ON TWO-SIDED $H^*$-ALGEBRAS

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We call a Banach algebra \( A \), whose norm is a Hilbert space norm, a two-sided \( H^* \)-algebra if for each \( x \in A \) there are elements \( x^r, x^l \) in \( A \) such that \((xy, z) = (y, x^l z)\) and \((yx, z) = (y, x^r z)\) for all \( y, z \in A \). A two-sided \( H^* \)-algebra is called discrete if each right ideal \( R \) such that \( \{x^r \mid x \in R\} = \{x^l \mid x \in R\} \) contains an idempotent \( e \) such that \( e^r = e^l = e \). The purpose of this paper is to obtain a structural characterization of those two-sided \( H^* \)-algebras \( M \) which consist of complex matrices \( x = (x_{ij} \mid i, j \in J) \) (\( J \) is any index set) for which

\[
\sum_{i,j} t_{ij} |x_{ij} |^2 
\]

converges. Here \( t_i \) is real and \( 1 \leq t_i \leq a \) for all \( i \in J \) and some real \( a \). The inner product in \( M \) is

\[
(x, y) = \sum_{i,j} t_{ij} x_{ij} \overline{y}_{ij}
\]

and

\[
x_{ij} = (t_{ij} |t_{ij}| x_{ij}, x_{ij} = (t_{ij} |t_{ij}| x_{ij}.
\]

Then every algebra \( M \) is discrete simple and proper (\( Mx = 0 \) implies \( x = 0 \)). Conversely every discrete simple and proper two-sided \( H^* \)-algebra is isomorphic to some algebra \( M \). An incidental result is that the radical of a two-sided \( H^* \)-algebra is the right (left) annihilator of the algebra.

In this paper we will refer to such an algebra \( M \) above as a canonical algebra. We studied two-sided \( H^* \)-algebras (and more general algebras) in two previous papers [4, 5]. When \( x^r = x^l \) for all \( x \) in \( A \) we have the \( H^* \)-algebras of Ambrose [1] and if we omit \( x^l \) we have the right \( H^* \)-algebra of Smiley [6]. Incidentally, in [4, Theorem 2] we proved that a proper right \( H^* \)-algebra is a two-sided \( H^* \)-algebra. So most of the theory of this paper applies to a right \( H^* \)-Algebra.

Our proof of the main result (Theorem 4) uses the technique of Ambrose [1] and the lemmas about existence of minimal two-sided projections (Theorem 3 and Lemma 6).

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2. A general theorem. The following theorem may be of an independent interest (compare with § 2 in [1]).

**Theorem 1.** The radical \( R \) of each two-sided \( H^* \)-algebra \( A \)
coincides with both the right and left annihilator of the algebra.

Proof. \( Ax = 0 \) gives \((xy, z) = (x, zy^r) = (x'x, y^r) = 0 \) for all \( y, z \in A \) so that \( xA = 0 \). Thus \( r(A) \), the right annihilator or \( A \), and \( l(A) \) coincide. Now consider \( B = r(A)^p \) which is easily seen to be a two-sided \( H^*\)-algebra which is proper in the sense that \( r(B) = l(B) = 0 \). The proof of Theorem 3.1 of [1] shows that each nonzero ideal of \( B \) contains a nonzero idempotent (see also [3], page 101). This means that \( B \cap \mathcal{R} = (0) \) since radical cannot contain idempotents [2, page 309]; thus \( \mathcal{R} = r(A) = l(A) \).

**Corollary.** The following conditions are equivalent in any two-sided \( H^*\)-algebra (each one of these conditions can be used to define a proper algebra):

(i) \( r(A) = 0 \)

(ii) \( l(A) = 0 \)

(iii) \( x^r \) is unique for each \( x \in A \)

(iv) \( x^l \) is unique for each \( x \in A \)

(v) \( A \) is semi-simple.

Proof. Equivalence of (i) and (iii) ((ii) and (iv)) can be established as in the proof of Theorem 2.1 of [1].

