

Pacific Journal of Mathematics

FACTORIZATIONS OF p -SOLVABLE GROUPS

JOHN GRIGGS THOMPSON

FACTORIZATIONS OF p -SOLVABLE GROUPS

JOHN G. THOMPSON

The object of this paper is to put in relief one of the ideas which has been very helpful in studying simple groups, viz. using factorizations of p -solvable groups to obtain information about the subgroups of a simple group which contain a given S_p -subgroup. Since the idea is so simple, it seems to deserve a simple exposition.

The group $J(\mathfrak{X})$ was introduced in [3]. In this paper, $J(\mathfrak{X})$ is again used, together with a similarly defined group, to obtain factorizations of some p -solvable groups which are of relevance in the study of simple groups.

As in [3], $m(\mathfrak{X})$ denotes the minimal number of generators of the finite group \mathfrak{X} , and $d(\mathfrak{X}) = \max\{m(\mathfrak{A})\}$, \mathfrak{A} ranging over all the abelian subgroups of \mathfrak{X} . For each nonnegative integer n , let $J_n(\mathfrak{X}) = \langle \mathfrak{A} \mid \mathfrak{A} \text{ is an abelian subgroup of } \mathfrak{X} \text{ with } m(\mathfrak{A}) \geq d(\mathfrak{X}) - n \rangle$. Thus $J_0(\mathfrak{X}) = J(\mathfrak{X})$ and $J_k(\mathfrak{X}) = \mathfrak{X}$ whenever $k \geq d(\mathfrak{X}) - 1$. Also $J_n(\mathfrak{X}) \subseteq J_{n+1}(\mathfrak{X})$ for $n = 0, 1, \dots$.

THEOREM 1. *Suppose \mathfrak{G} is a p -solvable finite group, p is a prime, and \mathfrak{G}_p is a S_p -subgroup of \mathfrak{G} . Suppose also that $O_p(\mathfrak{G}) = 1$ and that one of the following holds:*

- (a) $p \geq 5$.
- (b) $p = 3$ and $SL(2, 3)$ is not involved in \mathfrak{G} .
- (c) $p = 2$ and $SL(2, 2)$ is not involved in \mathfrak{G} .

Let $\mathfrak{H} = \bigcap_{\sigma \in \mathfrak{G}} C_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))^\sigma$. Then $\mathfrak{G} = \mathfrak{H} \cdot N_{\mathfrak{G}}(J(\mathfrak{G}_p))$ and if $p \geq 5$, then $\mathfrak{G} = \mathfrak{H} \cdot N_{\mathfrak{G}}(J_1(\mathfrak{G}_p))$. In particular, $\mathfrak{G} = C_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p)) \cdot N_{\mathfrak{G}}(J(\mathfrak{G}_p))$.

Proof. Let $\mathfrak{W}_1 = \mathbf{Z}(\mathfrak{G}_p)^{\mathfrak{G}}$, $\mathfrak{W} = \Omega_1(\mathfrak{W}_1)$. Then $\mathfrak{H} = C_{\mathfrak{G}}(\mathfrak{W}_1)$ and $\mathfrak{H} = O_p(\mathfrak{G} \text{ mod } \mathfrak{H})$. If $p \geq 5$, then since $J(\mathfrak{G}_p) \text{ char } J_1(\mathfrak{G}_p)$, it suffices to show that $J_1(\mathfrak{G}_p) \subseteq \mathfrak{H}$, while if $p \leq 3$, it suffices to show that $J(\mathfrak{G}_p) \subseteq \mathfrak{H}$.

Suppose the theorem is false and \mathfrak{G} is a minimal counterexample. Let \mathfrak{A} be an abelian subgroup of \mathfrak{G}_p , $\mathfrak{A} \not\subseteq \mathfrak{H}$, and $m(\mathfrak{A}) \geq d(\mathfrak{G}_p) - \delta$, where $\delta = 0$ if $p \leq 3$ and $\delta = 1$ if $p \geq 5$. Let $\mathfrak{K} = O_p(\mathfrak{G} \text{ mod } \mathfrak{H})$, $\mathfrak{L} = \mathfrak{K}\mathfrak{A}$. Since $\mathfrak{G}_p \cap \mathfrak{L}$ is a S_p -subgroup of \mathfrak{L} , it follows that the theorem is violated in \mathfrak{L} , so by induction, $\mathfrak{L} = \mathfrak{G}$. Minimality of \mathfrak{G} forces $\mathfrak{A}/\mathfrak{A} \cap \mathfrak{H}$ to be cyclic and forces $\mathfrak{K}/\mathfrak{H}$ to be a special q -group. On the other hand, since $m(\mathfrak{A}) \geq d(\mathfrak{G}_p) - \delta$, it follows that $|\mathfrak{W} : \mathfrak{W} \cap \mathfrak{A}| \leq p^{1+\delta}$. If $p \geq 5$, Theorem B of Hall-Higman [2] yields a contradiction, while if $p \leq 3$,

Received May 6, 1964. The author thanks the Sloan Foundation for its support.

(b) or (c) yields a contradiction, as in [3]. The proof is complete.

REMARKS. If the condition $O_{p'}(\mathfrak{G}) = 1$ is dropped, then $\mathfrak{G} = O_{p'}(\mathfrak{G})C_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))N_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$. This is so since $A(\overline{\mathfrak{G}_p}) = \overline{A(\mathfrak{G}_p)}$ where A is any one of the operations, \mathbf{Z} , \mathbf{J} , \mathbf{CZ} , \mathbf{NJ} and $-$ is any epimorphism of \mathfrak{G} with $\ker(-)$ a p' -group.

