoint operator. If $\mathcal{J}$ is the $(m+n) \times (m+n)$ matrix defined as in (2.7), then we may establish the following Lagrange identity by a simple inductive argument which does not use hypothesis (H).

**Lemma 3.1.** If $y \in \mathfrak{A}_n$ and $z \in \mathfrak{A}_m$, then

\begin{equation}
\bar{z}L[y] - L^*[z]y = (\bar{z}^* \mathcal{J} \tilde{y})'.
\end{equation}

**Theorem 3.1.** If $f \in \mathfrak{A}_n$, then there exists a $y$ in $\mathfrak{A}_n$ such that $L[y] = f$ if and only if $z$ in $\mathfrak{A}_m$ and $L^*[z] = 0$ implies that $(f, z) = 0$. 

Now if \( y \in \mathfrak{H}_0 \), \( L[y] = f \), \( z \in \mathfrak{H}_m \), and \( L^a[z] = 0 \), then, in view of Lemma 3.1,
\[
(f, z) = (L[y], z) - (y, L^a[z]) = \tilde{z}^* \mathcal{F} \tilde{y}^*|_{\mathfrak{H}_m} = 0 .
\]
On the other hand, suppose that \((f, z) = 0\) whenever \( z \in \mathfrak{H}_m \) and \( L^a[z] = 0 \), and let \( y \) be the function in \( \mathfrak{H}_m \) such that \( L[y] = f \) and \( \tilde{y}(a) = 0 \). If \( z_j, (j = 1, \cdots, m + n) \) are linearly independent solutions of \( L^a[z] = 0 \), then the \((m + n) \times (m + n)\) matrix \( \tilde{Z}(x) \) with column vectors \( \tilde{z}_j(x) \), \( (j = 1, \cdots, m + n) \), is nonsingular on \([a, b]\). From Lemma 3.1 we have the vector equation
\[
0 = [(f, z_j) - (y, L^a[z_j])] = \tilde{Z}^* \mathcal{F} \tilde{y}^*|_{\mathfrak{H}_m} = \tilde{Z}^*(b) \mathcal{F} \tilde{y}(b) ,
\]
and consequently \( \tilde{y}(b) = 0 \) also.

**Theorem 3.2.** If hypothesis (H) holds, then \( \mathfrak{D}_0^* = \mathfrak{H}_m \) and \( f_z = L^a[z] \) on \( \mathfrak{D}_0^* \).

That \( \mathfrak{H}_m \subset \mathfrak{D}_0^* \) follows from Lemma 3.1. Now let \( z_0 \in \mathfrak{D}_0^* \) and suppose \( f_{z_0} \) is a corresponding function in \( \mathfrak{H}_o \) such that \( (L[y], z_0) = (y, f_{z_0}) \) when \( y \in \mathfrak{H}_m^* \). Choose \( w_0 \) in \( \mathfrak{H}_m^* \) such that \( L^a[w_0] = f_{z_0} \), and suppose that \( z_i \in \mathfrak{H}_m^* \) are linearly independent solutions of \( L^a[z_i] = 0 \), with \( (z_i, z_j) = \delta_{ij}, (i, j = 1, \cdots, m + n) \). If \( w = w_0 + \sum_i \delta_i (z_0 - w_0, z_i) z_i \), then \( L^a[w] = f_{z_0} \) and \( (z_0 - w, z) = 0 \) when \( z \in \mathfrak{H}_m^* \) and \( L^a[z] = 0 \). It follows that if \( y \in \mathfrak{H}_m^* \), then
\[
(L[y], z_0) = (y, f_{z_0}) = (y, L^a[w]) = (L[y], w) ,
\]
so that \( (L[y], z_0 - w) = 0 \) when \( y \in \mathfrak{H}_m^* \). It but it follows from Theorem 3.1 that there is a function \( y \) in \( \mathfrak{H}_0^* \) such that \( L[y] = z_0 - w \). Consequently \( (z_0 - w, z_0 - w) = 0 \) and \( z_0 = w \in \mathfrak{H}_m^* \), so that \( \mathfrak{D}_0^* = \mathfrak{H}_m^* \) and \( f_{z_0} = L^a[z_0] \). This result extends Theorem 4.1 of Reid [7].

Now the operator \( T_o^* \) adjoint to \( T_o \) is defined to be the operator on \( \mathfrak{D}_0^* \) with value \( f_z \) at \( z \). In view of Theorem 3.2 we have \( \mathfrak{D}_0^* = \mathfrak{H}_m^* \) and \( T_o^*[z] = L^a[z] \).

**4. Extensions of the operator \( T_o \).** Let \( \mathcal{D} \) be a linear subspace of \( \mathfrak{H}_a \), containing \( \mathfrak{H}_m^* \), and denote by \( T \) the restriction of \( L \) to \( \mathcal{D} \). Denote by \( \mathcal{D}_a^* \) the class of functions \( z \) in \( \mathfrak{H}_o \) which are essentially bounded and for which there exists an \( f_z \) in \( \mathfrak{H}_o \) such that \( (L[y], z) = (y, f_z) \) for all \( y \) in \( \mathcal{D} \). It follows from Theorem 3.2 that \( \mathcal{D}_a^* \subset \mathfrak{H}_m \), and for each \( z \) in \( \mathcal{D}_a^* \) there is at most one \( f_z \), namely \( L^a[z] \), such that \( (L[y], z) = (y, f_z) \) for all \( y \) in \( \mathcal{D} \). The adjoint \( T_a^* \) of \( T \) is the
operator on $\mathcal{D}^*$ defined by the formula $T^*[z] = f_z$. The operator $T$ is said to be self-adjoint if and only if $\mathcal{D} = \mathcal{D}^*$ and $T = T^*$.

The following lemma will be helpful in describing $\mathcal{D}^*$. If $y_j \in \bar{\mathbb{A}}_n$, $(j = 1, \cdots, m + n)$, then $\bar{Y}$ will denote the matrix function defined by $\bar{Y}(x) = [\bar{y}_j(x)]$, $(j = 1, \cdots, m + n)$.

**Lemma 4.1.** If $\gamma$ and $\zeta$ are $(m + n)$-vectors, then there exists a function $y \in \bar{\mathbb{A}}_n$, $(z \in \bar{\mathbb{A}}_m)$, such that $\bar{y}(a) = \gamma$ and $\bar{y}(b) = \zeta$, $(\bar{z}(a) = \gamma$ and $\bar{z}(b) = \zeta)$.

Since $\bar{\mathbb{A}}_n$ is a vector space it is enough to show that there exist $m + n$ functions $y_j$ in $\bar{\mathbb{A}}_n$ such that $\bar{y}_j(a) = 0$, $(j = 1, \cdots, m + n)$ while $\bar{Y}(b)$ is nonsingular, and to show a corresponding result with $a$ and $b$ interchanged. To establish the existence of functions $y_j$ in $\bar{\mathbb{A}}_n$ such that $\bar{y}_j(a) = 0$, $(j = 1, \cdots, m + n)$, and $\bar{Y}(b)$ is nonsingular, suppose to the contrary that for each collection of $m + n$ functions $y_j$ in $\bar{\mathbb{A}}_n$ satisfying $\bar{y}_j(a) = 0$, $(j = 1, \cdots, m + n)$, we have $\bar{Y}(b)$ singular. Let $z_j$ be $m + n$ linearly independent solutions of $L^*[z] = 0$, and for $j = 1, \cdots, m + n$ let $y_j$ be the function in $\bar{\mathbb{A}}_n$ such that $L[y_j] = z_j$ and $\bar{y}_j(a) = 0$. Then there is a nonzero $(m + n)$-vector $\xi = (\xi_j)$ such that $\bar{Y}(b)\xi = 0$. If $y(x) = \sum_{j=1}^{m+n} y_j(x)\xi_j$ and $z(x) = \sum_{j=1}^{m+n} z_j(x)\xi_j$, then $L[y] = z$, $L^*[z] = 0$ and $z(x) \neq 0$, moreover, $y \in \bar{\mathbb{A}}_n$. Hence it follows from Lemma 3.1 that

$$0 = (L[y], z) - (y, L^*[z]) = (z, z),$$

which is impossible since $z(x) \neq 0$. The numbers $a$ and $b$ may be interchanged and the preceding argument remains valid. The result for $\bar{\mathbb{A}}_m$ follows by interchanging the roles of $\bar{\mathbb{A}}_n$ and $\bar{\mathbb{A}}_m$, that is, by replacing $[p_{ij}]$ with $[p_{ij}]^*$.\n
Denote by $\mathcal{D}$ the subspace of $2(m + n)$-dimensional complex space consisting of the end values $(\bar{y}(a), \bar{y}(a), \bar{y}(b), \bar{y}(b))$ for functions $y$ in $\mathcal{D}$. Similarly, $\mathcal{D}^*$ will denote the subspace of end values $(\bar{z}(a), \bar{z}(a), \bar{z}(b), \bar{z}(b))$ for functions $z$ in $\mathcal{D}^*$. If $k < 2m + 2n$ and the dimension of $\mathcal{D}$ is $2m + 2n - k$, then let $P$ and $Q$ be $(m + n) \times (2m + 2n - k)$ matrices such that the columns of $[-P^*Q^*]^*$ form a basis for $\mathcal{D}$. If $k > 0$ also, then let $M$ and $N$ be $k \times (m + n)$ matrices such that the $k \times 2(m + n)$ matrix $[MN]$ has rank $k$ and $MP - NQ = 0$. Then in view of Lemma 4.1 we have that $\mathcal{D}$ is characterized as the class of functions $y$ in $\bar{\mathbb{A}}_n$ with the property that

