

# Pacific Journal of Mathematics

**OPERATORS COMMUTING WITH TRANSLATIONS**

ROBERT E. EDWARDS

# OPERATORS COMMUTING WITH TRANSLATIONS

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This paper is concerned with the representation, in terms of convolutions with pseudomeasures, of continuous linear operators which commute with translations and which transform continuous functions with compact supports on a Hausdorff locally compact Abelian group  $G$  into restricted types of Radon measures on  $G$ . The two main theorems each assert that any such operator  $T$  is of the form  $Tf = s * f$  for a suitably chosen pseudomeasure  $s$  on  $G$ ; the assertions differ in detail in respect of the hypotheses imposed on the range of  $T$ . The second theorem is an extension of Proposition 2 of [1] from the case in which  $G$  is a finite product of lines and/or circles to the general situation.

**Preliminaries.** The notations are as described in §1 of [1], with  $G$  in place of  $X$ , and with the following additions. If  $K \subset G$ ,  $C_K(G)$  denotes the set of  $f \in C_c(G)$  satisfying  $\text{supp } f \subset K$ . The symbol  $M_b(G)$  will denote the set of all bounded Radon measures on  $G$ . Continuity of the operators  $T$  considered will, in the absence of any indication to the contrary, refer to the inductive limit topology on  $C_c(G)$  and the vague topology  $\sigma(M(G), C_c(G))$  on  $M(G)$  and its subsets. No distinction is drawn between a locally integrable function  $f$  on  $G$  and the associated measure  $f dx \in M(G)$ ,  $dx$  denoting the element of Haar measure on  $G$ . In this paper,  $X$  will denote the character group of  $G$ , the Haar measure  $d\xi$  on  $X$  being chosen so that the Fourier transformation is an isometry of  $L^2(G)$  onto  $L^2(X)$ .

Prior to stating the representation theorems, we make some remarks about pseudomeasures on  $G$ .

Let  $A(G)$  denote the space of functions  $u$  on  $G$  which are inverse Fourier transforms of functions  $v \in L^1(X)$ :

$$u(x) := \int_X v(\xi) \xi(x) d\xi;$$

$A(G)$  is a Banach space under the norm

$$\|u\|_A = \int_X |v(\xi)| d\xi \equiv \|v\|_1.$$

By a pseudomeasure on  $G$  is meant a continuous linear functional on  $A(G)$ , and we denote by  $P(G)$  the set of pseudomeasures on  $G$ . By  $\|\cdot\|_P$  is meant the usual norm on  $P(G)$  qua dual of  $A(G)$ . The Fourier transformation can be defined for pseudomeasures  $s$  in such a way that

$s \rightarrow \hat{s}$  is an isometric isomorphism of  $P(G)$  onto  $L^\infty(X)$ . There is an obvious sense in which  $M_b(G)$  can be regarded as a subset of  $P(G)$ .

If  $G$  is a finite product of lines and/or circles, one may think of  $P(G)$  as comprising exactly those temperate distributions on  $G$  whose Fourier-Schwartz transform is an essentially bounded function. It is this identification which provides the link between Proposition 2 of [1] and Theorem 2 below.

If  $s \in P(G)$ , the mapping  $f \rightarrow s * f$  is a continuous endomorphism of  $L^2(G)$ . In connection with Theorem 1 we shall be concerned with the case in which the restriction of this mapping to  $C_c(G)$  has a range lying in  $M_b(G)$ , i.e., equivalently, in  $L^1(G)$ . The pseudomeasures  $s$  having this property form a subset  $P^1(G)$  of  $P(G)$ . Naturally,  $P^1(G)$  contains the set  $P_c(G)$  of all pseudomeasures with compact supports (in particular,  $P^1(G) = P(G)$  when  $G$  is compact) and contains also  $M_b(G)$ . The closed graph theorem shows that, if  $s \in P^1(G)$ , then to each compact set  $K \subset G$  corresponds a number  $m_K > 0$  such that

$$(1.1) \quad \|s * f\|_1 \leq m_K \|f\| \quad (f \in C_K(G)),$$

where  $\|\cdot\|$  denotes the supremum norm. Further comments on  $P^1(G)$  are given in § 5 *infra*.

