HARNACK’S INEQUALITIES ON THE CLASSICAL CARTAN DOMAINS

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Recently an extensive work by L. K. Hua on harmonic analysis in Cartan domains, which are called the classical domains, has been translated into English. Here we give Harnack’s inequalities for the four main types of Cartan domains treated by Hua.

Harnack’s inequality on a type of Cartan domain was obtained [6] for the case of square matrix spaces. Some of these inequalities are application and extension of the results of [6]. I am grateful to Professor J. Mitchell for her encouragement and comments on writing this paper.

Let $z$ be a matrix of complex entries, $z^* = z'$ the complex conjugate of the transposed matrix $z'$ and $I$ the identity matrix. Also $H > 0$ means that a hermitian matrix $H$ is positive definite. The first three types of Cartan domains are defined by $D_k = \{z : I - zz^* > 0\}$, $k = 1, 2, 3$, where for $D_1 = D_1(m, n)$, $z$ is an $(m, n)$ matrix (Since the conditions $I - zz^* > 0$ and $I - z^*z > 0$ are equivalent we assume for definiteness that $m \leq n$), for $D_2 = D_2(n)$, $z$ is a symmetric matrix of order $n$ and for $D_3 = D_3(n)$, $z$ is a skew-symmetric matrix of order $n$. The fourth type, $D_4 = D_4(1, n)$, is the set of all $(1, n)$ matrices, or $n$-dimensional vectors ($n > 2$), of complex numbers satisfying the conditions

\[(1) \quad 1 + |zz'|^2 - 2zz^* > 0, \quad |zz'| < 1.\]

It is known that each of the domain $D_k$ possesses a distinguished boundary [1] or characteristic manifold [2, p. 6] $C_k : C_k = C_k(m, n)$ consists of the $(m, n)$ matrices $u$ satisfying the condition $uu^* = I$. $C_2 = C_2(n)$ consists of all symmetric unitary matrices of order $n$. $C_3 = C_3(n)$ [2, p. 71] consists of all matrices $u$ of the form $u = w'sw$, where $w$ is an $n$-rowed unitary matrix and

\[(2) \quad s_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for even } n \]

\[(2) \quad s_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + 0 \quad \text{for odd } n.\]

$C_4 = C_4(1, n)$ consists of all $(1, n)$ matrices $u$ of the form

\[(3) \quad u = e^{i\theta}x, \quad xx' = 1, \quad 0 \leq \theta \leq \pi\]

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where $x$ is a real vector.

We denote the Poisson kernel on $D_k$ by $P_k(z, u)$, its explicit forms [3, 4] being given in following sections. The following Dirichlet problem is solved on $D_k$ [2, 3]: Given a real-valued continuous function $f(u)$ on $C_k$, the Poisson integral

$$\phi(z) = \int_{D_k} f(u) P_k(z, u)\,d\hat{u}$$

where $d\hat{u}$ is Euclidean volume element on $C_k$, gives the unique function which is harmonic, in the sense given in [3], on the closure of $D_k$ and takes the given boundary values $f(u)$ on $C_k$. We obtain Harnack's inequality on each $D_k$ as a consequence of evaluating upper and lower bounds for $P_k(z, u)$.

2. Harnack's inequalities on $D_1$ and $D_2$. The Poisson kernel on $D_1$ [3, 4] is

$$P_1(z, u) = \frac{1}{V_1} \frac{[\det (I - zz^*)]^n}{|\det (I - zu^*)|^n}$$

where $z \in D_1$, $u \in C_1$ and $V_1$ is the Euclidean volume of $C_1$. It is known [4, p. 411] that $P_1 > 0$ and $\int_{D_1} P_1 \,d\hat{u} = 1$. In [6], we obtained bounds of the Poisson kernel (5) for the case $m = n$:

$$\frac{1}{V_1} \prod_{k=1}^{n} \left( \frac{1 - r_k}{1 + r_k} \right)^n \leq P_1(z, u) \leq \frac{1}{V_1} \prod_{k=1}^{n} \left( \frac{1 + r_k}{1 - r_k} \right)^n$$

where $z = u_0 R v_0 \in D_1(n, n)$, $u \in C_1(n, n)$, $u_0$ and $v_0$ are unitary matrices and $R = (\delta_{jk} r_k)$ is a diagonal matrix with $0 \leq r_k < 1$ for $k = 1, 2, \ldots, n$. In order to obtain (6), we proved the inequality

$$\prod_{k=1}^{n} (1 - r_k)^2 \leq |\det (I - z u^*)|^2 = |\det ((v - R)(v^* - R))| \leq \prod_{k=1}^{n} (1 + r_k)^2$$

where $u, v \in C_1(n, n)$, is any unitary matrix of order $n$, $z \in D_1(n, n)$ and $v = u_0 v_0^* v_0^*$. 