$$s(\bar{y}) = My(a) + N\bar{y}(b) = 0.$$\n
If $k = 0$, then by Lemma 4.1 we have $\mathcal{D} = \bar{\mathbb{A}}_n$.\n
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Theorem 4.1. Dim $\mathcal{B} + \dim \mathcal{B}^* = 2m + 2n$; if $\dim \mathcal{B} > 0$ and $P, Q$ are $(m + n) \times (2m + 2n - k)$ matrices such that the column vectors of $[-P^*Q^*]$ from a basis for $\mathcal{B}$, then $\mathcal{B}^*$ is the class of functions $z$ in $\mathfrak{M}_m$ for which

$$P^* \mathcal{J}^* \tilde{z}(a) + Q^* \mathcal{J}^* \tilde{z}(b) = 0.$$  

First note that if $\dim \mathcal{B} = 0$, then $\mathcal{B}^* = \mathfrak{M}_m$ by Theorem 3.2, and thus by Lemma 4.1 we have $\dim \mathcal{B}^* = 2m + 2n$. Now suppose that $\dim \mathcal{B} > 0$, $z \in \mathfrak{M}_m$, and (4.2) holds. Then for $y$ in $\mathcal{B}$ and $\xi$ a $(2m + 2n - k)$-vector chosen so that $\tilde{y}(a) = -P^* \xi$ and $\tilde{y}(b) = Q^* \xi$, it follows from Lemma 3.1 that

$$(L[y], z) = (y, L^*[z]) = \mathcal{J}^* \tilde{y} |^b_y = \{P^* \mathcal{J}^* \tilde{z}(a) + Q^* \mathcal{J}^* \tilde{z}(b)\} \xi = 0$$

and hence $z \in \mathcal{B}^*$. On the other hand, if $z \in \mathcal{B}^*$ then it follows from Theorem 3.2 that $z \in \mathfrak{M}_m$, since $\mathfrak{M}_n \subset \mathcal{B}$. Then (4.2) follows from Lemma 3.1, Lemma 4.1 and the choice of $P$ and $Q$. Therefore, in view of Lemma 4.1, it follows that $\dim \mathcal{B} + \dim \mathcal{B}^* = 2m + 2n$.

Corollary I. If $\dim \mathcal{B} > 0$, and $R$ and $S$ are $(2m + 2n - k) \times (m + n)$ matrices, then $\mathcal{B}^*$ is the collection of functions $z$ in $\mathfrak{M}_m$ for which

$$R \tilde{z}(a) + S \tilde{z}(b) = 0$$

if and only if the $(2m + 2n - k) \times 2(m + n)$ matrix $[R S]$ has rank $2m + 2n - k$ and $M \mathcal{J}^* R^* = N \mathcal{J}^* S^* = 0$.

Corollary II. The adjoint of $T^*$ is $T$.

The index of compatibility for a system $L[y] = 0, y \in \mathcal{D}$ is defined to be $\dim \{y : y \in \mathcal{D} \text{ and } L[y] = 0\}$. The next two theorems are consequences of the equivalence of the equations $L[y] = f$ and $L^*[z] = g$ to the systems (2.8) and (2.9), respectively, and corresponding theorems on first order systems. Analogous theorems for $n$th order linear differential equations are given in [2, Chapter 11], and those results may be extended to first order systems.

Theorem 4.2. If $\dim \mathcal{B}^* = k$ and the index of compatibility of the system $L[y] = 0, y \in \mathcal{D}$ is $r$, then $\rho = k + r - m - n$ is the index of compatibility for the system $L^*[z] = 0, z \in \mathcal{D}^*$.

Theorem 4.3. If $f \in \mathfrak{A}_o$, then there exists a function $y$ in $\mathcal{D}$ such that $L[y] = f$ if and only if $(f, z) = 0$ for all $z$ in $\mathcal{D}^*$ satisfying $L^*[z] = 0$. 

The next two theorems are analogues of Theorems 6.1 and 6.2 of Reid [7]. The second of the two gives necessary and sufficient conditions for the operator \( T \) to be self-adjoint when \( [p_i(x)] \) is Hermitian. If \( y_j \in \mathfrak{A}_n \) and \( \bar{Y} = [\bar{y}_j] \), \( (j = 1, \ldots, m + n) \), then the symbols \( s(\bar{Y}) \) and \( s^{-}(\bar{Y}) \) are used for the \( k \times (m + n) \) matrices \( M\bar{Y}(a) + N\bar{Y}(b) \) and \( MY(a) - N\bar{Y}(b) \), respectively. Similarly, if \( z_j \in \mathfrak{A}_m \) and \( \bar{Z} = [\bar{z}_j] \), \( (j = 1, \ldots, m + n) \), then \( t(\bar{Z}) \) and \( t^{-}(\bar{Z}) \) denote \( R\bar{Z}(a) + S\bar{Z}(b) \) and \( R\bar{Z}(a) - S\bar{Z}(b) \), respectively.

**THEOREM 4.4.** Suppose that \( 2(m + n) > \text{dim} \mathcal{B} > 0 \), \( y_j \) and \( z_j \), \( (j = 1, \ldots, m + n) \), are linearly independent solutions of \( L[y] = 0 \) and \( L^*[z] = 0 \), respectively, and let \( \Delta = (\bar{Z}^* \mathcal{S} \bar{Y})^{-1} \). Then \( \Delta \) is constant on \([a, b]\) and \( \mathcal{B}^* \) is the collection of functions \( z \) in \( \mathfrak{A}_m \) satisfying (4.3) if and only if the \( (2m + 2n - k) \times 2(m + n) \) matrix \( [RS] \) has rank \( 2m + 2n - k \) and

\[
(4.4) \quad s(\bar{Y})\Delta(t^{-}(\bar{Z}))^* + s^{-}(\bar{Y})\Delta(t(\bar{Z}))^* = 0.
\]

**THEOREM 4.5.** Suppose that \( m = n \), \( [p_{ij}(x)] \), \( (i, j = 0, \ldots, n; x \in [a, b]) \), is Hermitian and \( \text{dim} \mathcal{B} = 2n \). Let \( y_j \), \( (j = 1, \ldots, 2n) \), be linearly independent solutions of \( L[y] = 0 \), and let \( \Delta = (\bar{Y}^* \mathcal{S} \bar{Y})^{-1} \). Then \( \Delta \) is constant on \([a, b]\), and \( T \) is self-adjoint if and only if the \( 2n \times 2n \) matrix \( s^{-}(\bar{Y})\Delta(s(\bar{Y}))^* \) is Hermitian.

5. Generalized Green's functions. The subspaces \( \mathcal{D}, \mathcal{D}^* \) of \( \mathfrak{A}_n \) and \( \mathfrak{A}_m \), respectively, and the subspaces \( \mathcal{B}, \mathcal{B}^* \) of \( 2(m + n) \)-dimensional complex space are as defined in § 4. If \( 0 < \text{dim} \mathcal{B} < 2m + 2n \), then the matrices \( M, N, P, \) and \( Q \) are as specified in § 4.

If \( f \in \mathfrak{A}_0 \) then we are concerned with solutions of the quasi-differential system

\[
(5.1) \quad L[y] = f, \quad y \in \mathcal{D}.
\]

Of prime importance is the homogeneous system

\[
(5.2) \quad L[y] = 0 \quad y \in \mathcal{D},
\]

and its adjoint system

\[
(5.3) \quad L^*[z] = 0 \quad z \in \mathcal{D}^*.
\]

By definition a generalized Green's function for the system (5.2) is an essentially bounded and measurable function \( g \) on \( \square = \{(x, t) : a \leq x \leq b, a \leq t \leq b\} \) with the property that if \( f \) is a function in \( \mathfrak{A}_0 \) for which (5.1) has a solution, then a particular solution \( y \)
of (5.1) is given by

\begin{equation}
(5.4) \quad y(x) = \int_a^b g(x, t)f(t)dt .
\end{equation}

Reid [5] has shown the existence of a generalized Green’s matrix for a compatible first order system with two-point boundary conditions, where the number of independent boundary conditions is equal to the number of differential equations. If \( \text{dim} \mathcal{B} = m + n \), then Reid’s results could be used to obtain a generalized Green’s function for (5.2). In this section the existence and some properties of a generalized Green’s function will be shown when \( \text{dim} \mathcal{B} \) is not necessarily equal to \( m + n \). The technique used here may be modified to extend Reid’s results to the case where the number of independent boundary conditions is different from the number of differential equations.

For a \( \nu \)th order linear differential operator \( \sum_{j=0}^{\nu} q_j(x)y^{(j)} \) with \( q_j \in C_2 \), \( (j = 0, 1, \cdots, \nu) \), and \( q_\nu(x) \neq 0 \), the generalized Green’s function has been treated by Greub and Rheinboldt [4] and Wyler [10]; a more comprehensive treatment of an algebraic theory of operator solutions of boundary problems, which includes this case as a special instance, is given in Wyler [11].