We can now state the two main theorems.

**THEOREM 1.** *The continuous linear operators  $T$  from  $C_c(G)$  into  $M_b(G)$  which commute with translations are precisely those of the form*

$$(1.2) \quad Tf = s * f,$$

where  $s \in P^1(G)$ .

**THEOREM 2.** *The continuous linear operators  $T$  from  $C_c(G)$  into  $M_c(G)$  which commute with translations are precisely those of the form (1.2), where now  $s \in P_c(G)$ .*

Theorem 2, combined with the basic properties of pseudomeasures, shows that any continuous linear operator  $T$  from  $C_c(G)$  into  $M_c(G)$  which commutes with translations admits an extension which maps  $L_c^2(G)$  into  $L_c^2(G)$  and  $L^2(G)$  into  $L^2(G)$ ,  $L_c^2(G)$  denoting  $L^2(G) \cap M_c(G)$ , i.e., the set of functions in  $L^2(G)$  which vanish a.e. outside a compact subset of  $G$  (a property equivalent to saying that the associated measure has a compact support). In § 5(B) we shall see that, by virtue of Theorem 1, each continuous linear operator from  $C_c(G)$  into  $M_b(G)$  admits somewhat similar but less evident extensions.

In Theorem (3.2) of [3] G. I. Gaudry has shown that there is a valid analogue of Theorem 1 for the case in which  $M_b(G)$  is replaced by the space  $M(G)$  of all Radon measures on  $G$ , the pseudomeasure  $s$  being then replaced by a somewhat more general entity termed a "quasimeasure". Theorem 2 above is used in [3] as an aid in studying the local structure of quasimeasures.

2. In the proof of Theorem 1 we shall need a lemma.

LEMMA. *To each subset  $W$  of  $G$  containing interior points corresponds a number  $c = c_w > 0$  such that*

$$\|F\| < c \cdot \text{Sup} \{ \|F\hat{f}\| : f \in C_w(G), \|f\| \leq 1 \}$$

for all functions  $F$  on  $X$ .

*Proof.* Define

$$N(F) = \text{Sup} \{ \|F\hat{f}\| : f \in C_w(G), \|f\| \leq 1 \},$$

which is possibly  $\infty$ . If  $F$  is unbounded on  $X$ , the lemma on p. 281 of [1] shows that  $N(F) = \infty$ , so that in this case any value of  $c > 0$  will suffice (provided the usual conventions are adopted). Assume then that  $F \in B(X)$ , the space of bounded functions on  $X$ . The functional  $N$  is evidently a norm on  $B(X)$ . Moreover,  $B(X)$  is complete for  $N$ . For suppose that  $(F_n)$  is an  $N$ -Cauchy sequence in  $B(X)$ . Evidently, to each  $\xi \in X$  corresponds a number  $b_\xi > 0$  such that

$$(2.1) \quad |F(\xi)| \leq b_\xi \cdot N(F).$$

It follows that  $(F_n(\xi))$  is Cauchy for each  $\xi \in X$ , so that  $F = \lim F_n$  exists pointwise on  $X$ . For any  $\varepsilon > 0$  there exists  $n_0 = n_0(\varepsilon)$  such that

$$N(F_m - F_n) \leq \varepsilon \quad (m, n > n_0).$$

That is, for any  $f \in C_w(G)$  satisfying  $\|f\| \leq 1$ ,

$$\text{Sup}_{\xi \in X} |F_m(\xi) - F_n(\xi)| \|\hat{f}(\xi)\| \leq \varepsilon \quad (m, n > n_0).$$

On letting  $m \rightarrow \infty$  it appears that

$$\text{Sup}_{\xi \in X} |F(\xi) - F_n(\xi)| \|\hat{f}(\xi)\| \leq \varepsilon \quad (n > n_0)$$

and hence that

$$N(F - F_n) \leq \varepsilon \quad (n > n_0).$$

This shows first that  $N(F) < \infty$ , and hence that  $F \in B(X)$ , and then that  $F_n \rightarrow F$  in the sense of the norm  $N$ . Thus  $B(X)$  is  $N$ -complete.