For $z \in D_2(m, n)$ there are unitary matrices $u_0$ of order $m$ and $v_0$ of order $n$ such that $z = u_0 R v_0$, where $R$ is a diagonal submatrix $(\delta_{jk} r_k)$ of order $m$ and $0$ is the $(m, n-m)$ zero submatrix [3, p. 1049]. Hence

$$\det (I^{(m)} - zz^*) = \det (I^{(m)} - RR^*) = \prod_{k=1}^{m} (1 - r_k^2).$$

For the denominator of (5) we have

$$\det (I - z u^*) = \det \left( I - \begin{pmatrix} z \\ 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}^* \right)$$
where 0 is the \((n-m, n)\) zero submatrix and \(u_i\) is chosen so that \(U \equiv (u_i)\) becomes a unitary matrix of order \(n\) \([5, p. 190]\). If we also denote \(Z \equiv \begin{pmatrix} z \\ 0 \end{pmatrix}\), then \(Z \in D_i(n, n)\) and \(U \in C_i(n, n)\) for the case \(m = n\). Therefore from (7)

\[
\max_{U \in C_i} \left\{ \left| \det \left( I - ZU^* \right) \right|^2 \right\} \leq \prod_{k=1}^{m} (1 + r_k)^2 = \prod_{k=1}^{m} (1 + r_k)^2
\]

and

\[
\min_{U \in C_i} \left\{ \left| \det \left( I - ZU \right) \right|^2 \right\} \geq \prod_{k=1}^{m} (1 - r_k)^2 = \prod_{k=1}^{m} (1 - r_k)^2
\]

since \(r_{m+1} = \cdots = r_n = 0\) in \(Z\). Finally from (9), (10) and (11) we obtain

\[
\prod_{k=1}^{m} (1 - r_k)^2 \leq \left| \det \left( I - zU^* \right) \right|^2 \leq \prod_{k=1}^{m} (1 + r_k)^2
\]

where \(z = u_0(R, 0)v_0 \in D_i(m, n)\) and \(u \in C_i(m, n)\). This and (8) lead to

\[
\frac{1}{V_1} \prod_{k=1}^{m} \left( \frac{1 - r_k}{1 + r_k} \right)^n \leq P_i(z, u) \leq \frac{1}{V_1} \prod_{k=1}^{m} \left( \frac{1 + r_k}{1 - r_k} \right)^n
\]

for \(z \in D_i\) and \(u \in C_i\). Furthermore from (4) we obtain Harnack’s inequality

\[
\prod_{k=1}^{m} \left( \frac{1 - r_k}{1 + r_k} \right)^n \phi(0) \leq \phi(z) \leq \prod_{k=1}^{m} \left( \frac{1 + r_k}{1 - r_k} \right)^n \phi(0).
\]

For \(z \in D_3(n)\) it is known that there is a unitary matrix \(u_0\) such that \(u_0zv_0^* = (\delta_j k_j r_k)\) where \(r_1, \ldots, r_n\) are the positive square roots of the characteristic roots of \(z\). Since \(z \in D_3(n)\) implies \(z \in D_i(n, n)\) and the characteristic manifold \(C_2(n)\) is a subset of \(C_i(n, n)\) we know that (7) and (8) hold for \(z \in D_3(n)\) and \(u \in C_i(n, n)\), and it can be seen from the Poisson kernel

\[
P_3(z, u) = \frac{1}{V_3} \frac{[\det (I - zz^*)]^{(n+1)/2}}{[\det (I - zu^*)]^{n+1}}
\]

that Harnack’s inequality on \(D_i(n)\) is

\[
\prod_{k=1}^{n} \left( \frac{1 - r_k}{1 + r_k} \right)^{(n+1)/2} \phi(0) \leq \phi(z) \leq \prod_{k=1}^{n} \left( \frac{1 + r_k}{1 - r_k} \right)^{(n+1)/2} \phi(0).
\]