**Lemma 5.1.** If \( y_j \), \( (j = 1, \cdots, m + n) \), are linearly independent solutions of \( L[y] = 0 \), then there exist \( m + n \) linearly independent solutions \( z_j \) of \( L^*[z] = 0 \) such that

\begin{equation}
(5.5) \quad \tilde{Z}^* \mathcal{L} \tilde{Y} = E_{m+n} .
\end{equation}

This result follows from Lemma 3.1 and the existence and uniqueness theorems for the equations \( \mathcal{L}[y] = 0 \) and \( \mathcal{L}^*[z] = 0 \).

If \( y_j \in \tilde{A}_n \) and \( z_j \in \tilde{A}_m \), \( (j = 1, \cdots, m + n) \), then define matrix functions \( \tilde{Y} \), \( \tilde{Y} \), \( \tilde{Z} \), and \( \tilde{Z} \) as follows: \( \tilde{Y}(x) = [\tilde{y}_j(x)] \), \( \tilde{Y}(x) = [\tilde{y}_j(x)] \), \( \tilde{Z}(x) = [\tilde{z}_j(x)] \), and \( \tilde{Z}(x) = [\tilde{z}_j(x)] \), \( (j = 1, \cdots, m + n) \).

**Corollary.** If \( y_j \) and \( z_j \), \( (j = 1, \cdots, m + n) \), are as in Lemma 5.1, then

\begin{equation}
(5.6) \quad \tilde{Y}(x)\tilde{Z}^*(x) = 0_{nm} , \quad \tilde{Y}(x)\tilde{Z}^*(x) = E_n , \quad \tilde{Y}(x)\tilde{Z}^*(x) = -E_m , \quad \tilde{Y}(x)\tilde{Z}^*(x) = 0_{mn} .
\end{equation}

**Theorem 5.1.** If \( \tau \in [a, b] \), \( \xi_j \) is a constant, \( y \), and \( z_j \), \( (j = 1, \cdots, m + n) \), are as in Lemma 5.1, then the solution \( y \) of \( L[y] = f \) satisfying \( \tilde{y}(\tau) = \sum_{j=1}^{m+n} \tilde{y}_j(\tau)\xi_j \) is given by the first component of the vector
Indeed, if \( \xi = (\xi_j) \), \((j = 1, \ldots, m + n)\), and we set \( \tilde{y}(x) = \tilde{Y}(x)u(x) \), for \( u \) an \((m + n)\)-vector function, then \( \tilde{y} \) is a solution of \( \mathcal{L}[\tilde{y}] = fe^{(m+n,1)} \), \( \tilde{y}(\tau) = \tilde{Y}(\tau)\xi \) if and only if

\[
\mathcal{L} \tilde{Y}(x)u'(x) = e^{(m+n,1)} f(x), \quad u(\tau) = \xi.
\]

Hence \( u'(x) = \tilde{Z}^*(x)e^{(m+n,1)} f(x) \) and

\[
u(x) = \xi + \int_{\tau}^{x} \tilde{Z}^*(s)e^{(m+n,1)} f(s) ds,
\]

from which the theorem follows.

Now suppose that \( y_j \), \((j = 1, \ldots, m + n)\), are linearly independent solutions of \( L[y] = 0 \) and that \( z_j \), \((j = 1, \ldots, m + n)\), are chosen as in Lemma 5.1. If \( \dim \mathcal{B} = 2m + 2n - k \), \( k > 0 \), then \( s(\tilde{Y}) \) and \( s^- (\tilde{Y}) \) are \( k \times (m + n) \) matrices defined as \( s(\tilde{Y}) = M\tilde{Y}(a) + N\tilde{Y}(b) \) and \( s^- (\tilde{Y}) = M\tilde{Y}(a) - N\tilde{Y}(b) \). If \( r \) is the index of compatibility for (5.2), then \( s(\tilde{Y}) \) has rank \( m + n - r \). If \( r > 0 \), then let \( S \) be an \((m + n) \times r \) matrix with the property that \( S^* S = E_r \) and \( s(\tilde{Y}) S = 0 \). If \( r > m + n - k \), then \( T \) will represent a \( k \times (k - m - n + r) \) matrix such that \( T^* T = E_{k-m-n+r} \) and \( T^* s(\tilde{Y}) = 0 \). It follows that the \((k + r) \times (k + r)\) matrix

\[
(5.8)
\begin{bmatrix}
    s(\tilde{Y}) & T \\
    S^* & 0
\end{bmatrix}
\]

is nonsingular, and its inverse is of the form

\[
(5.9)
\begin{bmatrix}
    D & S \\
    T^* & 0
\end{bmatrix}.
\]

The \((m + n) \times k\) matrix \( D \) is the generalized reciprocal of \( s(\tilde{Y}) \) in the sense of E. H. Moore, (see [9, Section 14]). If \( r = 0 \), then the matrix \( S \) does not appear, if \( r = m + n - k \), then \( T \) does not appear.

Now if \( \dim \mathcal{B} < 2(m + n) \), let \( G(x, t) \) be the \((m + n) \times (m + n)\) matrix defined by

\[
G(x, t) = \frac{1}{2} \tilde{Y}(x) \left[ \frac{|x - t|}{x - t} E_{m+n} + Ds^-(\tilde{Y}) \right] \tilde{Z}^*(t), \quad x \neq t;
\]

\[
G(x, x) = \frac{1}{2} \tilde{Y}(x)Ds^-(\tilde{Y})\tilde{Z}^*(x), \quad x \in [a, b].
\]

If \( \dim \mathcal{B} = 2(m + n) \), let \( G(x, t) \) be defined by
\[ G(x, t) = \frac{1}{2} \frac{|x - t|}{x - t} \tilde{X}(x) \tilde{Z}^*(t), \quad x \neq t; \]

\[ G(x, x) = 0, \quad x \in [a, b]. \]

Let \( g_0 \) be the function with domain \( [a, b] \) whose value at \((x, t)\) is the element in the first row and first column of \( G(x, t) \), that is

\[
g_0(x, t) = g_{0,1}(x, t) + g_{0,2}(x, t) \quad \text{if } \dim \mathcal{B} < 2(m + n),
\]

\[
g_0(x, t) = g_{0,1}(x, t) \quad \text{if } \dim \mathcal{B} = 2(m + n),
\]

where

\[
g_{0,1}(x, t) = \frac{1}{2} \operatorname{sgn}(x - t) \sum_{i=1}^{m+n} y_i(x) \tilde{z}_i(t),
\]

\[
g_{0,2}(x, t) = \frac{1}{2} \sum_{i,j=0}^{m+n} y_i(x) \mathcal{X}_{ij} \tilde{x}_j(t),
\]

provided \([\mathcal{X}_{ij}]\) is the matrix \( Ds^{-1}(\tilde{Y}) \) and \( \operatorname{sgn} u = |u|/u \) for \( u \neq 0 \), \( \operatorname{sgn} 0 = 0 \).

**Theorem 5.2.** The function \( g_0 \) defined above is a generalized Green's function for (5.2).

If \( \dim \mathcal{B} = 2(m + n) \), then this result follows directly from Theorem 5.1. Now suppose that \( \dim \mathcal{B} < 2(m + n) \), and \( f \) is an integrable function for which (5.1) has a solution. If \( y \) is a solution of \( L[y] = f \), then for a suitable vector \( \xi \) one has

\[
\bar{y}(x) = \frac{1}{2} \left[ \bar{Y}(x)\xi + \int_s^b \bar{Y}(x) \bar{Z}^*(t)e^{(m+n,1)}f(t)dt - \int_s^b \bar{Y}(x) \bar{Z}^*(t)e^{(m+n,1)}f(t)dt \right].
\]

Thus, since (5.9) is the inverse of (5.8), it follows that \( y \) is a solution of (5.1) if and only if

\[
T^{*s^{-1}}(\bar{Y}) \int_s^b \bar{Z}^*(t)e^{(m+n,1)}f(t)dt = 0,
\]

and for some \( r \)-vector \( \eta \) we have

\[
\xi = Ds^{-1}(\bar{Y}) \int_s^b \bar{Z}^*(t)e^{(m+n,1)}f(t)dt + S\eta.
\]

Therefore,

\[
\bar{y}(x) = \frac{1}{2} \left[ \bar{Y}(x)S\eta + \bar{Y}(x)Ds^{-1}(\bar{Y}) \int_s^b \bar{Z}^*(t)e^{(m+n,1)}f(t)dt \right.
\]

\[
+ \int_s^b \bar{Y}(x) \frac{|x - t|}{x - t} \bar{Z}^*(t)e^{(m+n,1)}f(t)dt \bigg],
\]
from which the theorem follows since \( \eta \) may be chosen to be zero.

The symbol \( g_0^{(i,j)} \) will be used to signify the partial derivative \( \partial^{i+j} g \big/ \partial t^i \partial x^j \). Generalized partial derivatives \( g_0^{(\alpha,\beta)} \) will now be defined for \( g_0 \). If \( \alpha < n \) and \( \beta < m \), then \( g_0^{(\alpha,\beta)}(x, t) = g_0^{(\alpha,\beta)}(x, t) \). If \( \alpha < n \), then \( g_0^{(\alpha,\beta + j)} \) (\( j = 0, \ldots, n - 1 \)), is defined as follows:

\[
g_0^{(\alpha,\beta + j)}(x, t) = \sum_{i=0}^{m} \bar{p}_i(t) g_0^{(\alpha,\beta + i)}(x, t)
\]

if \( g_0^{(\alpha,\beta + j)} \) is a.c. in its second argument, then

\[
g_0^{(\alpha,\beta + j)}(x, t) = \sum_{i=0}^{m} p_i(t) g_0^{(\alpha,\beta + i)}(x, t) - \delta \partial g_0^{(\alpha,\beta + j)}(x, t)
\]

\( (j = 1, \ldots, n - 1) \).

