

# Pacific Journal of Mathematics

## **UNIQUE FACTORIZATION IN POWER SERIES RINGS AND SEMIGROUPS**

DON DECKARD AND LINCOLN KEARNEY DURST

## UNIQUE FACTORIZATION IN POWER SERIES RINGS AND SEMIGROUPS

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In this note a short proof is given for a theorem due originally to Deckard and to Cashwell and Everett. The theorem states that every ring of power series over an integral domain  $R$  is a unique factorization domain if and only if every ring of power series over  $R$  in a finite set of indeterminates is a unique factorization domain. The proof is based on a study of the structure of the multiplicative semigroups of such rings. Much of the novelty and most of the brevity of this argument may be accounted for by the fact that Dilworth's theorem on the decomposition of partially ordered sets is invoked at a crucial point in the proof.

Suppose  $R$  is a ring and that  $R^+$  is the additive group of  $R$ . Suppose  $I$  is a nonvoid set well ordered by  $<$ . If  $N$  is the additive semigroup of the nonnegative integers and  $M$  is the weak product (Chevalley [3])  $\prod_{i \in I} N$ , let  $R_I = \prod_{m \in M} R^+$ . For  $f, g$  in  $R_I$  define

$$(fg)(m) = \sum_{k+l=m} f(k)g(l);$$

the resulting ring is called a *ring of power series* over  $R$ , and is an integral domain if  $R$  is. An  $f$  in  $R_I$  is a unit if and only if  $f(0)$  is a unit of  $R$ .

Deckard [5] and Cashwell and Everett [2] proved that  $R_I$  is a unique factorization domain for every  $I$  if  $R_J$  is a unique factorization domain for every finite set  $J$ . (Thus it follows, from results of Samuel [8] and Buchsbaum [1], that  $R_I$  is a unique factorization domain if  $R$  is a regular unique factorization domain.)

It suffices to consider either the multiplicative semigroup  $S_I$  of the nonzero members of  $R_I$  or the quotient of  $S_I$  obtained by identifying associates; factorization in a cancellative semigroup being unique if and only if (Clifford [4]) both

- (1) The divisor chain condition holds.
- (2) All irreducibles are primes.

This note contains a study of factorization in a class of semigroups which, on specialization, yields a direct and concise proof of the Deckard-Cashwell and Everett theorem.

Extensions of semigroups. With  $I$  as above, suppose  $\{S_i: i \in I\}$

is a family of commutative cancellative semigroups indexed by  $I$ , and that

(a) For each pair  $i, j$  of members of  $I$ , if  $i < j$ ,  $S_j$  contains a nonvoid prime coideal (MacKenzie [7])  $C_i^j$  and  $C_i^k \subseteq C_j^k$  if  $i < j < k$ .

(b) If  $i, j$  are in  $I$  and  $i < j$ , there is an epimorphism  $h_i^j: C_i^j \rightarrow S_i$  with  $h_i^k = h_i^j \circ h_j^k$  if  $i < j < k$ .

(c)  $(h_i^j)^{-1}U_i = U_j$  for  $i < j$ , where  $U_i$  is the group of units of  $S_i$ . Let  $S$  be the set of all  $f$  in  $\bigcup_{i \in I} \prod_{j \in I}^{i \leq j} S_j$  for which

(d)  $f(j) \in C_i^j$  for  $i < j$  if and only if  $I = \text{Dom } f$  or  $\text{Inf Dom } f < i \in \text{Dom } f$ .

(e)  $f(i) = h_i^j f(j)$  if  $i \in \text{Dom } f$  and  $i < j$ .

With the definition  $(fg)(i) = f(i)g(i)$  for  $i \in (\text{Dom } f) \cap (\text{Dom } g)$ ,  $S$  is a semigroup. Suppose, finally, that

(f) For each  $k$  in  $I$ , if there exist  $u_i$  in  $U_i$  for  $i < k$  satisfying  $u_i = h_i^j u_j$  whenever  $i < j < k$ , then there is a  $u$  in  $S$  such that  $u(i) = u_i$  for  $i < k$ .

In view of condition (c) the  $u$  in condition (f) clearly belongs to  $U$ , the group of units of  $S$ . In the power series case a much stronger version of condition (f) holds. However, for present purposes, it is sufficient to assume (f) only for units.

It will be convenient to refer to  $S$  as the *extension* of the family  $\{S_i: i \in I\}$ .

**Divisor chain condition.** If  $J \subseteq I$  and if  $J$  has no upper bound in  $I$ ,  $J$  will be said to contain "arbitrarily large" members of  $I$ .