3. Invariant ideals. Unless otherwise stated \( A \) will denote a simple proper two-sided complex \( H^*\)-algebra. Note that both involutions \((x \rightarrow x^r\) and \( x \rightarrow x^l)\) in \( A \) are continuous (This follows from the closed graph theorem).

**Lemma 1.** If \( x, y \in A \) then \((x, y) = (y^l, x^r) = (y^r, x^l)\).

Proof. The set \( I \) of linear combinations of products of members of \( A \) is dense in \( A \) (because \( I \) is a two-sided ideal). If \( x = uv \) for some \( u, v \in A \) then \((x, y) = (uv, y) = (u, yy^r) = (y^l u, v^r) = (y^l, v^r v^r) = (y^l, x^r)\). Hence \((x, y) = (y^l, x^r)\) (and similarly \((x, y) = (y^r, x^l)\)) holds if \( x \in I \). The lemma now follows from the continuity of the involutions.

**Corollary.** If \( S \) is any subset of \( A \), then \( S^r^p = S^l^p \) and \( S^l^p = S^r^p \) (as in [4] \( S^p \) denoted the set of elements of \( A \) orthogonal to \( S \) and \( S^r \) \( (S^l) \) denotes the image of \( S \) under the involution \( x \rightarrow x^r \) \( (x \rightarrow x^l)\)).

**Lemma 2.** If \( B \) is a closed right (left) ideal of \( A \), then \( l(B) = B^r^p = B^l^p \) \( (r(B) = B^l^l = B^r^r)\).
Proof. From \((B^{r}B, A) = (B^{r}, AB^{r}) = A^{r}B^{r}, B^{r}) = (B^{r}, B^{r}) = 0\) we conclude that \(B^{r}B = 0\). Thus \(B^{r} \subset l(B)\). If \(xB = 0\), then \(0 = (xB, A) = (x, AB^{r}) = (A^{r}x, B^{r}) = (Ax, B^{r}), Ax \subset B^{r}\) by Lemma 1 of [6]. This simple means that \(l(B) \subset B^{r}\).

**Definition.** An ideal \(I\) in \(A\) is said to be **invariant** if \(I^{r} = I^{l}\).

**Lemma 3.** A closed (right, left) ideal \(I\) in \(A\) is invariant if and only if \(I^{r}\) is invariant.

**Proof.** Direct verification: \(I^{rl} = I^{rl} = I^{rl} = I^{rl}\).

**Corollary.** A closed right (left) ideal \(R\) (\(L\)) is invariant if and only if \(l(R^{r}) = l(R^{r}) (r(L^{r}) = r(L^{r}))\).

**Definition.** An idempotent in \(A\) which is both left and right self-adjoint will be called a **two-sided projection**.

**Lemma 4.** If \(e \in A\) is a left projection and \(eA\) is invariant, then \(e\) is a two-sided projection.

**Proof.** From \(Ae = Ae^{r}\) we have \(ee^{r} = e\) which shows that \(e^{r} = e\) also.

**Theorem 2.** A proper two-sided H*-algebra \(A\) is an H*-algebra if and only if each closed right (left) ideal of \(A\) is invariant.

**Proof.** In view of the first structure theorem (Theorem 1 in [4]) we may assume (without loss of generality) that \(A\) is simple. Now the condition of the theorem implies that each left projection is a right projection (Lemma 4) an vice-versa. From this it is not difficult to show that both involutions coincide. This could be done either by proving the second structure theorem (Theorem 4.3 of [1]) or by showing that the set \(S\) of all linear combinations of products of projections is dense in \(A\) (using the arguments in proofs of Lemma 8 in [4] and Theorem 1 in [5] one can show that \(S\) is a two-sided ideal).