If \( \beta < m \), then \( g_0^{(\alpha,\beta + j)} \) (\( i = 0, \ldots, m - 1 \)), is defined as follows:

\[
g_0^{(\alpha,\beta + j)}(x, t) = \sum_{j=0}^{n} p_{m-j}(x) g_0^{(\alpha,\beta + j)}(x, t)
\]

if \( g_0^{(\alpha,\beta + j)} \) is a.c. in its first argument, then

\[
g_0^{(\alpha,\beta + j)}(x, t) = \sum_{j=1}^{n} p_{m-j}(x) g_0^{(\alpha,\beta + j)}(x, t) - \partial g_0^{(\alpha,\beta + j)}(x, t)
\]

\( (i = 1, \ldots, m - 1) \).

**Theorem 5.3.** If \( \alpha + \beta \leq m + n - 2 \), and \( g \) is the function of Theorem 5.2, then \( g_0^{(\alpha,\beta)} \) exists and is continuous on \( \square \).

This result clearly holds for \( g_{0,2} \), hence one need only consider specifically \( g_{0,1} \). Let \( \alpha + \beta \leq m + n - 2 \), and suppose first that \( \alpha < n \). If \( \beta < m \), then the theorem follows from the fact that \( \tilde{Y}(x)\tilde{Z}^*(x) = 0 \). If \( \beta = m - 1 + j \) (\( j = 1, \ldots, n - \alpha - 1 \)), then use the identity \( \tilde{Y}(x)\tilde{Z}^*(x) = E_m \). On the other hand, if \( \beta < m \) and \( \alpha = n - 1 + i \) (\( i = 1, \ldots, m - \beta - 1 \)), then use the identity \( \tilde{Y}(x)\tilde{Z}^*(x) = -E_m \).

**Theorem 5.4.** The generalized Green's function for the system (5.2) is not unique. If \( u_1, \ldots, u_r \) form a basis for the solutions of (5.2), \( v_1, \ldots, v_p \) form a basis for the solutions of (5.3), and \( g_0 \) is one generalized Green's function for (5.2) then a function \( g \) on \( \square \) is also a generalized Green's function for (5.2) if and only if there exist essentially bounded and measurable functions \( \psi_1, \ldots, \psi_r, \Phi_1, \ldots, \Phi_p \) such that if \( (x, t) \in \square \), then

\[
g(x, t) = g_0(x, t) + \sum_{i=1}^{r} u_i(x) \psi_i(t) + \sum_{j=1}^{p} \Phi_j(x) \Phi_j(t)
\]

If \( g \) is a function on \( \square \) satisfying (5.10), then in view of Theorem
it follows that $g$ is a generalized Green's function for (5.2).

To establish the converse we may assume without loss of generality that $(u_i, u_j) = \delta_{ij}, (i, j = 1, \cdots, r)$, and $(v_\alpha, v_\beta) = \delta_{\alpha\beta}, (\alpha, \beta = 1, \cdots, \rho)$. If $w \in H$ and $f(x) = w(x) - \sum_{j=1}^{r} (w, v_j)v_j(x)$, then $(f, v_\alpha) = 0, (\alpha = 1, \cdots, \rho)$. Thus for this choice of $f$ it follows from Theorem 4.3 that (5.1) has a solution. Suppose that $g$ is a second generalized Green’s function for (5.2) and let $d(x, t) = g(x, t) - g_0(x, t)$. Then there are constants $\xi_1, \cdots, \xi_r$ such that

$$\int_a^b d(x, t)f(t)dt = \sum_{i=1}^{r} u_i(x)\xi_i ,$$

and if $\Phi(x, t) = d(x, t) - \sum_{i=1}^{r} \bar{v}_i(t)\int_a^b d(x, s)v_i(s)ds$, then

$$\int_a^b \Phi(x, t)f(t)dt = \sum_{i=1}^{r} u_i(x)\xi_i .$$

Multiplying (5.11) by $\bar{u}_i(x)$, and integrating with respect to $x$, we have

$$\int_a^b \int_a^b \bar{u}_i(x)\Phi(x, t)f(t)dt dx = \xi_i , \quad (i = 1, \cdots, r) ,$$

and consequently

$$\int_a^b \left[ \Phi(x, t) - \sum_{i=1}^{r} u_i(x)\int_a^b \bar{u}_i(s)\Phi(s, t)ds \right] w(t)dt = 0 .$$

But $w$ is an arbitrary integrable function, and hence

$$\Phi(x, t) - \sum_{i=1}^{r} u_i(x)\int_a^b \bar{u}_i(s)\Phi(s, t)ds = 0 \quad \text{on } \square ,$$

and

$$d(x, t) = \sum_{i=1}^{r} u_i(x)\int_a^b \bar{u}_i(s)\Phi(s, t)ds + \sum_{j=1}^{\rho} \bar{v}_j(t)\int_a^b d(x, s)v_j(s)ds .$$

Hence (5.10) holds with $\psi_i$ and $\varphi_j$ defined by $\psi_i(t) = \int_a^b u_i(s)\varphi(s, t)ds$, and $\varphi_j(x) = \int_a^b d(x, s)v_j(s)ds$, $(i = 1, \cdots, r; j = 1, \cdots, \rho)$, and clearly these functions are essentially bounded and measurable.

We now show that a generalized Green's function $g$ for (5.2) has the property that the function $h$ defined by $h(x, t) = \bar{g}(t, x)$ is a generalized Green's function for the adjoint system (5.3). Preliminary to this result we shall prove the following theorem.

**Theorem 5.5.*** Suppose that $u_1, \cdots, u_r$ form a basis for the solutions of (5.2), $v_1, \cdots, v_\rho$ from a basis for the solutions of (5.3), and $\Theta = \{\theta_1, \cdots, \theta_r\}$, $\Omega = \{\omega_1, \cdots, \omega_\rho\}$ are sets of integrable functions
with the property that the matrices \([u_{ij}, \theta_j]\), \((i, j = 1, \ldots, r)\), and \([v_{\alpha}, \omega_{\beta}]\), \((\alpha, \beta = 1, \ldots, \rho)\), are nonsingular. Then there exists a unique generalized Green's function \(g_{\alpha}(x; \theta, \Omega)\) for (5.2) satisfying the conditions

\[
\int_a^b g_{\alpha}(x, t; \theta, \Omega) \omega_{\alpha}(t) dt = 0, \quad (\alpha = 1, \ldots, \rho),
\]

\[
\int_a^b \bar{\varphi}_i(x) g_{\alpha}(x, t; \theta, \Omega) dx = 0, \quad (\alpha = 1, \ldots, r).
\]

Without any loss of generality we can assume that \([u_{ij}, \theta_j] = E_r\) and \([v_{\alpha}, \omega_{\beta}] = E_\rho\). Let \(g_{\alpha}\) be the generalized Green's function for (5.2) described in Theorem 5.2. We now determine functions \(\psi_1, \ldots, \psi_r\) and functions \(\phi_1, \ldots, \phi_\rho\) such that the generalized Green's function given by (5.10) satisfies conditions (5.12). Such a generalized Green's function \(g\) will satisfy the conditions (5.12) if and only if the functions \(\psi_i, (i = 1, \ldots, r)\), and \(\phi_\alpha, (\alpha = 1, \ldots, \rho)\), satisfy the equations

\[
\psi_i(x) + \int_a^b \sum_{\beta=1}^\rho \bar{\varphi}_i(s) \phi_\beta(s) v_\beta(x) ds + \int_a^b \bar{\varphi}_i(s) g_{\alpha}(s, x) ds = 0, \quad (i = 1, \ldots, r),
\]

\[
\phi_\alpha(x) + \int_a^b \sum_{i=1}^r u_i(x) \psi_i(s) \omega_{\alpha}(s) ds + \int_a^b g_{\alpha}(x, s) \omega_{\alpha}(s) ds = 0, \quad (\alpha = 1, \ldots, \rho).
\]

A particular set of solutions for equations (5.13) is

\[
\phi_\alpha(x) = -\int_a^b g_{\alpha}(x, s) \omega_{\alpha}(s) ds, \quad (\alpha = 1, \ldots, \rho),
\]

\[
\psi_i(x) = \int_a^b \int_a^b \sum_{\beta=1}^\rho \bar{\varphi}_i(t) g_{\beta}(t, s) \omega_\beta(s) \bar{v}_\beta(x) ds dt
\]

\[-\int_a^b \bar{\varphi}_i(t) g_{\alpha}(t, x) dt, \quad (i = 1, \ldots, r).
\]

Moreover, if \(\psi_i\) and \(\phi_\alpha\), \((i = 1, \ldots, r; \alpha = 1, \ldots, \rho)\), is any collection of solutions of (5.13), then after substituting the value of \(\psi_i(x)\) given by the first equation into the second equation of (5.13) it can be shown by straightforward computation that the value of

\[
\sum_{i=1}^r u_i(x) \psi_i(t) + \sum_{\alpha=1}^\rho \phi_\alpha(x) \bar{v}_\alpha(t)
\]

is independent of the particular \(\psi_i\) and \(\phi_\alpha\). Hence there is a unique generalized Green's function for (5.2) satisfying (5.12).