Reference to (2.1) shows that the supremum norm is lower semi-continuous relative to  $N$ . Therefore, this supremum norm is actually continuous relative to  $N$ , which is precisely what the lemma asserts.

**3. Proof of Theorem 1.** The inequality (1.1) makes it plain that, if  $s \in P^1(G)$ , then (1.2) defines  $T$  as a continuous linear operator from  $C_c(G)$  into  $L^1(G) \subset M_b(G)$  which commutes with translations. Actually  $T$ , thus defined, maps  $C_c(G)$  into  $L^2(G)$  and is continuous for the  $L^2$ -topologies.

Turning to the converse, let us first show that the seminorm  $f \rightarrow \int_G d(|Tf|)$  is continuous on  $C_c(G)$ . Indeed, integration theory shows that

$$\int_G d(|Tf|) = \text{Sup} \left| \int_G f d(Tf) \right|,$$

the supremum being taken with respect to those  $f \in C_c(G)$  satisfying  $\|f\| \leq 1$ . It thus appears that the seminorm  $f \rightarrow \int_G d(|Tf|)$  is lower semicontinuous on the barrelled space  $C_c(G)$ , and is therefore continuous.

Accordingly, if  $K \subset G$  is compact, there exists a number  $m_K > 0$  such that

$$(3.1) \quad \int_G d(|Tf|) \leq m_K \|f\| \quad (f \in C_K(G)).$$

Take now a net  $(e_i)$  of nonnegative functions in  $C_c(G)$  such that  $\int_G e_i dx = 1$  and  $\text{supp } e_i \subset N_i$ , where the  $N_i$  form a neighbourhood base at the origin in  $G$ . We may assume that all the  $N_i$  are contained in some compact set  $N$ . If  $f \in C_K(G)$ , then  $\lim e_i * f = f$  uniformly on  $G$  and  $\text{supp } (e_i * f) \subset N + K$ . Since  $T$  is continuous and commutes with translations,  $T(e_i * f) = T e_i * f$  for  $e, f \in C_c(G)$ . So, if  $\mu_i = T e_i$ , it follows from (3.1) that

$$(3.2) \quad Tf = \lim T(e_i * f) = \lim \mu_i * f \text{ in } M_b(G),$$

and that

$$\int_G d(|\mu_i * f|) \leq m_{N+K} \|f\|.$$

Taking the Fourier transform of this relation, it follows that for  $f \in C_K(G)$  we have

$$\|\hat{\mu}_i * \hat{f}\| \leq m_{N+K} \|f\|.$$

Fixing  $K$  as any compact set with interior points, and applying the lemma in § 2, we conclude that

$$\text{Sup}_i \|\hat{\mu}_i\| < \infty .$$

This in turn ensures that the net  $(\mu_i)$  has a weak limiting point  $s \in P(G)$ . The net  $(\mu_i * f)$  then has  $s * f$  as a weak limiting point in  $L^2(G)$  and a comparison with (3.2) shows that  $Tf$  must coincide with  $s * f$ , i.e., that (1.2) must hold. Since  $T$  maps  $C_c(G)$  into  $M_b(G)$ ,  $s$  must belong to  $P^1(G)$ . The proof is complete.

4. **Proof of Theorem 2.** Once again it is evident that, if  $s \in P_c(G)$ , then (1.2) defines  $T$  as a continuous linear map of  $C_c(G)$  into  $M_c(G)$  which commutes with translations.