**LEMMA 1.**  *$S$  satisfies the divisor chain condition if, for arbitrarily large  $i$ ,  $S_i$  satisfies the divisor chain condition.*

Assume, indeed, that  $f^{(k)} \in S$  and  $f^{(k+1)} \mid f^{(k)}$  for  $k = 0, 1, 2, \dots$ . If  $f^{(l)} g^{(l,n)} = f^{(n)}$  for  $l > n$ , then, for  $i$  sufficiently large,  $f^{(l)}(i) g^{(l,n)}(i) = f^{(n)}(i)$  if  $l > n$ . Thus for  $n$  sufficiently large,  $g^{(l,n)}(i) \in U_i$  for some  $i$  if  $l > n$ . Hence, by (c),  $g^{(l,n)} \in U$  if  $l > n$  and  $n$  is sufficiently large.

**Interpolation and primes.** Suppose  $J$  is a subset of  $I$  containing arbitrarily large members of  $I$  and that for each  $i$  in  $J$  there is an  $f_i$  such that  $f_i \in C_i^j$  and  $f_i = h_i^j f_j$  if  $i, j$  in  $J$  and  $i < j$ . Using  $h_i^k = h_i^j \circ h_j^k$  if  $i < j < k$ ,  $f_i$  can be defined on an upper segment of  $I$  containing  $J$ , determining a unique  $f$  of  $S$  such that  $f(i) = f_i$  for  $i \in J$ .

**LEMMA 2.** *If  $p \in S$  and if, for arbitrarily large  $i$ ,  $p(i)$  is a prime in  $S_i$ , then  $p$  is a prime in  $S$ .*

If  $f, g$  are in  $S$  and  $p \mid fg$ , then for sufficiently large  $i$ ,  $p(i) \mid f(i)g(i)$ ; and for arbitrarily large  $i$ ,  $p(i) \mid f(i)$  or  $p(i) \mid g(i)$ . If  $p(i)q_i = f(i)$  for arbitrarily large  $i$ , then  $h_i^j q_j = q_i$  if  $i < j$  for  $i, j$  in an unbounded subset of  $I$ , so there is a  $q$  in  $S$  for which  $pq = f$ .

**Identification of associates.** If  $\bar{S}_i$  and  $\bar{C}_i^j$  are, respectively, the quotients of  $S_i$  and  $C_i^j$  obtained by identifying associates and if  $\bar{h}_i^j: \bar{C}_i^j \rightarrow \bar{S}_i$  are the corresponding homomorphisms generated by the  $h_i^j$ , it follows that conditions (a)-(f) remain valid for these "reduced" semi-groups. Let  $\bar{S}$  be the quotient of  $S$  and let  $S'$  be the extension of the  $\bar{S}_i$ .

**LEMMA 3.**  $\bar{S}$  and  $S'$  are isomorphic.

For  $f$  in  $S$ , write  $\bar{f}$  for the member of  $\bar{S}$  containing  $f$  and  $\bar{f}(i)$  for the member of  $\bar{S}_i$  containing  $f(i)$ , and define the map  $\varphi: \bar{S} \rightarrow S'$  by  $(\varphi\bar{f})(i) = \bar{f}(i)$ ,  $i \in \text{Dom } f$ . It is easy to see that  $\varphi$  is a monomorphism. An induction argument shows that  $\varphi$  is surjective. Indeed, suppose  $g \in S'$ : to find  $f$  in  $S$  satisfying  $\bar{f}(i) = g(i)$  for  $i \in \text{Dom } g$ . Let  $i_g = \text{Inf Dom } g$ , pick  $f_{i_g}$  in  $g(i_g)$  and suppose that for  $i_g \leq i < k$  there are  $f_i$  in  $g(i)$  satisfying  $h_i^j f_j = f_i$  whenever  $i_g \leq i < j < k$ . If  $s \in g(k)$ , then, for  $i_g \leq i < k$ ,  $h_i^k s = u_i f_i$  where  $u_i \in U_i$ . Here  $u_i = h_i^j u_j$  for  $i < j < k$ , and by condition (f) there is a  $u$  in  $U$  satisfying  $h_i^k(u(k)) = u_i$  for  $i < k$ . If  $f_k = [u(k)]^{-1}s$ , then  $h_i^k(f_k) = u_i^{-1}h_i^k(s) = f_i$ . By induction, there is an  $f$  in  $S$  satisfying  $f(i) = f_i$  and  $\varphi\bar{f} = g$ .

**Reducibility.** If  $f \in S$  say that  $f$  is *normal* if, for arbitrarily large  $i$ ,  $f(i)$  has a unique representation in  $S_i$  as a product of irreducibles.