The conditions of Theorem 5.5 are clearly satisfied by the sets \(\theta_i = u_i, (i = 1, \ldots, r)\), and \(\omega_\alpha = v_\alpha, (\alpha = 1, \ldots, \rho)\); in particular, for linear homogeneous differential operators whose coefficients satisfy
suitable differentiability conditions, the treatment of Greub and Rheinboldt [4] is limited to this specification.

It is to be remarked that, in view of the definition of \( g_0 \), if \( \psi_i \) and \( \varphi_{\alpha} \), \( (i = 1, \ldots, r; \alpha = 1, \ldots, \rho) \), is any collection of solutions of (5.13), then \( \varphi_{\alpha} \in \mathcal{H}_n \), \( (\alpha = 1, \ldots, \rho) \), and \( \psi_i \in \mathcal{H}_m \), \( (i = 1, \ldots, r) \).

Correspondingly, there exists a unique generalized Green’s function \( g_L(\cdot, \cdot, \partial, \Omega) \) for the system (5.3) which satisfies the conditions

\[
\begin{align*}
\int_a^b \bar{\omega}_\alpha(x)g_L(\cdot, t; \Omega, \theta)dx &= 0, \quad (\alpha = 1, \ldots, \rho), \\
\int_a^b g_L(\cdot, t; \Omega, \theta)\partial_i(t)dt &= 0, \quad (i = 1, \ldots, r).
\end{align*}
\] (5.15)

For brevity, denote by \( b_\theta \) and \( b_\theta \) the functions defined on \( \square \) by the formulas

\[
b_\theta(x, t) = \sum_{i=1}^r \omega_i(x)v_i(t), \quad b_\theta(x, t) = \sum_{i=1}^r \theta_i(x)\bar{u}_i(t).
\]

**Theorem 5.6.** If \( g_L(\cdot, \cdot, \theta, \Omega) \) is the unique generalized Green’s function satisfying (5.12), then the following conditions (5.16)–(5.20) are satisfied:

(5.16) \( g_L^{(\partial, \omega)}(\cdot, \theta, \Omega), (j = 0, \ldots, m + n - 2), \) exists and is continuous on \( \square \) while \( g_L^{(m+n-1, \partial)}(x, t; \theta, \Omega) \) and \( \partial/\partial x g_L^{(m+n-1, \partial)}(x, t; \theta, \Omega) \) exist on the individual domains \( a \leq t < x, a < x < b \) and \( a \leq x < b, x < t \leq b; \)

(5.17) if \( t \in [a, b] \), then the function whose value at \( x \neq t \) is \( g_L^{(m+n-1, \partial)}(t+, t; \theta, \Omega) \) has a right and a left limit at \( t \), denoted by \( g_L^{(m+n-1, \partial)}(t+, t; \theta, \Omega) \) and \( g_L^{(m+n-1, \partial)}(t-, t; \theta, \Omega) \), respectively, and

\[
g_L^{(m+n-1, \partial)}(t-, t; \theta, \Omega) - g_L^{(m+n-1, \partial)}(t+, t; \theta, \Omega) = 1;
\]

(5.18) if \( t \in [a, b] \), then \( L[g_L(t, \theta, \Omega)] = b\Omega(t, t) \) on \( [a, t) \) and \( (t, b] \);

(5.19) if \( t \in (a, b) \), then the function whose value at \( x \) is \( g_L(x, t; \theta, \Omega) \) satisfies the boundary conditions which characterize the set \( \mathcal{D} \);

(5.20) \[
\int_a^b \bar{\psi}_i(x)g_L(x, t; \Omega, \theta)dx = 0, \quad (i = 1, \ldots, r; \ t \in [a, b]).
\]

Conditions (5.16)–(5.18) may be verified directly using the properties of \( g_L \) and the remark following the proof of Theorem 5.5. Condition (5.20) is merely one of the conditions in (5.12). If \( \mathcal{D} = \mathcal{H}_n \), then (5.19) is trivially satisfied. Otherwise, let \( w \) be any integrable function, and define \( f \) by

\[
f(x) = w(x) - \sum_{\alpha=1}^\rho \omega_\alpha(x)w_\alpha(x) = w(x) - \int_a^b b_\theta(x, t)w(t)dt.
\]
In view of the assumption that \([(v_\alpha, \phi_\beta)] = E_\rho\), it follows that \((f, v_\alpha) = 0\), \((\alpha = 1, \cdots, \rho)\), and therefore the function \(u\) defined by

\[
u(x) = \int_a^b g_L(x, t; \Theta, \Omega)f(t)dt
\]
is a solution of (5.1). But it follows from (5.12) that

\[
\int_a^b g_L(x, t, \Theta, \Omega)f(t)dt = \int_a^b g_L(x, t; \Theta, \Omega)w(t)dt.
\]

Therefore,

\[
0 = M\tilde{u}(a) + N\tilde{u}(b)
= \int_a^b (Mg_L(a, t; \Theta, \Omega) + Ng_L(b, t; \Theta, \Omega))w(t)dt,
\]
from which (5.19) follows in view of the arbitrariness of the function \(w\).

**COROLLARY.** If \(w \in \mathfrak{A}_0\) and \(y\) is defined by

\[
y(x) = \int_a^b g_L(x, t; \Theta, \Omega)w(t)dt,
\]
then

\[
L[y] = w - \int_a^b b_\alpha(, t)w(t)dt,
\]

\[y \in \mathfrak{D}, \quad (y, \theta_i) = 0, \quad (i = 1, \cdots, r).
\]

It should be noted that the unique generalized Green's function \(g_L^*(, ; \Theta, \Omega)\) for (5.3) which satisfies (5.15) also satisfies conditions analogous to (5.16)-(5.20).

**THEOREM 5.7.** If \(x, t \in [a, b]\), then \(g_L(x, t; \Theta, \Omega) = g_L(t, x; \Theta, \Omega)\).

Let \(w\) and \(h\) be arbitrary integrable functions and define \(y\) and \(z\) by

\[
y(x) = \int_a^b g_L(x, t; \Theta, \Omega)w(t)dt,
\]

\[
z(x) = \int_a^b g_L^*(x, t; \Theta, \Omega)h(t)dt,
\]
respectively. Then it follows from the corollary to Theorem 5.6 and its analogue that \(y \in \mathfrak{D}, z \in \mathfrak{D}^*\), and therefore

\[
(L[y], z) - (y, L^*[z]) = 0.
\]

But it also follows from the corollary to Theorem 5.6 that \(L[y] = \)
\[ w - \int_a^b b_\omega(s, t)w(t)dt, \quad L^\omega[z] = h - \int_a^b b_\omega(s, t)h(t)dt, \text{ and therefore in view of (5.12), (5.15), and the definition of } b_\omega \text{ and } b_\nu, \] we have

\[ \int_a^b \int_a^b \bar{h}(x)[\bar{g}_{L^\nu}(t, x; \Omega, \Theta) - g_L(x, t; \Theta, \Omega)]w(t)dtdx = 0, \]

from which the theorem follows since \( w \) and \( h \) are arbitrary integrable functions.

**COROLLARY I.** The function \( g_L(, ; \Theta, \Omega) \) is characterized by conditions (5.16)–(5.20), and the function \( g_{L^\nu}(, ; \Theta, \Omega) \) is characterized by analogous conditions.

As a consequence of Theorems 5.4 and 5.7 one has the following result:

**COROLLARY II.** If \( g \) is a generalized Green's function for (5.2), then the function \( h \) defined by \( h(x, t) = g(t, x) \) is a generalized Green's function for (5.3).

6. A canonical form for boundary conditions. Let \( [f_{ij}] \) and \( [g_{ij}] \), \((i = 0, \ldots, m; j = 0, \ldots, n) \), be \((m+1) \times (n+1)\) integrable matrix functions. Suppose that the matrix function \([f_{ij}]\), \((i = 0, \ldots, m; j = 0, \ldots, n)\), satisfies hypothesis \((H)\), and \( g_m(x) = g_m(x) = 0 \), \((i = 0, \ldots, m; j = 0, \ldots, n)\).