4. Finite-dimensional algebras.

**Lemma 5.** For each right projection \(f\) in \(A\) there exist a left projection \(e \in A\) such that \((e, f - e) = 0\) and \(ef = e, fe = f\). If \(f\) is minimal then \(e\) is minimal also. A similar statement holds for a left projection.
Proof. Consider the closed right ideal \( R = \{ x - fx \mid x \in A \} = r(f) \) and write \( f = e + u \) with \( e \in R^a, u \in R \). Then by Lemma 2 in [4] \( e \) is a left projection such that \( R^a = eA \) and \( R = r(e) = \{ x \in A \mid ex = 0 \} \). Also \( (e, f - e) = (e, u) = 0, \) \( ef = e(e + u) = e \) and \( fe = f(f - u) = f \). If \( f \) is minimal then minimality of \( e \) follows from the fact that \( Af = Ae \).

REMARK. The algebra \( A \) in Lemma 5 does not have to be finite-dimensional.

THEOREM 3. Every finite-dimensional proper two-sided \( H^* \)-algebra \( A \) contains a minimal two-sided projection.

Proof. We may assume that \( A \) is simple. By Lemma 5 there exists a sequence \( \{ f_1, f_2, \ldots, f_n, \ldots \} \) of minimal right projections and a sequence \( \{ e_1, e_2, \ldots, e_n, \ldots \} \) of minimal left projections such that \( ||f_n||^2 = ||e_n||^2 + ||f_n - e_n||^2; ||e_n||^2 = ||f_{n+1}||^2 + ||e_n - f_{n+1}||^2 \) (and \( e_n f_n = e_n, f_n e_n = f_n, e_n f_{n+1} = f_{n+1}, f_{n+1} e_n = e_n \)). Also \( ||f_n|| \leq ||f_1|| \geq ||e_n|| \) for each \( n \). By the Bolzano-Weierstrass theorem there exists a subsequence \( \{ g_k \} \) of \( \{ f_n \} \) (for simplicity we write \( g_k \) instead of \( f_{n_k} \)) and some \( g \in A \) such that \( g = \lim g_k \). Then \( g \) is right self-adjoint and idempotent. From

\[
||f_1||^2 = ||f_1 - e_1||^2 + ||e_1 - f_1||^2 + ||f_2 - e_2||^2 + \cdots \\
+ ||f_n - e_n||^2 + ||e_n - f_{n+1}||^2 + ||f_{n+1}||^2
\]

and \( ||f_{n+1}|| \geq ||f_{n+p}|| \geq ||g|| \) it follows that \( ||f_n - e_n|| \to 0 \). Therefore \( g = \lim g_k \) also and so \( g \) is left self-adjoint.

It remains to show that \( g \) is minimal. If \( x \in A \) then for each \( k \) there exists a complex number \( \lambda_k \) such that \( g_k x g_k = \lambda_k g_k \) ([4], page 52 and [1], page 380). Then \( \lambda_k g_k \) tends to \( gxg \). From \( ||\lambda_k g_k - g|| = \sum ||g_k x g_k|| \leq ||g_k||^2 ||x|| \leq ||g||^2 ||x|| \) it follows that \( \lambda_k \) has a subsequence converging to some complex number \( \lambda \). Then \( gxg = \lambda g \) and so \( Ag \) is isomorphic to the complex number field, from which we may conclude that \( g \) is minimal.

Later (corollary to Theorem 4) we will see that each finite-dimensional proper simple two-sided \( H^* \)-algebra is isomorphic to a canonical algebra \( M \). In fact each such an algebra is discrete in the sense of the next definition.

5. Discrete algebras.

DEFINITION. A two-sided \( H^* \)-algebra \( A \) is said to be discrete if
each invariant ideal in $A$ contains an invariant ideal of the form $eA$
where $e$ is a left projection.

Because of Lemma 4 this definition is equivalent to the corre-
sponding definition in the introduction.

**LEMMA 6.** Each invariant closed right ideal $R$ in a discrete two-
sided $H^*$-algebra $A$ contains a minimal two-sided projection.

**Proof.** By Lemma 4 $R$ contains a two-sided projection $e$. The
set $eAe$ is a finite-dimensional proper two-sided $H^*$-algebra included in
$R$. The lemma now follows from Theorem 3.

**COROLLARY.** Each discrete proper two-sided $H^*$-algebra $A$
contains a (maximal) family \{\(g_i\)\} of mutually orthogonal minimal two-sided
projections such that $A = \sum_i g_iA = \sum_i A g_i = \sum_{i,j} g_i A g_j$.