**LEMMA 4.** If  $f$  is normal then  $f$  is reducible in  $S$  if, for arbitrarily large  $i$ ,  $f(i)$  is reducible in  $S_i$ .

In view of Lemma 3 associates may be identified. Suppose, then, that  $J$  is an unbounded subset of  $I$  and that for each  $j$  in  $J$ ,  $f(j)$  has a unique representation

$$f(j) = a_j^{(1)} \cdots a_j^{(n_j)}$$

where each  $a_j^{(k)}$  is an irreducible in  $S_j$ . If  $i \in J$  and  $i < j$

$$f(i) = \prod_1^{n_i} a_i^{(l)} = \prod_1^{n_j} h_i^j a_j^{(l)};$$

so  $n_i \geq n_j$  and for sufficiently large  $j$  in  $J$ ,  $n_j$  is a constant  $n$  depend-

ing only on  $f$ . By hypothesis,  $n \geq 2$ . Hence, for each  $i$  in some unbounded subset  $K$  of  $I$ ,  $f(i)$  is a product of exactly  $n$  irreducibles. In the set  $P$  of all irreducible factors of  $f(i)$  with  $i \in K$  define

$$a < b \text{ if } a = h_i b \text{ for some } i, j \text{ in } K, i < j.$$

The relation  $<$  is transitive by condition (b) and thus defines a partial order in  $P$ . However each subset of  $P$  with more than  $n$  members contains a comparable pair and thus, by a theorem of Dilworth [6],  $P$  is the union of  $n$  or fewer chains. At least one such chain is unbounded and determines in  $S$  a proper factor of  $f$ .

**THEOREM.** *In  $S$  factorization into irreducibles is unique if, for arbitrarily large  $i$ , factorization into irreducibles in  $S_i$  is unique.*

*Proof.* By Lemma 1,  $S$  satisfies the divisor chain condition. It remains to show that in  $S$  all irreducibles are primes. Suppose  $p \in S$  and  $p$  is irreducible. Then, by Lemma 4,  $p(i)$  is irreducible in  $S_i$  for some  $i$ , since every member of  $S$  is normal. In view of condition (c),  $p(i)$  is irreducible in  $S_i$  for all sufficiently large  $i$ . But then,  $p(i)$  is prime in  $S_i$  for  $i$  arbitrarily large and, by Lemma 2,  $p$  is prime in  $S$ .

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# Pacific Journal of Mathematics

Vol. 16, No. 2

December, 1966

Loren N. Argabright, <i>Invariant means on topological semigroups</i> . . . . .	193
William Arveson, <i>A theorem on the action of abelian unitary groups</i> . . . . .	205
John Spurgeon Bradley, <i>Adjoint quasi-differential operators of Euler type</i> . . . . .	213
Don Deekard and Lincoln Kearney Durst, <i>Unique factorization in power series rings and semigroups</i> . . . . .	239
Allen Devinatz, <i>The deficiency index of ordinary self-adjoint differential operators</i> . . . . .	243
Robert E. Edwards, <i>Operators commuting with translations</i> . . . . .	259
Avner Friedman, <i>Differentiability of solutions of ordinary differential equations in Hilbert space</i> . . . . .	267
Boris Garfinkel and Gregory Thomas McAllister, Jr., <i>Singularities in a variational problem with an inequality</i> . . . . .	273
Seymour Ginsburg and Edwin Spanier, <i>Semigroups, Presburger formulas, and languages</i> . . . . .	285
Burrell Washington Helton, <i>Integral equations and product integrals</i> . . . . .	297
Edgar J. Howard, <i>First and second category Abelian groups with the <math>n</math>-adic topology</i> . . . . .	323
Arthur H. Kruse and Paul William Liebnitz, Jr., <i>An application of a family homotopy extension theorem to ANR spaces</i> . . . . .	331
Albert Marden, I. Richards and Burton Rodin, <i>On the regions bounded by homotopic curves</i> . . . . .	337
Willard Miller, Jr., <i>A branching law for the symplectic groups</i> . . . . .	341
Marc Aristide Rieffel, <i>A characterization of the group algebras of the finite groups</i> . . . . .	347
P. P. Saworotnow, <i>On two-sided <math>H^*</math>-algebras</i> . . . . .	365
John Griggs Thompson, <i>Factorizations of <math>p</math>-solvable groups</i> . . . . .	371
Shih-hsiung Tung, <i>Harnack's inequalities on the classical Cartan domains</i> . . . . .	373
Adil Mohamed Yaqub, <i>Primal clusters</i> . . . . .	379