3. Harnack’s inequality on \(D_3\). The Poisson kernel on \(D_3\) is

\[
P_3(z, u) = \frac{1}{V_3} \frac{[\det (I - zz^*)]^a}{[\det (I - zu^*)]^{2a}}
\]
where \( a = (n - 1)/2 \) for even \( n \) and \( a = n/2 \) for odd \( n \). For \( z \in D_3(n) \), it is known [2, p. 67] that there is a unitary matrix \( u_0 \) such that

\[
(13) \quad u_0 z u_0^* = s = \begin{pmatrix}
0 & r_1 & \cdots & r_b \\
-r_1 & 0 & \cdots & 0 \\
0 & r_2 & \cdots & 0 \\
& -r_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

for even \( n \)

\[
= \begin{pmatrix}
0 & r_1 & \cdots & r_b \\
-r_1 & 0 & \cdots & 0 \\
0 & r_2 & \cdots & 0 \\
& -r_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix} + \begin{pmatrix}
0 & r_{[b]} \\
-r_{[b]} & 0
\end{pmatrix}
\]

for odd \( n \)

where \( b = n/2 \) and \( r_1, r_2, \cdots \) are characteristic roots of \( zz^* \). Thus for even \( n \)

\[
u_0 zz^* u_0^* = [r_1^2, r_2^2, \cdots, r_b^2].
\]

And for all \( n \)

\[
det(I - zz^*) = det(I - u_0 zz^* u_0^*) = \prod_{k=1}^{b} (1 - r_k^2)^2.
\]

We denote the right hand side of (2) by \( u_1 \) with the change that the last term 0 be replaced by 1 in the case of odd \( n \). Hence \( u_1 \) is a unitary matrix of order \( n \) and from (13) \( su_1^* \) is a diagonal matrix \( R = [r_1, r_1, \cdots, r_{[b]}, r_{[b]}] \) for even \( n \) and \( R_0 = [R, 0] \) for odd \( n \). First we consider the case of even \( n \). We notice from (2) that \( u \in C_3 \) is skew-symmetric unitary. Therefore for \( z \in D_3 \) and \( u \in C_3 \)

\[
|det(I - z u^*)| = |det(I - u_0^* su_1^* u_1 u_0 u^*)| = |det(v - R)|
\]

where \( v = u_0 uu_0^* u_1^* \) is a unitary matrix of order \( n \). Hence from (7) we have

\[
(14) \quad \prod_{k=1}^{b} (1 - r_k)^2 \leq |det(I - z u^*)| \leq \prod_{k=1}^{b} (1 + r_k)^2.
\]

For the case of odd \( n \), we notice from (2) that \( det(u) = 0 \). It is known [3, p. 1073] that any \( v \in C_3(n + 1) \) can be written in the form

\[
v = \begin{pmatrix}
u & w'h' \\
-hw & 0
\end{pmatrix}, \quad h = (0, \cdots, 0, e^{i\theta})
\]

where \( u = w'sw \in C_3(n) \). Hence for odd \( n \)

\[
det(I - z u^*) = det \left( I^{(n+1)} - \begin{pmatrix}
z & 0 \\
0 & w'h'
\end{pmatrix}
\right)
\]

and from (13)

\[
\begin{pmatrix}
u_0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
z & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
u_0 & 0 \\
0 & 1
\end{pmatrix}' = R + \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

Since \( n + 1 \) is even we can apply (14) and obtain
\[ \prod_{k=1}^{[k]} (1 - r_k)^2 \leq |\det (I - zu^*)| \leq \prod_{k=1}^{[k]} (1 + r_k)^2. \]

As the result, this inequality is good for both odd and even \( n \). Thus, as in § 2 we can obtain both bounds for the Poisson kernel \( P_8(z, u) \) and finally Harnack's inequality on \( D_3(n) \) as
\[
\prod_{k=1}^{[k]} \left( \frac{1 - r_k}{1 + r_k} \right)^{2a} \phi(0) \leq \phi(z) \leq \prod_{k=1}^{[k]} \left( \frac{1 + r_k}{1 - r_k} \right)^{2a} \phi(0)
\]
where \( b = n/2 \) and \( a = (n - 1)/2 \) for even \( n \) and \( a = n/2 \) for odd \( n \).