For the converse, note that Theorem 1 implies the existence of a pseudomeasure  $s$  such that (1.2) holds. The proof of Theorem 1 shows moreover that  $s$  is a weak limiting point in  $P(G)$  of the measures  $\mu_i = Te_i$ . Now  $\text{supp } e_i \subset N$ , a compact subset of  $G$ . Lemmas 2 and 3 of [2] show that accordingly there is a compact subset  $K'$  of  $G$  such that  $\text{supp } \mu_i \subset K'$  for all  $i$ . But then it follows that  $\text{supp } s \subset K'$  too, showing that  $s \in P_c(G)$ .

REMARK. In Theorem 4.2 of [3] it is remarked that Theorem 2 entails that every quasimeasure with a compact support is a pseudomeasure. Theorem 1 leads to an analogous result, as we now show.

Reference to the proof of Theorem 4.5 of [3] confirms that if  $q$  is a quasimeasure on  $G$ , then  $f \rightarrow q * f$  maps  $L_c^2(G)$  continuously into  $L_{loc}^2(G)$ . Let us write

$$\|h\|_1 = \int_G^* |h(x)| dx \quad (\leq \infty)$$

for an arbitrary complex-valued function  $h$  on  $G$ , so that  $h \in L^1(G)$  if and only if  $h$  is measurable and  $\|h\|_1 < \infty$ . Then we have the

COROLLARY. *If  $q$  is a quasimeasure on  $G$  such that*

$$(4.1) \quad \|q * f\|_1 < \infty \quad (f \in C_c(G)) ,$$

*then  $q$  is a pseudomeasure belonging to  $P^1(G)$ .*

*Proof.* Since  $q * f \in L_{loc}^2(G)$ , (4.1) shows that  $q * f \in L^1(G)$ . The preceding remarks show that the mapping  $f \rightarrow q * f$  has a graph which is closed in  $C_c(G) \times L^1(G)$  and is therefore continuous. The assertion therefore follows from Theorem 1.

5. **Concerning  $P^1(G)$ .** We collect a few results about  $P^1(G)$  and its elements.

(A) When  $G$  is compact,  $P^1(G) = P(G)$  (see § 1). The situation is

much more complex when  $G$  is noncompact, and we know of no effective and direct characterisation of  $P^1(G)$  as a subset of  $P(G)$ . It is easy to see that if  $s \in P^1(G)$ , then  $\hat{s}$  coincides l.a.e. on each compact subset  $H$  of  $X$  with the transform of an ( $H$ -dependent) function in  $L^1(G)$ ; in particular,  $\hat{s}$  is equal l.a.e. on  $X$  to a continuous function on  $X$ . This shows that  $P^1(G)$  is dense in  $P(G)$  if and only if  $G$  is compact. More elaborate arguments (based on properties of Helson subsets of  $X$ ; see [4], Chapter 5) will show also that  $P^1(G)$  is closed in  $P(G)$  if and only if  $G$  is compact.

We turn next to a positive assertion which adds interest and weight to Theorem 1.

(B) Suppose that  $s \in P^1(G)$ , that  $2 \leq p \leq \infty$ , and that  $p'$  is defined by  $1/p + 1/p' = 1$ . Let  $W$  be any relatively compact open subset of  $G$ ,  $(a_r)_{r=1}^\infty$  any sequence of points of  $G$ . Put  $e_r$  for the characteristic function of  $a_r \bar{W}$ . If  $f$  is a measurable function on  $G$  vanishing outside a compact subset of  $E = \bigcup \{a_r \bar{W} : r = 1, 2, \dots\}$  and such that

$$(5.1) \quad \|f\|_{*p} \equiv \sum_{r=1}^\infty \|f e_r\|_p < \infty ,$$

then  $s * f \in L^{p'}(G)$ , and furthermore there exists a number  $m'_W > 0$  such