For a complex number \( \lambda \) let \( p_{ij}(; \lambda) \) be the function defined on \([a, b]\) by

\[ p_{ij}(x; \lambda) = f_{ij}(x) + \lambda g_{ij}(x), \quad (i = 0, \ldots, m; j = 0, \ldots, n). \]

It follows that for each number \( \lambda \) hypothesis \((H)\) holds for the matrix function \([p_{ij}(; \lambda)]\). For suitable \( y \) in \( \mathcal{U}_n \) let \( \bar{y}_i(, \lambda), \ldots, \bar{y}_m(, \lambda) \) be defined on \([a, b]\) as follows:

\[ \bar{y}_n(x; \lambda) = \sum_{j=0}^{\nu} p_m(x; \lambda)y^{(j)}(x) = \sum_{j=0}^{\nu} f_{mj}(x)y^{(j)}(x); \]

\[ \text{if } \bar{y}_i(, \lambda) \in \mathcal{U}_i, \text{ then } \bar{y}_i(x; \lambda) = \sum_{j=0}^{\nu} p_{ij}(x; \lambda)y^{(j)}(x) - \bar{y}_{i+1}(x; \lambda), \]

\[ (i = m - 1, \ldots, 1). \]

The class of functions \( y \) in \( \mathcal{U}_n \) for which \( \bar{y}_1(, \lambda), \ldots, \bar{y}_m(, \lambda) \) are a.c. will be denoted by \( \mathcal{A}_n(\lambda) \), and \( L[; \lambda] \) will be the operator with domain \( \mathcal{A}_n(\lambda) \), and defined by

\[ L[y; \lambda] = \sum_{j=0}^{\nu} p_{ij}(, \lambda)y^{(j)} - \bar{y}_1(, \lambda). \]
The vector function \( \tilde{y}(; \lambda) \), \( (i = 1, \cdots, m) \), will be represented by \( \tilde{y}(; \lambda) \), and \( \tilde{y}(; \lambda) \) will signify the \((n + m)\)-vector function \( (y, \cdots, y^{(n-1)}, \tilde{y}_1(\lambda), \cdots, \tilde{y}_m(\lambda)) \). For a complex number \( \nu \) let \( p^\nu_i(\cdot; \nu) \) be the function on \([a, b]\) defined by

\[
p^\nu_i(x; \nu) = \tilde{f}_i(x) + \nu \tilde{g}_i(x), \quad (i = 0, \cdots, m; j = 0, \cdots, n).
\]

For suitable \( z \) in \( \mathcal{A}_m \) define \( \tilde{z}_i(\cdot; \nu), \cdots, \tilde{z}_n(\cdot; \nu) \) by

\[
\tilde{z}_n(x; \nu) = \sum_{i=1}^m p^\nu_i(x; \nu)z^{(i)}(x) = \sum_{i=1}^m \tilde{f}_i(x)z^{(i)}(x); \\
\text{if } \tilde{z}_{j+1}(\cdot; \nu) \in \mathcal{A}_i, \text{ then } \tilde{z}_j(x; \nu) = \sum_{i=1}^m p^\nu_i(x; \nu)z^{(i)}(x) - \tilde{z}_{j+1}(x; \nu);
\]

\( (j = n - 1, \cdots, 1) \).

The class of functions \( z \) in \( \mathcal{A}_m \) for which \( \tilde{z}_i(\cdot; \nu), \cdots, \tilde{z}_n(\cdot; \nu) \) are a.c. will be denoted by \( \tilde{\mathcal{A}}(\nu) \) and \( \mathcal{L}^\nu[z; \nu] \) will be operator with domain \( \tilde{\mathcal{A}}(\nu) \), and defined by

\[
\mathcal{L}^\nu[z; \nu] = \sum_{i=1}^m p^\nu_i(\cdot; \nu)z^{(i)} - \tilde{z}(\cdot; \nu).
\]

The vector function \( \tilde{z}_j(\cdot; \nu), (j = 1, \cdots, n) \), will be represented by \( \tilde{z}(\cdot; \nu) \), and \( \tilde{z}(\cdot; \nu) \) will denote the vector function \( (z, \cdots, z^{(m-1)}, \tilde{z}_1(\cdot; \nu), \cdots, \tilde{z}_n(\cdot; \nu)) \). Let \( A_{10}, A_{11}, A_{20}, \) and \( A_{21} \) be \( k \times n \) matrices, and let \( B_1 \) and \( B_2 \) be \( k \times m \) matrices, \( (1 \leq k \leq 2m + 2n - 1) \), such that for each number \( \lambda \) the \( k \times 2(m + n) \) matrix

\[
[A_1(\lambda) - B_1 A_2(\lambda) B_2]
\]

has rank \( k \), where \( A_1(\lambda) = A_{10} + \lambda A_{11} \) and \( A_2(\lambda) = A_{20} + \lambda A_{21} \). Let \( \mathcal{D}(\lambda) \) be the collection of functions \( y \) in \( \tilde{\mathcal{A}}(\lambda) \) for which

\[
A_1(\lambda)\tilde{y}(a) - B_1\tilde{y}(a; \lambda) + A_2(\lambda)\tilde{y}(b) + B_2\tilde{y}(b; \lambda) = 0.
\]

This section is concerned with the particular Euler type quasi-differential system

\[
L[y; \lambda] = 0, \quad y \in \mathcal{D}(\lambda).
\]

It follows from Theorem 3.2 that the system adjoint to (6.6) is

\[
\mathcal{L}^\nu[z; \lambda] = 0, \quad z \in \mathcal{D}^*(\lambda),
\]

where \( \mathcal{D}^*(\lambda) \subset \tilde{\mathcal{A}}(\lambda) \). The following assumption is made about \( \mathcal{D}^*(\lambda) \):

**Hypoththesis (H).** There exist \((2m + 2n - k) \times m\) matrices \( A_2(\nu) = A_{20} + \nu A_{21}, A_1(\nu) = A_{10} + \nu A_{11} \) and \((2m + 2n - k) \times n\) matrices \( B_3, B_4 \) such that for arbitrary \( \lambda \) the set \( \mathcal{D}^*(\lambda) \) is the collection of function \( z \) in \( \tilde{\mathcal{A}}_m(\lambda) \) for which
\[(6.8) \quad A_3(\lambda)\dot{z}(a) - B_3\dot{z}(a; \lambda) + A_4(\lambda)\dot{z}(b) + B_4\dot{z}(b; \lambda) = 0.\]

It should be noted that the assumption used by Zimmerberg to obtain Theorem 2.1 of [13] does not imply that hypothesis (H_i) holds. For if \(m = n = 1\) and \(k = 2n\), then let the matrices \(A_{10}, A_{11}, B_1, A_{20}, A_{21}, B_2\) be defined as

\[
\begin{align*}
A_{10} &= [1 \ 1], & A_{11} &= [0 \ 1], & B_1^* &= [2 \ 1],
A_{20} &= [1 \ 0], & A_{21} &= [0 \ 1], & B_2^* &= [0 \ 1].
\end{align*}
\]

Then the hypothesis of Theorem 2.1 of [13] is satisfied, but hypothesis (H_i) does not hold.

If hypothesis (H_i) holds then for each complex number \(\nu\) the \((2m + 2n - k) \times 2(m + n)\) matrix

\[(6.9) \quad [A_{\nu}(\nu) B_{\nu}] \]

has rank \(2m + 2n - k\). Moreover, by a proof quite analogous to that used by Reid to obtain (11.11') of [6] one may establish the following result.

**Lemma 6.1.** If hypothesis (H_i) holds, then \(\mathcal{D}(\lambda)\) is the collection of functions \(y\) in \(\mathcal{A}_w(\lambda)\) for which there is a \((2m + 2n - k)\)-vector \(e_0\) such that

\[
\begin{align*}
\dot{y}(a) &= B_2^* e_0, & \ddot{y}(a; \lambda) &= A_3^*(\lambda)e_0, \\
\dot{y}(b) &= B_1^* e_0, & \ddot{y}(b; \lambda) &= -A_4^*(\lambda)e_0,
\end{align*}
\]

and \(\mathcal{D}^*(\lambda)\) is the collection of functions \(z\) in \(\mathcal{A}_m(\lambda)\) for which there is a \(k\)-vector \(e_1\) such that

\[
\begin{align*}
\dot{z}(a) &= B_1^* e_1, & \ddot{z}(a; \lambda) &= A_1^*(\lambda)e_1, \\
\dot{z}(b) &= B_2^* e_1, & \ddot{z}(b; \lambda) &= -A_2^*(\lambda)e_1,
\end{align*}
\]

where \(A_i^*(\nu) = (A_i(\nu))^*\), \((i = 1, 2, 3, 4)\).

Now let \(K_{10} = A_{10}B_3^* + A_{20}B_2^*, \ K_{11} = A_{11}B_3^* + A_{21}B_2^*, \ K_1(\lambda) = K_{10} + \lambda K_{11}, \ K_{20} = A_{20}B_1^* + A_{30}B_2^*, \ K_{21} = A_{21}B_1^* + A_{31}B_2^*, \) and \(K_2(\lambda) = K_{20} + \lambda K_{21}\). Then the next result follows from Lemma 6.1 and Lemma 3.1.