4. Harnack's inequality on \( D_4 \). The Poisson kernel on \( D_4 \) is
\[
P_4(z, u) = \frac{1}{V_4} \frac{(1 + |zz'|^2 - 2zz^*)^{n/2}}{|1 + zz'u'u' - 2zz^*|^n}
\]
where \( z \in D_4 \) and \( u \in C_4 \). For every fixed \( z \in D_4 \) there is a real orthogonal matrix \( t \) such that [3, p. 1037]
\[
z = (z_1, z_2, 0, \cdots, 0)t.
\]
Thus we have
\[
1 + |zz'|^2 - 2zz^* = 1 + |z_1|^2 + |z_2|^2 - 2(|z_1|^2 + |z_2|^2).
\]
Here by denoting \( z_1 - iz_2 = w_1 = r_1e^{i\theta_1} \) and \( z_1 + iz_2 = w_2 = r_2e^{i\theta_2} \), from (1)
\[
0 < 1 + |zz'|^2 - 2zz^* = (1 - |w_1|^2)(1 - |w_2|^2) = (1 - r_1^2)(1 - r_2^2)
\]
and
\[
0 < 1 - |zz'| = 1 - |z_1|^2 + |z_2|^2 = 1 - |w_1w_2| = 1 - r_1r_2,
\]
hence
\[
0 \leq r_k < 1, \quad k = 1, 2.
\]
Next, from \( z \in D_4 \) in (15) and \( u \in C_4 \) in (3)
\[
zz'u'u' = (z_1^2 + z_2^2)e^{-2i\theta} = w_1w_2e^{-2i\theta}
\]
and
\[
2zu^* = 2(z_1, z_2, 0, \cdots, 0)t(x')e^{-i\theta}.
\]
By denoting \( t = (t_{jk}) \) we have \( tx' = (a_1, a_2, \cdots, a_n)' \) where \( a_j = \sum_{k=1}^n t_{jk}x_k \) and since
\[
\sum_{j=1}^n a_j^2 = (tx')(tx') = xt'tx' = xx' = 1
\]
we have \( a_1^2 + a_2^2 \leq 1 \). Now with \( z_1 = (w_1 + w_2)/2 \) and \( z_2 = i(w_1 - w_2)/2 \), we have \( 2zu* = (aw_1 + \bar{a}w_2)e^{-i\theta} \) where \( a = a_1 + ia_2 \) and \( |a|^2 \leq 1 \). Thus

\[
(17) \quad 1 + zz'u' - 2zu* = 1 + w_1w_2e^{-2i\theta} - (aw_1 + \bar{a}w_2)e^{-i\theta}.
\]

We wish to find upper and lower bounds for the absolute values of expression (17). We consider the image of the closed unit disk \( |a| \leq 1 \) under the mapping \( f(a) = aw_1 + \bar{a}w_2 \) for \( a = re^{i\alpha} \). Here \( f(a) \) can be written in the form

\[
f(a) = r[r_1(e^{i\beta} + e^{-i\beta}) + (r_2 - r_1)e^{-i\beta}]e^{i(\theta_1 + \theta_2)/2}
\]

where \( \beta = \alpha + (\theta_1 - \theta_2)/2 \). For the case \( r_1 = r_2 \), \( f(a) \) maps the closed unit disk onto a line segment of length \( 4r_1 \). When \( r_1 \neq r_2 \), the image is a simple closed connected region. Furthermore the image of the unit circle is the line segment when \( r_1 = r_2 \) and is the boundary of the region when \( r_1 \neq r_2 \). Hence from the fact that \( 1 + zz'u' - 2zu* \neq 0 \) [3, p. 1079] we know that the maximum and the minimum of the absolute values of (17) can be found in the case \( |a| = 1 \). Thus

\[
1 + w_1w_2e^{-2i\theta} - (aw_1 + \bar{a}w_2)e^{-i\theta} = (1 - aw_1e^{-i\theta})(1 - \bar{a}w_2e^{-i\theta})
\]

and therefore

\[
(1 - r_1)(1 - r_2) \leq |1 + zz'u' - 2zu*| \leq (1 + r_1)(1 + r_2).
\]

This and (16) give us

\[
\frac{1}{V_4} \prod_{k=1}^{2} \left( \frac{1 - r_k}{1 + r_k} \right)^{n/2} \leq P_4(z, u) \leq \frac{1}{V_4} \prod_{k=1}^{2} \left( \frac{1 + r_k}{1 - r_k} \right)^{n/2}
\]

and the corresponding Harnack's inequality

\[
\prod_{k=1}^{2} \left( \frac{1 - r_k}{1 + r_k} \right)^{n/2} \phi(0) \leq \phi(z) \leq \prod_{k=1}^{2} \left( \frac{1 + r_k}{1 - r_k} \right)^{n/2} \phi(0).
\]

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