**THEOREM 4.** Each simple discrete proper two-sided $H^*$-algebra
$A$ is isomorphic to a canonical algebra.

**Proof.** Consider the family \{\(g_i\)\} of the last corollary and select
\(g_{ij} \in g_iA g_j\) such that $g_{ij} = g_{ji}$, $g_{ik} g_{jk} = g_{ik}$ and $g_{ii} = g_i$ for each $i$, $j$, $k$
(as in [1], page 381). Then the $g_{ij}$'s are mutually orthogonal. We set\n\(t_i = ||g_i||; \) then $1 \leq t_i$ for each $i$ and also $||g_{ij}||^2 = (g_{ij}, g_{ji}) = ||g_i||^2 = t_i^2$
for each $j$ (and a fixed $i$). Also one can show that $g_{ij} = t_i^{-2} t_j^2 g_{ji}$ (note
that $(g_{ij}, g_{ji}) = (g_{ij} g_{ji}, g_{ji}) = (g_{ij}, g_{ji})$ and that $g_{ij}$ is a scalar multiple
of $g_{ji}$). Let $e_{ij} = t_i^2 t_j^2 g_{ji}$, then $(e_{ij}, e_{ij}) = t_i t_j$, $e_{ij} = (t_i/t_j) e_{ji}$ and
$e_{ji} = (t_j/t_i) e_{ij}$. The theorem now is easy to complete (see for example
the proof of Theorem 4.3 in [1]). Boundedness of the set \{\(t_i\)\} follows
from continuity of the right involutions: take a fixed $k$ and consider
\(x_i = g_{ik} x_k\), then $||x_i|| = t_i^{-2} t_k ||g_{ki}|| = t_i^{-1} t_k$ and $||x_i|| = t_k$.

**COROLLARY.** Each finite-dimensional proper simple two-sided
$H^*$-algebra is isomorphic to a canonical algebra $M$ for some finite
set $J$.

6. **Remark on the algebra $M$.** To complete the paper we show
that the canonical algebra $M$ in the introduction is discrete. For each $k$
let $e_k$ be the matrix $x_{ij} = \delta_i^j \delta_k^j$ ($\delta_i^j$ are Kronecker deltas). Then
\{\(e_i\)\} is a maximal family of mutually orthogonal minimal two-sided
projections in $M$. Let $R$ be an invariant closed right ideal in $M$. Let
$e$ in \{\(e_i\)\} be such that $eR \neq 0$. Let $R_e = (eM)^p = r(e)$; then $R_e =
R \cap (R \cap R_e)^p$ is an invariant closed nonzero right ideal (note that
$R_e = 0$ would imply $R \subset R_e = r(e)$ since $R_e$ is the orthogonal comple-
Suppose that $R_2$ is not minimal. Let $e_1, e_2$ be two orthogonal left projections in $R_2$. Let $x = \lambda e_1 + \mu e_2$ ($\lambda$, $\mu$ are scalars) be such that $(x, e) = 0$. If $xe = 0$ then $ex = 0$ and so $R_1 \cap R_2 \neq 0$ (note that $x^\perp = \lambda e_1 + \mu e_2$ belongs to $R_2$). If $xe \neq 0$ then $xeM$ contains a left projection $e_3$ ([4], Lemma 5), $e_3 = xey$ for some $y \in M$. Then $(e_3, e) = (xey, e) = (x, ey'e) = 0$ (since $ey'e$ is a scalar multiple of $e$) from which it follows that $e_3e = 0 ((e_3e, e_3e) = (e_3, e) = 0)$. But then $e_3 = 0$ since $e_3$ and $e$ are both left self-adjoint. So we see that also in this case there exists a nonzero element $z$ in $R_3 \cap R_4$. But this implies $z \in R \cap R_4$ and $z \in (R \cap R_4)^\perp$, which is impossible.

Thus $R_2$ is minimal and so it is of the form $R_2 = gM$ for some (minimal) left projection $g$.

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