**Lemma 6.2.** If hypothesis (H_i) holds, then \(K_2^*(\lambda) = K_2(\lambda)\).

**Lemma 6.3.** Suppose that hypothesis (H_i) holds, the \(k \times 2m\) matrix \([B_1 B_2]\) has rank \(k - p\), and the \((2m + 2n - k) \times 2n\) matrix \([B_3 B_4]\) has rank \(2m + 2n - k - q\). Then there exist \(p \times n\) matrices \(\psi_1, \psi_2\) and \(q \times m\) matrices \(\psi_3, \psi_4\) such that the \(p \times 2n\) matrix \([\psi_1 \psi_2]\) has rank \(p\), the \(q \times 2m\) matrix \([\psi_3 \psi_4]\) has rank \(q\), and
Suppose that $R$ is a $p \times k$ matrix of rank $p$ such that $R[B_1 B_2] = 0$, and define $\psi_1$ and $\psi_2$ as $\psi_1 = RA_{10}$, $\psi_2 = RA_{20}$. In view of Lemma 6.2 and the fact that for arbitrary complex $\lambda$ the $k \times 2(m + n)$ matrix $[A_i(\lambda) B_i A_3(\lambda) B_3]$ has rank $k$ it follows that there exists a $p \times p$ matrix $V$ such that

$$[RA_1(\lambda) RA_2(\lambda)] = (E_p + \lambda V)R[A_{10} A_{20}] .$$

Hence $E_p + \lambda V$ is nonsingular and the equation (6.12) is equivalent to

$$RA_i(\lambda) \hat{y}(a) + RA_4(\lambda) \hat{y}(b) = 0 .$$

If $R_0$ is a $q \times (2m + 2n - k)$ matrix of rank $q$ such that $R_0[B_3 B_4] = 0$, and $\psi_3, \psi_4$ are defined as $\psi_3 = R_0 A_{30}$, $\psi_4 = R_0 A_{40}$, then equation (6.13) may be verified in a similar fashion. The conclusion concerning the ranks of $[\psi_1 \psi_3]$ and $[\psi_2 \psi_4]$ is clear.

From Lemma 6.2 it then follows that $[B_1 B_2][\psi_3 \psi_4]^* = 0$ and $[B_3 B_4][\psi_1 \psi_2]^* = 0$, so that $q \leq 2m - (k - p)$ and $p \leq 2n - [2m + 2n - k - q] = k + q - 2m$, from which one has the following result.

**Lemma 6.4.** If hypothesis (H1) holds, then the columns of $[\psi_3 \psi_4]^*$ form a basis for the null space of $[B_1 B_2]$ and the columns of $[\psi_1 \psi_2]^*$ form a basis for the null space of $[B_3 B_4]$.

The following theorem gives a simultaneous canonical representation of the boundary conditions for (6.6) and (6.7) in terms of parameter matrices $\psi_i, Q_i, G_i, (i = 1, 2, 3, 4)$, and is the central result of this section.

**Theorem 6.1.** Suppose that hypothesis (H1) holds. Then there exist $m \times n$ matrices $Q_i$ and $G_i, (i = 1, 2, 3, 4)$, such that $y \in \mathcal{D}(\lambda)$ if and only if there exists a $q$-vector $\eta_i$ such that

$$\psi_1 \hat{y}(a) + \psi_2 \hat{y}(b) = 0 ,$$

(6.14)  

$$(Q_1 - \lambda G_1) \hat{y}(a) + (Q_3 - \lambda G_3) \hat{y}(b) + \psi_i^* \eta_i - \hat{y}(a; \lambda) = 0 ,$$

$$(Q_2 - \lambda G_2) \hat{y}(a) + (Q_4 - \lambda G_4) \hat{y}(b) + \psi_i^* \eta_i + \hat{y}(b; \lambda) = 0 .$$

Moreover, $z \in \mathcal{D}^*(\lambda)$ if and only if there exists a $p$-vector $\eta_2$ such that

$$\psi_3 \hat{z}(a) + \psi_4 \hat{z}(b) = 0 ,$$

(6.15)  

$$(Q_1^* - \bar{\lambda} G_1^*) \hat{z}(a) + (Q_3^* - \bar{\lambda} G_3^*) \hat{z}(b) + \psi_i^* \eta_2 - \hat{z}(a; \lambda) = 0 ,$$

$$(Q_2^* - \bar{\lambda} G_2^*) \hat{z}(a) + (Q_4^* - \bar{\lambda} G_4^*) \hat{z}(b) + \psi_i^* \eta_2 + \hat{z}(b; \bar{\lambda}) = 0 .$$
Suppose that the matrices $K_{10}$ and $K_{11}$ have ranks $q_0$ and $q_1$, respectively. Let $D_{20}$ and $D_{21}$ be $(2m + 2n - k) \times (2m + 2n - k - q_0)$ and $(2m + 2n - k) \times (2m + 2n - k - q_1)$ matrices, respectively, whose individual column vectors form orthonormal bases for the null spaces of $K_{10}$ and $K_{11}$, that is, $K_{20}D_{10} = 0$ and $K_{21}D_{11} = 0$. As $K_{20} = K_{10}^*$ and $K_{21} = K_{11}^*$ by Lemma 6.2, there exist matrices $D_{20}$ and $D_{21}$ of respective orders $k \times (k - q_0)$ and $k \times (k - q_1)$ whose individual column vectors form orthonormal bases for the null spaces of $K_{20}$ and $K_{21}$. Then

\begin{align}
\begin{bmatrix}
K_{10} & D_{20} \\
D_{20}^* & 0
\end{bmatrix}
\begin{bmatrix}
K_{11} & D_{21} \\
D_{21}^* & 0
\end{bmatrix}
\begin{bmatrix}
K_{20} & D_{10} \\
D_{20} & 0
\end{bmatrix}
\begin{bmatrix}
K_{21} & D_{11} \\
D_{21} & 0
\end{bmatrix}
\end{align}

are nonsingular and have inverses of the form

\begin{align}
\begin{bmatrix}
H_{10} & D_{10} \\
D_{10}^* & 0
\end{bmatrix}
\begin{bmatrix}
H_{11} & D_{11} \\
D_{11}^* & 0
\end{bmatrix}
\begin{bmatrix}
H_{20} & D_{20} \\
D_{20} & 0
\end{bmatrix}
\begin{bmatrix}
H_{21} & D_{21} \\
D_{21} & 0
\end{bmatrix}
\end{align}

respectively. The matrices $H_{1i}$, $H_{2i}$, $H_{3i}$, and $H_{4i}$ are generalized reciprocals of the respective matrices $K_{1i}$, $K_{1i}$, $K_{20} = K_{10}^*$, and $K_{21} = K_{11}^*$. Let $Q_i$ and $G_i$, $(i = 1, 2, 3, 4)$, be defined as $Q_1 = A_{20}^*H_{10}A_{10}$, $Q_2 = A_{20}^*H_{10}A_{20}$, $Q_3 = A_{20}^*H_{10}A_{10}$, $Q_4 = A_{20}^*H_{10}A_{20}$, $G_1 = -A_{20}^*H_{10}A_{10}$, $G_2 = -A_{20}^*H_{10}A_{20}$, $G_3 = -A_{20}^*H_{11}A_{11}$, and $G_4 = -A_{20}^*H_{11}A_{21}$.

Now if $y \in \mathcal{B}(\lambda)$ then in view of Lemma 6.3 we need only verify the last two equations of (6.14). Suppose that $e_0$ is determined by (6.10). Then it follows from (6.10) and the fact that the matrices (6.17) are the inverses of the matrices (6.16) that

\begin{align}
e_0 = H_{10}A_{10}\hat{y}(a) + H_{10}A_{20}\hat{y}(b) + D_{10}D_{10}^*e_0, \\
e_0 = H_{11}A_{11}\hat{y}(a) + H_{11}A_{21}\hat{y}(b) + D_{11}D_{11}^*e_0.
\end{align}

Now it follows from (6.10) and (6.18) that

\begin{align}
(Q_1 - \lambda G_1)\hat{y}(a) + (Q_2 - \lambda G_2)\hat{y}(b) + (A_{20}^*D_{20}D_{10}^* + \lambda A_{21}^*D_{11}D_{11}^*)e_0 \\
- \hat{y}(a; \lambda) = 0,
\end{align}

\begin{align}
(Q_3 - \lambda G_3)\hat{y}(a) + (Q_4 - \lambda G_4)\hat{y}(b) + (A_{20}^*D_{10}D_{10}^* + \lambda A_{21}^*D_{11}D_{11}^*)e_0 \\
+ \hat{y}(b; \lambda) = 0.
\end{align}

But $B_1(A_{20}^*D_{20}D_{10}^* + \lambda A_{21}^*D_{11}D_{11}^*) + B_2(A_{20}^*D_{20}D_{10}^* + \lambda A_{21}^*D_{11}D_{11}^*) = K_{20}D_{10}D_{10}^* + \lambda K_{21}D_{11}D_{11}^* = 0$, and consequently the two equations of (6.19) may be written in the form of the last two equations of (6.14) involving the parameter vector $\gamma_1$.

On the other hand, suppose that $y \in \mathcal{B}_a(\lambda)$ and (6.14) holds. Now the first equation of (6.14) implies that there is a $(2m + 2n - k)$-vector $e_0$ such that $\hat{y}(a) = B_{11}^*e_0$ and $\hat{y}(b) = B_{11}^*e_0$. Hence it follows from (6.16) and (6.17) that (6.18) holds for this value of $e_0$. Solving the equations
(6.18) for $H_{10}A_{10}y(a) + H_{11}A_{11}y(b)$ and $H_{11}A_{11}y(a) + H_{11}A_{11}y(b)$, multiplying the first equation on the left by $A^*_1$ and $A^*_0$, and the second equation on the left by $\lambda A^*_0$ and $\lambda A^*_1$, respectively, and adding it can be shown that the last two equations of (6.14) may be written as

$$
\begin{align*}
A^*_0(e_0 - D_{10}^*D_{10}e_0) + \lambda A^*_0(e_0 - D_{11}^*D_{11}e_0) + \psi^*_1\gamma_1 - \tilde{y}(a; \lambda) &= 0 , \\
A^*_0(e_0 - D_{10}^*D_{10}e_0) + \lambda A^*_0(e_0 - D_{11}^*D_{11}e_0) + \psi^*_1\gamma_1 + \tilde{y}(b; \lambda) &= 0 .
\end{align*}
$$

In view of Lemma 6.2, the definition of the matrices $D_{10}, D_{11}$, and the choice of the vector $e_0$, one sees after multiplying the first equation of (6.20) by $B_1$, the second equation by $B_2$, and adding the two equations, that $y$ satisfies the boundary conditions of (6.6). The conclusion concerning $D^*(\lambda)$ may be established in a similar manner.