$$(5.2) \quad \|s * f\|_{p'} \leq m'_W \cdot \|f\|_{*p} .$$

*Proof.* Consider first the case in which  $f$  is essentially bounded and vanishes outside  $\bar{W}$ . There exists then a sequence  $(f_n)_{n=1}^\infty$  of functions in  $C_{\bar{W}}(G)$  such that  $\|f_n\| \leq \|f\|_\infty$  and  $f_n \rightarrow f$  a.e. By (1.1),  $\|s * f_n\|_1 \leq m_{\bar{W}} \|f\|_\infty$  and so the  $s * f_n$  have a weak limiting point  $\mu \in M_b(G)$ . On the other hand, since  $f_n \rightarrow f$  in  $L^2(G)$ ,  $s * f_n \rightarrow s * f$  in  $L^2(G)$ . It follows that  $\mu = s * f \in M_b(G) \cap L^2(G) \subset L^1(G)$  and

$$(5.3) \quad \|s * f\|_1 \leq \lim_{n \rightarrow \infty} \|s * f_n\|_1 \leq m_{\bar{W}} \|f\|_\infty .$$

We also know that

$$(5.4) \quad \|s * f\|_2 \leq \|s\|_{P'} \cdot \|f\|_2 .$$

Now (5.3) and (5.4) and the Riesz convexity theorem combine to show that, for some number  $m'_W > 0$  and all  $p \geq 2$ , one has

$$(5.5) \quad \|s * f\|_{p'} \leq m'_W \cdot \|f\|_p$$

whenever  $f \in L^p(G)$  vanishes outside  $\bar{W}$ . By translation, (5.5) remains valid whenever  $f \in L^p(G)$  vanishes outside a translated set  $a \bar{W}$ , where  $a \in G$  is arbitrary.

Now suppose that  $f$  vanishes outside a compact subset of  $E$  and and satisfies (5.1). Then  $f = \sum_{r=1}^\infty f_r$ , where  $f_r = f e_r$  and where the series converges in  $L^p(G)$  and a fortiori in  $L^2(G)$ . By (5.5),

$$(5.6) \quad \|s * f_r\|_{p'} \leq m'_w \cdot \|f_r\|_p,$$

so that in particular  $\sum_{r=1}^{\infty} (s * f_r)$  is convergent in  $L^{p'}(G)$ . This latter series is, however, convergent in  $L^2(G)$  to  $s * f$ , whence it appears that  $s * f \in L^{p'}(G)$  and, from (5.6), that (5.2) is true. This completes the proof.

REMARKS. (1) In the statement of (B) we assumed that  $f$  vanishes outside a compact subset of  $E$  merely to ensure that  $s * f$  is defined *a priori*. Actually, the proof furnishes a method of extending the definition of  $s * f$  to all cases in which  $f$  vanishes outside  $E$  and satisfies (5.1).

Notice that if  $G = R^n$ , we can always arrange that the  $a_r \bar{W}$  form a covering of  $R^n$  by nonoverlapping congruent closed  $n$ -dimensional cubes; this is indeed one of the most natural choices of the  $a_r \bar{W}$  in this case. Taking  $n = 1$ , we see that  $s \in P^1(R)$  if and only if the operator  $f \rightarrow s * f$  maps the Wiener class  $M_1$  ([5], p. 73) into  $L^1(R)$ ; and that any continuous linear operator from  $M_1$  into  $L^1(R)$  which commutes with translations is of the form  $f \rightarrow s * f$  for a suitably chosen  $s \in P^1(R)$ .

(2) By virtue of Theorem 1, (B) expresses some nontrivial extension properties possessed by all continuous linear operators from  $C_c(G)$  into  $M_b(G)$  which commute with translations.

(C) In case  $G = R^n = X$ , it is simple to specify smoothness conditions on  $\hat{s}$  ensuring that a given  $s \in P(R^n)$  shall belong to  $P^1(R^n)$ . In fact, if we define  $m_n$  to be 1 if  $n = 1$  and to be  $2[n/4] + 2$  if  $n > 1$  (square brackets denoting the integral part), it is sufficient that each partial derivative of  $\hat{s}$  of order at most  $m_n$  be expressible as the sum of a function in  $L(R^n)$  and a function in  $L^2(R^n)$ . (The partial derivatives are here understood in the distributional sense.)

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