It would appear that the hypothesis of p -solvability in Theorem 1 is not the proper one and that some more general family of groups will admit exploitable factorizations. However, our meagre knowledge of simple groups makes it impossible at present to guess the shape of the factorization.

THEOREM 2. *Suppose \mathfrak{G} is a finite group, p is a prime, \mathfrak{G}_p is a S_p -subgroup of \mathfrak{G} and $p \geq 5$. Suppose also that the following hold:*

(a) *1 is the only p -signalizer of \mathfrak{G} .¹*

(b) *$C_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))$, $N_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$, and $N_{\mathfrak{G}}(\mathbf{Z}(\mathbf{J}_1(\mathfrak{G}_p)))$ are p -solvable. Then $C_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p)) \cdot N_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$ is a subgroup of \mathfrak{G} which contains every p -solvable subgroup of \mathfrak{G} which contains \mathfrak{G}_p .*

Proof. Let $\mathfrak{N}_1 = C_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))$, $\mathfrak{N}_2 = N_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$, $\mathfrak{N}_3 = N_{\mathfrak{G}}(\mathbf{Z}(\mathbf{J}_1(\mathfrak{G}_p)))$, $\mathfrak{N}_{ij} = \mathfrak{N}_i \cap \mathfrak{N}_j$. By Theorem 1 with \mathfrak{N}_2 in the role of \mathfrak{G} , we have $\mathfrak{N}_2 = \mathfrak{N}_{21}\mathfrak{N}_{23}$; and similarly, $\mathfrak{N}_3 = \mathfrak{N}_{31}\mathfrak{N}_{32}$. The factorization $\mathfrak{N}_1 = \mathfrak{N}_{12}\mathfrak{N}_{13}$ is easily obtained, as in Lemmas 24.4 and 7.7 of [1], for example, so by Lemma 8.6 of [1], $\mathfrak{N}_1\mathfrak{N}_2$ is a subgroup of \mathfrak{G} . If \mathfrak{K} is a p -solvable subgroup of \mathfrak{G} which contains \mathfrak{G}_p , then by Theorem 1, $\mathfrak{K} = (\mathfrak{K} \cap \mathfrak{N}_1)(\mathfrak{K} \cap \mathfrak{N}_2) \subseteq \mathfrak{N}_1\mathfrak{N}_2$. The proof is complete.

REMARK. It is clear that Theorem 2 may be used to shorten some of the proofs in [1] which deal with π_4 .

REFERENCES

1. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13**, (1963).
2. P. Hall and G. Higman, *The p -length of a p -soluble group, and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3), **7** (1956), 1-42.
3. J. G. Thompson, *Normal p -complements for finite groups*, Jour. of Alg., vol. 1, no. 1, pp. 43-46.

THE UNIVERSITY OF CHICAGO

¹ The subgroup \mathfrak{A} of \mathfrak{G} is a p -signalizer if and only if $|\mathfrak{A}|$ and $|\mathfrak{G} : N_{\mathfrak{G}}(\mathfrak{A})|$ are prime to p .

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California

R. M. BLUMENTHAL

University of Washington
Seattle, Washington 98105

*J. DUGUNDJI

University of Southern California
Los Angeles, California 90007

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$ 8.00; single issues, \$ 3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$ 4.00 per volume; single issues \$ 1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

* Paul A. White, Acting Editor until J. Dugundji returns.

Pacific Journal of Mathematics

Vol. 16, No. 2

December, 1966

Loren N. Argabright, <i>Invariant means on topological semigroups</i>	193
William Arveson, <i>A theorem on the action of abelian unitary groups</i>	205
John Spurgeon Bradley, <i>Adjoint quasi-differential operators of Euler type</i>	213
Don Deekard and Lincoln Kearney Durst, <i>Unique factorization in power series rings and semigroups</i>	239
Allen Devinatz, <i>The deficiency index of ordinary self-adjoint differential operators</i>	243
Robert E. Edwards, <i>Operators commuting with translations</i>	259
Avner Friedman, <i>Differentiability of solutions of ordinary differential equations in Hilbert space</i>	267
Boris Garfinkel and Gregory Thomas McAllister, Jr., <i>Singularities in a variational problem with an inequality</i>	273
Seymour Ginsburg and Edwin Spanier, <i>Semigroups, Presburger formulas, and languages</i>	285
Burrell Washington Helton, <i>Integral equations and product integrals</i>	297
Edgar J. Howard, <i>First and second category Abelian groups with the n-adic topology</i>	323
Arthur H. Kruse and Paul William Liebnitz, Jr., <i>An application of a family homotopy extension theorem to ANR spaces</i>	331
Albert Marden, I. Richards and Burton Rodin, <i>On the regions bounded by homotopic curves</i>	337
Willard Miller, Jr., <i>A branching law for the symplectic groups</i>	341
Marc Aristide Rieffel, <i>A characterization of the group algebras of the finite groups</i>	347
P. P. Saworotnow, <i>On two-sided H^*-algebras</i>	365
John Griggs Thompson, <i>Factorizations of p-solvable groups</i>	371
Shih-hsiung Tung, <i>Harnack's inequalities on the classical Cartan domains</i>	373
Adil Mohamed Yaqub, <i>Primal clusters</i>	379