The next theorem is an application of Theorem 6.1, where it is to be noticed that if $m = n$ and $[f_{ij}(x)], [g_{ij}(x)]$ are Hermitian, then $\tilde{\mathfrak{H}}_n(\lambda) = \tilde{\mathfrak{H}}_n(\lambda)$; in particular, if $z \in \tilde{\mathfrak{H}}_n(\lambda)$, then $\tilde{z}(\lambda) = z(\lambda)$.

**Theorem 6.2.** Suppose that $m = n$, $[f_{ij}(x)]$ and $[g_{ij}(x)]$ are Hermitian on $[a, b]$, $k = 2n$, and $D^*(\lambda) = D^*(\lambda)$. Then the system (6.6) is equivalent to the Euler-Lagrange equations and transversality conditions for minimizing the functional

$$
\tilde{y}^*(a)[Q_1\tilde{y}(a) + Q_2\tilde{y}(b)] + \tilde{y}^*(b)[Q_2^*\tilde{y}(a) + Q_1^*\tilde{y}(b)] + \sum_{\alpha=0}^{\beta} \sum_{\beta=0}^{n} \tilde{y}^{(\alpha)}f_{\alpha\beta}\tilde{y}^{(\beta)},
$$

subject to the restraints

$$
\begin{align*}
\psi_1\tilde{y}(a) + \psi_2\tilde{y}(b) &= 0 , \\
\tilde{y}^*(a)[G_1\tilde{y}(a) + G_2\tilde{y}(b)] + \tilde{y}^*(b)[G_2^*\tilde{y}(a) + G_1^*\tilde{y}(b)] + \sum_{\alpha=0}^{\beta} \sum_{\beta=0}^{n} \tilde{y}^{(\alpha)}g_{\alpha\beta}\tilde{y}^{(\beta)} &= \text{const} .
\end{align*}
$$

If $m = n$, the problem is restricted to the field of real numbers, $g_{ij}(x) \equiv f_{ij}(x) = 0$ for $i \neq j$, and if $f_{ii}, g_{ii} \in \mathbb{C}_i$, $(i, j = 0, \cdots, n)$, then the results of this section are the same as obtained by Zimmerberg [12], provided that the formula (2.4) of that paper is corrected by replacing $f_i, f_{i+1}, \cdots, f_{n-1}$ by $f_i - \lambda g_i$, $f_{i+1} - \lambda g_{i+1}, \cdots, f_{n-1} - \lambda g_{n-1}$, respectively. If, moreover, $g_{ii}(x) \equiv 0$ for $i \geq 1$, then these are the same results as obtained by Reid [6, Section 11].

7. **An application.** In this section the results of Section 6 and a theorem of Reid [7] will be used to show that the boundary conditions for a rather large class of linear $\nu$th order differential operators may be written in the form given by Theorem 6.1. Reid [7] has considered $\nu$th order linear differential operators $L$ of the form
with integrable coefficients. Functions $A_i(y; p)$, $(i = 0, 1, 2, \cdots)$, were defined as

$$A_0(y; p) = p(x)y, \quad A_2(y; p) = (p(x)y(r))^{(r)},$$
$$A_{2r-1}(y; p) = \frac{1}{2}[(p(x)y^{(r-1)})^{(r)} + (p(x)y(r))^{(r-1)}], \quad (r = 1, 2, \cdots),$$

with the understanding that $p \in \mathcal{A}$ in the definition of $A_{2r}$ and $A_{2r-1}$.

The primary result of that paper, and the one of most interest here, is Theorem 3.2, to the effect that if the polynomials $1, x, \cdots, x^n/n!$, where $n = \nu/2$ or $n = (\nu + 1)/2$ according as $\nu$ is even or odd, belong to the domain of the adjoint operator $T^*_0$, then there exist functions $\pi_j$, $(j = 1, \cdots, \nu)$, with $\pi_0 \in \mathcal{A}_0$, $\pi_{2n-1} \in \mathcal{A}_n$, $\pi_{2n} \in \mathcal{A}_a$ such that $L[y]$ is given by

$$L[y] = \sum_{j=0}^{\nu} A_j(y; \pi_j),$$

while $\mathcal{A}_\nu$ is contained in the domain of the adjoint operator $T^*_0$ and

$$T^*_0(z) = L^*[z] = \sum_{j=0}^{\nu} A_j(z; (-1)^j\pi_j) \quad \text{for } z \in \mathcal{A}_\nu.$$

In view of the differentiability properties of $\pi_j$, $(j = 1, \cdots, \nu)$, it follows that (7.2) and (7.3) are of the form (6.2) and (6.4), respectively, which in turn reduce to (2.2) and (2.4), respectively, provided that $m = n$, $g_{ij}(x) = 0$ when $i \geq 1$ or $j \geq 1$, and for $i, j = 0, \cdots, n$ one defines $f_{ij}(x)$ as follows: $f_{ii}(x) = (-1)^i\pi_{2i}(x)$; $f_{i-1,j}(x) = (-1)^i(1/2)\pi_{2i-1}(x)$, $(i = 1, \cdots, n)$; $f_{i,i+1}(x) = (-1)^i(1/2)\pi_{2i+1}(x)$, $(i = 0, \cdots, n - 1)$; $f_{ij}(x) = 0$, $(j < i - 1$ and $j > i + 1)$.

In particular, if $\nu = 2n$ and $\pi_{2n}(x) \neq 0$, then the vector $\tilde{y}(x)$ consists of $y(x)$ and its first $n - 1$ derivatives. Similarly, $\tilde{z}(x)$ consists of $z(x)$ and its first $n - 1$ derivatives. The coordinates $\tilde{y}_i(x)$ of the $n$-vector $\tilde{y}(x)$ are defined by (2.1), and may be expressed in terms of $y(x)$ and its first $2n - j$ derivatives, $(j = 1, \cdots, n - 1)$, and similarly for the coordinates of $\tilde{z}(x)$, defined by (2.3). Consequently, $L[y]$ and $L^*[z]$ are defined for $y, z \in \mathcal{A}_\nu$.

If $\nu = 2n - 1$, and $\pi_2(x) \neq 0$, then $L$ is an operator of odd order and we modify the above defined matrix $[f_{ij}(x)]$ in the following way: delete the last row, replace $f_{n-1,n}(x)$ with $(-1)^{n-1}\pi_{2n-1}(x)$, and replace $f_{n-1,n-1}(x)$ with $(-1)^{n-1}(\pi_{2n-3}(x) + (1/2)\pi_{2n-2}(x))$. This change from an $(n + 1) \times (n + 1)$ matrix $[f_{ij}(x)]$ to the $n \times (n + 1)$ matrix $[f^*_0]$ changes neither the value of $L[y]$ nor the value of $L^*[z]$. Now if $\pi_{2n-1} \in \mathcal{A}_n$, 

$$L[y] = \sum_{j=0}^{\nu} q_j(x)y^{(j)}, \quad \nu \geq 1,$$
then $\pi_{2n-1} \in \mathfrak{A}_{n-1}$ so that $\tilde{y}_j(x)$ may still be differentiated out and written in terms of $y$ and its first $2n - j$ derivatives, $(j = 1, \cdots, n - 2)$, and similarly $\tilde{z}_i(x)$, $(i = 1, \cdots, n - 1)$, may be written in terms of $z(x)$ and its first $2n - i$ derivatives. Consequently we still have that $L$ and $L^*$ have the common domain $\mathfrak{A}_n$.

If now it is assumed that there is an $\varepsilon > 0$ such that $|q_\varepsilon(x)| \geq \varepsilon$ almost everywhere, then it follows from Theorem 3.2, or Theorem 4.1 of [7], that the domain of the adjoint operator $T_0^*$ is $\mathfrak{A}_n$. Moreover, in view of the formulas which give the canonical variables $\tilde{y}_j(x)$ and $\tilde{z}_i(x)$ in terms of $y(x), \cdots, y^{(n-1)}(x)$ and $z(x), \cdots, z^{(m-1)}(x)$, respectively, we see that there exist nonsingular linear transformations $T$ and $T_1$ which transform the vector functions $(y, y', \cdots, y^{(n-1)})$ and $(z, z', \cdots, z^{(m-1)})$ into the vector functions $(y, y', \cdots, y^{(n-1)}, \tilde{y}_1, \cdots, \tilde{y}_m)$ and $(z, z', \cdots, z^{(m-1)}, \tilde{z}_1, \cdots, \tilde{z}_n)$, respectively. Therefore, in view of Theorem 3.2 of Reid [7] and Theorem 6.1, it follows that boundary conditions for a $\nu$th order differential operator of the type described above which involve linearly $y$ and its first $\nu - 1$ derivatives at two points may be written as (6.14), and the adjoint boundary conditions may be written as (6.15).

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